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BernoulliB2

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Notations

Traditional name

Bernoulli polynomial

Traditional notation

$B_n(z)$

Mathematica StandardForm notation

`BernoulliB[n, z]`

Primary definition

05.14.02.0001.01

$$B_n(z) = n! \left([t^n] \frac{t e^{z t}}{e^t - 1} \right) /; n \in \mathbb{N}$$

Specific values

Specialized values

For fixed *n*

05.14.03.0001.01

$$B_n(0) = B_n$$

05.14.03.0002.01

$$B_n(0) = -\frac{2 n!}{(2 \pi)^n} \cos\left(\frac{n \pi}{2}\right) \zeta(n) /; n - 1 \in \mathbb{N}^+$$

05.14.03.0003.01

$$B_{2n}\left(\frac{1}{6}\right) = \frac{1}{2} \left(1 - 2^{1-2n}\right) \left(1 - 3^{1-2n}\right) B_{2n} /; n \in \mathbb{N}$$

05.14.03.0004.01

$$B_n\left(\frac{1}{4}\right) = -n E_{n-1} 4^{-n} - 2^{-n} \left(1 - 2^{1-n}\right) B_n /; n \in \mathbb{N}^+$$

05.14.03.0005.01

$$B_{2n}\left(\frac{1}{3}\right) = -\frac{1}{2} \left(1 - 3^{1-2n}\right) B_{2n} /; n \in \mathbb{N}$$

05.14.03.0006.01

$$B_n\left(\frac{1}{2}\right) = -\left(1 - 2^{1-n}\right) B_n$$

05.14.03.0007.01

$$B_{2n}\left(\frac{2}{3}\right) = -\frac{1}{2}\left(1 - 3^{1-2n}\right) B_{2n} \text{ /; } n \in \mathbb{N}$$

05.14.03.0008.01

$$B_n\left(\frac{3}{4}\right) = (-1)^{n-1}\left(2^{-n}\left(1 - 2^{1-n}\right) B_n + 4^{-n}\, n\, E_{n-1}\right) \text{ /; } n \in \mathbb{N}^+$$

05.14.03.0009.01

$$B_{2n}\left(\frac{5}{6}\right) = \frac{1}{2}\left(1 - 2^{1-2n}\right)\left(1 - 3^{1-2n}\right) B_{2n} \text{ /; } n \in \mathbb{N}$$

05.14.03.0010.01

$$B_n(1) = B_n \text{ /; } n - 1 \in \mathbb{N}^+$$

05.14.03.0012.01

$$B_n(m) = B_n - \delta_{m-1}\,\delta_{n-1} - \delta_m\,\delta_{n-1} + \delta_{n-1} + n \sum_{k=1}^{m-1} k^{n-1} \text{ /; } m \in \mathbb{N}$$

05.14.03.0013.01

$$B_n\left(\frac{p}{q}\right) = \frac{n\, i^n}{(2\pi q)^n} \sum_{k=0}^{q-1} c_n\left(\frac{k}{q}\right) \exp\left(\frac{2\pi i\, p\, k}{q}\right) \text{ /; }$$

$$c_n(z) = \frac{\partial^n \log(\sin(\pi z))}{\partial z^n} \bigwedge c_n(0) = -2\,\delta_{n \bmod 2,0}\,\zeta(n)\,(n-1)! \bigwedge n - 1 \in \mathbb{N}^+ \bigwedge p \in \mathbb{N} \bigwedge q \in \mathbb{N}^+ \bigwedge p \le q$$

05.14.03.0014.01

$$B_n\left(\frac{p}{q}\right) = -\frac{2\, n!}{(2\pi q)^n} \sum_{k=1}^{q} \zeta\left(n, \frac{k}{q}\right) \cos\left(\frac{2\, k\, p\, \pi}{q} - \frac{\pi n}{2}\right) \text{ /; } n - 1 \in \mathbb{N}^+ \land p \in \mathbb{N} \land q \in \mathbb{N}^+ \land p \le q$$

