# BellB



## Notations

### Traditional name

Bell numbers

### Traditional notation

$B_n$

### *Mathematica* StandardForm notation

BellB[n]

## Primary definition

04.23.02.0001.01

$B_n = n!\,\left([t^n]\, e^{e^t-1}\right) /;\, n \in \mathbb{N}$

$B_n$ is the number of partitions of a set with n elements (the number of ways a set of $n$ elements can be partitioned into nonempty subsets). Not to be confused with the Bernoulli number, which is also commonly denoted $B_n$.

Examples: 1) There are five ways the elements $\{a, b, c\}$ can be partitioned: $\{\{a, b, c\}\}$; $\{\{a, b\}, \{c\}\}$, $\{\{a, c\}, \{b\}\}$, $\{\{a\}, \{b, c\}\}$, and $\{\{a\}, \{b\}, \{c\}\}$; so $B_3 == \mathcal{S}_3^{(3)} + \mathcal{S}_3^{(2)} + \mathcal{S}_3^{(1)} == 1 + 3 + 1 == 5$.

2) There are 15 ways the elements $\{a, b, c, d\}$ can be partitioned: $\{\{a, b, c, d\}\}$;$\{\{a, b\}, \{c, d\}\}$,$\{\{a, c\}, \{b, d\}\}$,$\{\{a, d\}, \{b, c\}\}$,$\{\{a, b, c\}, \{d\}\}$,$\{\{a, b, d\}, \{c\}\}$,$\{\{a, c, d\}, \{b\}\}$,$\{\{b, c, d\}, \{a\}\}$; $\{\{a, b\}, \{c\}, \{d\}\}$,$\{\{a, c\}, \{b\}, \{d\}\}$,$\{\{a, d\}, \{b\}, \{c\}\}$,$\{\{b, c\}, \{a\}, \{d\}\}$,$\{\{b, d\}, \{a\}, \{c\}\}$,$\{\{c, d\}, \{a\}, \{b\}\}$, and $\{\{a\}, \{b\}, \{c\}, \{d\}\}$; so $B_4 == 1 + 7 + 6 + 1 == 15$.

## Specific values

### Values at fixed points

04.23.03.0001.01

$B_0 = 1$

04.23.03.0002.01

$B_1 = 1$

04.23.03.0003.01

$B_2 = 2$

04.23.03.0004.01

$$B_3 = 5$$

04.23.03.0005.01

$$B_4 = 15$$

04.23.03.0006.01

$$B_5 = 52$$

04.23.03.0007.01

$$B_6 = 203$$

04.23.03.0008.01

$$B_7 = 877$$

04.23.03.0009.01

$$B_8 = 4140$$

04.23.03.0010.01

$$B_9 = 21\,147$$

04.23.03.0011.01

$$B_{10} = 115\,975$$

## General characteristics

### Domain and analyticity

$B_n$ is a nonanalytical function which is defined only for nonnegative integer $n$.

04.23.04.0001.01

$$n \longrightarrow B_n :: \mathbb{Z} \longrightarrow \mathbb{Z}$$

### Symmetries and periodicities

#### Symmetry

No symmetry

#### Periodicity

No periodicity

## Series representations

### Generalized power series

04.23.06.0001.01

$$B_n = \sum_{k=1}^{n} \frac{k^n}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} /; n \in \mathbb{N}^+$$

04.23.06.0002.01

$$B_n = \left\lceil \frac{1}{e} \sum_{k=1}^{2n} \frac{k^n}{k!} \right\rceil /; n \in \mathbb{N}^+$$

### Asymptotic series expansions

04.23.06.0003.01

$$B_n \propto \frac{1}{\sqrt{n}} \left(\frac{n}{W(n)}\right)^{n+\frac{1}{2}} e^{\frac{n}{W(n)}-n-1} /; (n \to \infty)$$

04.23.06.0004.01

$$B_n \propto \frac{1}{\sqrt{n}} e^{W(n)\left(n+\frac{1}{2}\right)} e^{-n+e^{W(n)}-1} /; (n \to \infty)$$

04.23.06.0005.01

$$B_n \propto \frac{n!}{\sqrt{2\pi W(n)^2 e^{W(n)}}} e^{e^{W(n)}-1} W(n)^{-n} /; (n \to \infty)$$

04.23.06.0006.01

$$B_n \propto n^n \log^{-n}(n)\, e^{\frac{n\log^2(\log(n))}{2\log^2(n)}-n+\frac{n(\log(\log(n))+1)}{\log(n)}} /; (n \to \infty)$$

04.23.06.0007.01

$$\log(B_n) \propto n\left(\log(n) - \log(\log(n)) - 1 + \frac{\log(\log(n))+1}{\log(n)} + \frac{1}{2}\left(\frac{\log(\log(n))}{\log(n)}\right)^2\right) /; (n \to \infty)$$

### Other series representations

04.23.06.0008.01

$$B_n = \sum_{k=0}^{n} \mathcal{S}_n^{(k)} /; n \in \mathbb{N}$$

04.23.06.0009.01

$$B_n = e^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!} /; n \in \mathbb{N}$$

## Integral representations

### On the real axis

04.23.07.0001.01

$$B_n = \frac{2\,n!}{\pi\, e} \int_0^{\pi} e^{e^{\cos(t)}\cos(\sin(t))} \sin\left(e^{\cos(t)}\sin(\sin(t))\sin(n\,t)\right) dt$$

## Limit representations

04.23.09.0001.01

$$B_n = \lim_{z \to 0} \frac{\partial^n e^{e^z-1}}{\partial z^n}$$

## Generating functions

04.23.11.0001.01

$$B_n = n!\left([t^n]\, e^{e^t-1}\right) /;\, n \in \mathbb{N}$$

# Identities

## Functional identities

04.23.17.0001.01

$$B_{n+1} = \sum_{k=0}^{n} B_k \binom{n}{k} /;\, n \in \mathbb{N}$$

## Identities involving determinants

04.23.17.0002.01

$$\left| (B_{k+l})_{0\le k\le n \atop 0\le l\le n} \right| = \prod_{k=1}^{n} k!$$

# Complex characteristics

## Real part

04.23.19.0001.01

$$\operatorname{Re}(B_n) = B_n$$

## Imaginary part

04.23.19.0002.01

$$\operatorname{Im}(B_n) = 0$$

## Absolute value

04.23.19.0003.01

$$|B_n| = B_n$$

## Argument

04.23.19.0004.01

$$\arg(B_n) = 0$$

## Conjugate value

04.23.19.0005.01

$$\overline{B_n} = B_n$$

## Signum value

04.23.19.0006.01

$$\operatorname{sgn}(B_n) = 1$$

# Summation

### Finite summation

04.23.23.0001.01

$$\sum_{k=0}^{n} \binom{n}{k} B_k = B_{n+1}$$

### Infinite summation

04.23.23.0002.01

$$\sum_{n=0}^{\infty} \frac{B_n z^n}{n!} = e^{e^z - 1}$$

## Representations through more general functions

### Through other functions

04.23.26.0001.01

$$B_n = \sum_{k=0}^{n} \mathcal{S}_n^{(k)} \; /; \; n \in \mathbb{N}$$

## Representations through equivalent functions

### With related functions

04.23.27.0001.01

$$B_n = B_n(1)$$

## Inequalities

04.23.29.0001.01

$$B_n \geq 1$$

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