For fixed *z*

05.14.03.0015.01

$$B_0(z) = 1$$

05.14.03.0016.01

$$B_1(z) = z - \frac{1}{2}$$

05.14.03.0017.01

$$B_2(z) = z^2 - z + \frac{1}{6}$$

05.14.03.0018.01

$$B_3(z) = z^3 - \frac{3 z^2}{2} + \frac{z}{2}$$

05.14.03.0019.01

$$B_4(z) = z^4 - 2 z^3 + z^2 - \frac{1}{30}$$

05.14.03.0020.01

$$B_5(z) = z^5 - \frac{5z^4}{2} + \frac{5z^3}{3} - \frac{z}{6}$$

05.14.03.0021.01

$$B_6(z) = z^6 - 3z^5 + \frac{5z^4}{2} - \frac{z^2}{2} + \frac{1}{42}$$

05.14.03.0022.01

$$B_7(z) = z^7 - \frac{7z^6}{2} + \frac{7z^5}{2} - \frac{7z^3}{6} + \frac{z}{6}$$

05.14.03.0023.01

$$B_8(z) = z^8 - 4z^7 + \frac{14z^6}{3} - \frac{7z^4}{3} + \frac{2z^2}{3} - \frac{1}{30}$$

05.14.03.0024.01

$$B_9(z) = z^9 - \frac{9z^8}{2} + 6z^7 - \frac{21z^5}{5} + 2z^3 - \frac{3z}{10}$$

05.14.03.0025.01

$$B_{10}(z) = z^{10} - 5z^9 + \frac{15z^8}{2} - 7z^6 + 5z^4 - \frac{3z^2}{2} + \frac{5}{66}$$

General characteristics

Domain and analyticity

$B_n(z)$ is a polynomial of z and as such an analytical function of z. $B_n(z)$ is defined in the whole complex z-plane and for $n \in \mathbb{N}$.

05.14.04.0001.01

$$(n * z) \longrightarrow B_n(z) :: (\mathbb{Z} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

05.14.04.0002.01

$$B_n(\bar{z}) = \overline{B_n(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

The function $B_n(z)$ has a pole of order n at $z = \tilde{\infty}$.

05.14.04.0003.01

$$\mathcal{Sing}_z(B_n(z)) = \{\{\tilde{\infty}, n\}\}$$

Branch points

With respect to *z*

The function $B_n(z)$ does not have branch points.

05.14.04.0004.01

$$\mathcal{BP}_z(B_n(z)) == \{\}$$

Branch cuts

With respect to *z*

The function $B_n(z)$ does not have branch cuts.

05.14.04.0005.01

$$\mathcal{BC}_z(B_n(z)) == \{\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

05.14.06.0012.01

$$B_n(z) \propto B_n(z_0) + n\,B_{n-1}(z_0)\,(z-z_0) + \frac{1}{2}(n-1)\,n\,B_{n-2}(z_0)\,(z-z_0)^2 + \ldots \;/;\; (z \to z_0)$$

05.14.06.0013.01

$$B_n(z) \propto B_n(z_0) + n\,B_{n-1}(z_0)\,(z-z_0) + \frac{1}{2}(n-1)\,n\,B_{n-2}(z_0)\,(z-z_0)^2 + O\big((z-z_0)^3\big)$$

05.14.06.0014.01

$$B_n(z) == \sum_{k=0}^{n} \frac{(n-k+1)_k\,B_{n-k}(z_0)}{k!}\,(z-z_0)^k$$

05.14.06.0015.01

$$B_n(z) \propto B_n(z_0)\,(1 + O(z-z_0))$$

Expansions at $z == 0$

For the function itself

05.14.06.0016.01

$$B_n(z) \propto B_n + n\,B_{n-1}\,z + \frac{(n-1)\,n}{2}\,B_{n-2}\,z^2 + \ldots \;/;\; (z \to 0)$$

05.14.06.0005.01

$$B_n(z) \propto \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\,z^k \;/;\; (z \to 0)$$

05.14.06.0006.01

$$B_n(z) \propto B_n\,(1+O(z))\;/;\;(z\to 0)$$

Exponential Fourier series

05.14.06.0002.01

$$B_n(x) == -\frac{n!}{(2\pi i)^n}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{e^{2ik\pi x}}{k^n}\;/;\;0<x<1\bigwedge n\in\mathbb{N}^+$$

05.14.06.0001.01

$$B_n(x) == -\frac{2\,n!}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{1}{k^n}\cos\left(2\pi k x-\frac{\pi n}{2}\right)\;/;\;0<x<1\bigwedge n\in\mathbb{N}^+$$

05.14.06.0003.01

$$B_{2n-1}(x) == \frac{(-1)^n\,2\,(2n-1)!}{(2\pi)^{2n-1}}\sum_{k=1}^{\infty}\frac{\sin(2\pi k x)}{k^{2n-1}}\;/;\;-1\leq x\leq 1\bigwedge n-1\in\mathbb{N}^+$$

05.14.06.0004.01

$$B_{2n}(x) == \frac{(-1)^{n-1}\,2\,(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos(2\pi k x)}{k^{2n}}\;/;\;-1\leq x\leq 1\bigwedge n\in\mathbb{N}^+$$

Asymptotic series expansions

05.14.06.0007.02

$$B_n(z) \propto z^n\sum_{k=0}^{n}\binom{n}{k}B_k\,z^{-k}\;/;\;(|z|\to\infty)$$

05.14.06.0008.02

$$B_n(z) \propto z^n\left(1+O\left(\frac{1}{z}\right)\right)$$

05.14.06.0009.02

$$B_n(z) \propto z^n\left(1-\frac{n}{2z}+O\left(\frac{1}{z^2}\right)\right)\;/;\;(|z|\to\infty)$$

05.14.06.0010.02

$$B_n(z) \propto z^n\left(1-\frac{n}{2z}+\frac{(n-1)\,n}{12\,z^2}+O\left(\frac{1}{z^3}\right)\right)\;/;\;(|z|\to\infty)$$

05.14.06.0017.01

$$B_n(z) \propto z^n\;/;\;(|z|\to\infty)$$

Other series representations

05.14.06.0011.01

$$B_n(z) == \sum_{k=0}^{n}\binom{n}{k}B_{n-k}\,z^k$$

Integral representations

On the real axis

Of the direct function

05.14.07.0001.01

$$B_n(z) = (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor} n \int_0^\infty \frac{t^{n-1}}{\cosh(2\pi t) - \cos(2\pi z)} \left((-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \cos\left(\frac{\pi n}{2} + 2\pi z\right) - \left(2 \left\lfloor \frac{n}{2} \right\rfloor - n + 1\right) e^{-2\pi t} \right) dt \;/;\; 0 < \operatorname{Re}(z) < 1 \bigwedge n \in \mathbb{N}^+$$

05.14.07.0002.01

$$B_n(x) = -\frac{2n}{(2\pi)^n} \int_0^1 \frac{1}{t^2 - 2\cos(2x\pi)\,t + 1} \left(\cos\left(2x\pi - \frac{\pi n}{2}\right) - t\cos\left(\frac{\pi n}{2}\right) \right) \log^{n-1}\left(\frac{1}{t}\right) dt \;/;\; 0 < x < 1 \bigwedge n \in \mathbb{N}^+$$

05.14.07.0003.01

$$B_{2n}(x) = \frac{(-1)^n\, 2n\,(2n-1)}{(2\pi)^{2n}} \int_0^1 \frac{\log^{2n-2}(t) \log\left(t^2 - 2\cos(2\pi x)\,t + 1\right)}{t} dt \;/;\; 0 \le x \le 1 \bigwedge n \in \mathbb{N}^+$$

05.14.07.0004.01

$$B_{2n+1}(x) = \frac{(-1)^{n+1}\, 2\,(2n+1) \sin(2\pi x)}{(2\pi)^{2n+1}} \int_0^1 \frac{\log^{2n}(t)}{t^2 - 2\cos(2\pi x)\,t + 1} dt \;/;\; 0 \le x \le 1 \wedge n \in \mathbb{N}$$

Generating functions

05.14.11.0001.01

$$B_n(z) = n! \left([t^n] \frac{t\, e^{zt}}{e^t - 1} \right) /;\; n \in \mathbb{N}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.14.16.0001.01

$$B_n(1+z) = n\, z^{n-1} + B_n(z)$$

05.14.16.0002.01

$$B_n(1+z) = \sum_{k=0}^{n} \binom{n}{k} B_k(z)$$

05.14.16.0003.01

$$B_n(1-z) = (-1)^n B_n(z)$$

05.14.16.0004.01

$$B_n(-z) = (-1)^n \left(B_n(z) + n\, z^{n-1}\right)$$

05.14.16.0005.01

$$B_n(z-1) = B_n(z) - n\,(z-1)^{n-1}$$

05.14.16.0006.01

$$B_n(z+m) = B_n(z) + n \sum_{k=0}^{m-1} (k+z)^{n-1} \;/;\; m \in \mathbb{N}^+$$

05.14.16.0007.01

$$B_n(z-m) = B_n(z) - n\sum_{k=0}^{m-1}(k-m+z)^{n-1} /; m \in \mathbb{N}^+$$

Addition formulas

05.14.16.0008.01

$$B_n(w+z) = \sum_{k=0}^{n}\binom{n}{k} B_k(z)\, w^{n-k}$$

Multiple arguments

05.14.16.0009.01

$$B_n(2z) = 2^{n-1}\left(B_n(z) + B_n\left(z+\frac{1}{2}\right)\right)$$

05.14.16.0010.01

$$B_n(2z) = 2^{n-1}\left(B_n(z) + (-1)^n B_n\left(\frac{1}{2}-z\right)\right)$$

05.14.16.0011.01

$$B_n(m z) = m^{n-1}\sum_{k=0}^{m-1} B_n\left(\frac{k}{m}+z\right) /; m \in \mathbb{N}^+$$

Identities

Recurrence identities

Consecutive neighbors

05.14.17.0001.01

$$B_n(z) = B_n(z+1) - n\, z^{n-1}$$

05.14.17.0002.01

$$B_n(z) = B_n(z-1) + n\,(z-1)^{n-1}$$

Distant neighbors

05.14.17.0003.01

$$B_n(z) = B_n(m+z) - n\sum_{k=0}^{m-1}(k+z)^{n-1} /; m \in \mathbb{N}^+$$

05.14.17.0004.01

$$B_n(z) = B_n(z-m) + n\sum_{k=0}^{m-1}(k-m+z)^{n-1} /; m \in \mathbb{N}^+$$

Functional identities

Relations of special kind

05.14.17.0005.01

$$B_n(z+1) - B_n(z) = n\, z^{n-1}$$

05.14.17.0006.01

$$B_n(m) = B_n(0) + n \sum_{k=1}^{m-1} k^{n-1} \;/;\; m \in \mathbb{N} \bigwedge n - 1 \in \mathbb{N}^+$$

05.14.17.0007.01

$$B_n(z+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(z)$$

Complex characteristics

Real part

05.14.19.0001.01

$$\operatorname{Re}(B_n(x + i\, y)) = \frac{1}{2} \left(B_n\left(x - x \sqrt{-\frac{y^2}{x^2}} \right) + B_n\left(x + x \sqrt{-\frac{y^2}{x^2}} \right) \right)$$

Imaginary part

05.14.19.0002.01

$$\operatorname{Im}(B_n(x + i\, y)) = \frac{x}{2\, y} \sqrt{-\frac{y^2}{x^2}} \left(B_n\left(x - x \sqrt{-\frac{y^2}{x^2}} \right) - B_n\left(x + x \sqrt{-\frac{y^2}{x^2}} \right) \right)$$

Absolute value

05.14.19.0003.01

$$|B_n(x + i\, y)| = \sqrt{B_n\left(x - x \sqrt{-\frac{y^2}{x^2}} \right) B_n\left(x + \sqrt{-\frac{y^2}{x^2}}\; x \right)}$$

Argument

05.14.19.0004.01

$$\arg(B_n(x + i\, y)) = \tan^{-1}\left(\frac{1}{2} \left(B_n\left(x - x \sqrt{-\frac{y^2}{x^2}} \right) + B_n\left(x + x \sqrt{-\frac{y^2}{x^2}} \right) \right),\; \frac{x}{2\, y} \sqrt{-\frac{y^2}{x^2}} \left(B_n\left(x - x \sqrt{-\frac{y^2}{x^2}} \right) - B_n\left(x + x \sqrt{-\frac{y^2}{x^2}} \right) \right) \right)$$

Conjugate value

05.14.19.0005.01

$$\overline{B_n(x + i\, y)} = \frac{1}{2} \left(B_n\left(x - x \sqrt{-\frac{y^2}{x^2}} \right) + B_n\left(x + x \sqrt{-\frac{y^2}{x^2}} \right) \right) - \frac{i\, x}{2\, y} \sqrt{-\frac{y^2}{x^2}} \left(B_n\left(x - x \sqrt{-\frac{y^2}{x^2}} \right) - B_n\left(x + x \sqrt{-\frac{y^2}{x^2}} \right) \right)$$

Signum value

05.14.19.0006.01

$$\operatorname{sgn}(B_n(x+i\,y)) = \frac{\frac{i\,x\sqrt{-\frac{y^2}{x^2}}\left(B_n\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)-B_n\left(\sqrt{-\frac{y^2}{x^2}}\,x+x\right)\right)}{y}+B_n\left(\sqrt{-\frac{y^2}{x^2}}\,x+x\right)+B_n\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)}{2\sqrt{B_n\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)B_n\left(\sqrt{-\frac{y^2}{x^2}}\,x+x\right)}}$$

Differentiation

Low-order differentiation

05.14.20.0001.01

$$\frac{\partial B_n(z)}{\partial z} = n\,B_{n-1}(z)$$

05.14.20.0002.01

$$\frac{\partial^2 B_n(z)}{\partial z^2} = n\,(n-1)\,B_{n-2}(z)$$

Symbolic differentiation

05.14.20.0003.02

$$\frac{\partial^m B_n(z)}{\partial z^m} = (n-m+1)_m\,B_{n-m}(z)\;/;\;m\in\mathbb{N}$$

Fractional integro-differentiation

05.14.20.0004.01

$$\frac{\partial^\alpha B_n(z)}{\partial z^\alpha} = n!\sum_{k=0}^{n}\frac{z^{k-\alpha}}{(n-k)!\,\Gamma(k-\alpha+1)}\,B_{n-k}$$

Integration

Indefinite integration

Involving only one direct function

05.14.21.0001.01

$$\int B_n(a\,z)\,dz = \frac{B_{n+1}(a\,z)}{a(n+1)}$$

05.14.21.0002.01

$$\int B_n(z)\,dz = \frac{B_{n+1}(z)}{n+1}$$

Definite integration

For the direct function itself

05.14.21.0003.01

$$\int_0^1 B_n(t)\, B_m(t)\, dt == \frac{(-1)^{n-1}\, m!\, n!}{(m+n)!}\, B_{m+n} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$$

05.14.21.0004.01

$$\int_0^{a\,b} \tilde{B}_m\left(\frac{t}{a}\right) \tilde{B}_n\left(\frac{t}{b}\right) dt == \frac{(-1)^{n-1}\, m!\, n!}{(m+n)!\, a^{m-1}\, b^{n-1}}\, \gcd(a, b)^{m+n}\, B_{m+n} \;/;$$
$$\tilde{B}_n(x) == B_n(x \bmod 1) \bigwedge a-1 \in \mathbb{N}^+ \bigwedge b-1 \in \mathbb{N}^+ \bigwedge m-1 \in \mathbb{N}^+ \bigwedge n-1 \in \mathbb{N}^+$$

Integral transforms

Fourier exp transforms

05.14.22.0001.01

$$\mathcal{F}_t[B_n(t)]\,(x) == \sqrt{2\pi}\, \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, (-i)^k\, \delta^{(k)}(x)$$

Inverse Fourier exp transforms

05.14.22.0002.01

$$\mathcal{F}_t^{-1}[B_n(t)]\,(x) == \sqrt{2\pi}\, \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\, i^k\, \delta^{(k)}(x)$$

Fourier cos transforms

05.14.22.0003.01

$$\mathcal{F}c_t[B_n(t)]\,(x) == \sqrt{2\pi}\, \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{2k} B_{n-2k}\, \delta^{(2k)}(x) - \sqrt{\frac{2}{\pi}}\, \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k\, (2k+1)!}{x^{2k+2}} \binom{n}{2k+1} B_{n-2k-1}$$

Fourier sin transforms

05.14.22.0004.01

$$\mathcal{F}s_t[B_n(t)]\,(x) == \sqrt{\frac{2}{\pi}}\, \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k\, (2k)!}{x^{2k+1}} \binom{n}{2k} B_{n-2k} - \sqrt{2\pi}\, \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{2k+1} B_{n-2k-1}\, \delta^{(2k+1)}(x)$$

Laplace transforms

05.14.22.0005.01

$$\mathcal{L}_t[B_n(t)]\,(z) == \sum_{k=0}^{n} k! \binom{n}{k} B_{n-k}\, z^{-k-1} \;/;\; \mathrm{Re}(z) > 0$$

Summation

Finite summation

05.14.23.0001.01

$$\sum_{k=0}^{n} \binom{n}{k} B_k(z) == B_n(z+1)$$

05.14.23.0002.01

$$\sum_{k=0}^{n} \binom{n}{k} B_k(z)\, w^k == w^n\, B_n\left(z + \frac{1}{w}\right)$$

05.14.23.0003.01

$$\sum_{k=0}^{m} B_n\left(z + \frac{k}{m+1}\right) == (m+1)^{1-n}\, B_n(z\,m + m) \,/;\, m \in \mathbb{N}$$

05.14.23.0004.01

$$\sum_{k=0}^{m} (-1)^k\, B_n\left(z + \frac{k}{m+1}\right) == -\frac{n}{2\,(m+1)^{n-1}}\, E_{n-1}(m\,z + z) \,/;\, \frac{m+1}{2} \in \mathbb{N}^+$$

05.14.23.0005.01

$$\sum_{k=0}^{n} \binom{n}{k} B_k(z)\, B_{n-k}(w) == n\,(w+z-1)\, B_{n-1}(w+z) - (n-1)\, B_n(w+z)$$

Infinite summation

05.14.23.0006.01

$$\sum_{n=0}^{\infty} \frac{w^n}{n!}\, B_n(z) == \frac{w\, e^{z\,w}}{e^w - 1} \,/;\, |w| < 2\,\pi$$

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

05.14.26.0001.01

$$B_m(n) - B_m(0) == m\, H_{n-1}^{(1-m)} \,/;\, m - 1 \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$$

Involving polylog

05.14.26.0002.01

$$B_n(z) == -\frac{n!}{(2\,\pi\,i)^n} \left((-1)^n\, \mathrm{Li}_n\left(e^{-2\,i\,\pi\,z}\right) + \mathrm{Li}_n\left(e^{2\,i\,\pi\,z}\right)\right) \,/;\, \mathrm{Re}(z) > 0$$

Involving Stirling numbers

05.14.26.0003.01

$$B_n(z) - B_n == n \sum_{k=1}^{n} \frac{(-1)^k\, (-z)_k}{k}\, \mathcal{S}_{n-1}^{(k-1)}$$

Involving zeta functions

05.14.26.0004.01

$$B_n(z) == -n\, \zeta(1-n,\, z) \,/;\, n \in \mathbb{N}^+$$

Representations through equivalent functions

With related functions

05.14.27.0001.01

$$B_n(z) == \sum_{k=0}^{n} \binom{n}{k} B_k z^{n-k}$$

05.14.27.0002.01

$$B_n(z) == 2^{-n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k} E_k(2 z)$$

05.14.27.0003.01

$$B_n(z) == E_n(z) + \sum_{k=2}^{n} \binom{n}{k} B_k E_{n-k}(z)$$

Inequalities

05.14.29.0001.01

$$|B_{2 n}(x)| < |B_{2 n}| /; n \in \mathbb{N}^+ \bigwedge 0 < x < 1$$

05.14.29.0002.01

$$0 < (-1)^{n+1} B_{2 n+1}(x) < \frac{2 (2 n + 1)!}{(2 \pi)^{2 n+1} \left(1 - 2^{-2 n}\right)} /; n \in \mathbb{N}^+ \bigwedge 0 < x < \frac{1}{2}$$

Zeros

For each $n \in \mathbb{N}^+$ the equation $B_n(z) == 0$ has exactly n different real roots. If $(n - 1)/2 \in \mathbb{N}^+$, the set of roots include 0, $1/2$, and 1.

For each $n \in \mathbb{N}$ the equation $B_{2 n+1}(z) == 0$ has in the interval $(0, 1)$ only the zero $z = 1/2$.

Other identities

Congruence properties

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$$q^n \left(B_n\left(\frac{p}{q}\right) - B_n(0)\right) \in \mathbb{Z} /; n \in \mathbb{N}^+ \bigwedge p \in \mathbb{Z} \bigwedge q \in \mathbb{Z} \bigwedge q \neq 0$$

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$$q^{2 n+1} B_{2 n+1}\left(\frac{p}{q}\right) \in \mathbb{Z} /; n \in \mathbb{N}^+ \bigwedge p \in \mathbb{N} \bigwedge q \in \mathbb{N}^+ \bigwedge p \leq q$$

Theorems

Eigenfunctions of the Fröbenius-Perron operator

Bernoulli polynomials are the right eigenfunctions of the Fröbenius-Perron operator of the r -adic map $x_{n+1} == r\, x_n \bmod 1$.

History

– Jac. Bernoulli (1705, 1713)
– T. Seki (1712)
– L. Euler (1738, 1755)
– .L. Raabe (1848, 1851) used the name "Bernoulli polynomials"
– C. Hermite (1875); C. J. Malmstén (1884)
– G. Peano (1903) introduced modern notations

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