EllipticE

Notations

Traditional name

Complete elliptic integral of the second kind

Traditional notation

$E(z)$

Mathematica StandardForm notation

`EllipticE[z]`

Primary definition

08.01.02.0001.01

$$E(z) \equiv E\left(\frac{\pi}{2} \,\middle|\, z\right)$$

Specific values

Values at fixed points

08.01.03.0001.01

$$E(0) \equiv \frac{\pi}{2}$$

08.01.03.0002.01

$$E(1) \equiv 1$$

08.01.03.0009.01

$$E\left(\frac{1}{2}\right) \equiv \frac{2\,\Gamma\left(\frac{3}{4}\right)^4 + \pi^2}{4\sqrt{\pi}\;\Gamma\left(\frac{3}{4}\right)^2}$$

08.01.03.0010.01

$$E(-1) = \frac{2\,\Gamma\left(\frac{3}{4}\right)^4 + \pi^2}{2\sqrt{2\pi}\;\Gamma\left(\frac{3}{4}\right)^2}$$

Values at infinities

08.01.03.0004.01

$$E(\infty) = i\,\infty$$

08.01.03.0005.01

$$E(-\infty) = \infty$$

08.01.03.0006.01

$$E(i\,\infty) = -(-1)^{3/4}\,\infty$$

08.01.03.0007.01

$$E(-i\,\infty) = (-1)^{1/4}\,\infty$$

08.01.03.0008.01

$$E(\tilde{\infty}) = \tilde{\infty}$$

General characteristics

Domain and analyticity

$E(z)$ is an analytical function of z which is defined over the whole complex z-plane.

08.01.04.0001.01

$$z \longrightarrow E(z) :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

08.01.04.0002.01

$$E(\bar{z}) = \overline{E(z)} \;/;\; z \notin (1, \infty)$$

Periodicity

The function $E(z)$ is not periodic.

Poles and essential singularities

The function $E(z)$ does not have poles and essential singularities.

08.01.04.0003.01

$$\mathcal{S}ing_z(E(z)) = \{\}$$

Branch points

The function $E(z)$ has two branch points: $z = 1$, $z = \tilde{\infty}$.

08.01.04.0004.01

$$\mathcal{BP}_z(E(z)) = \{1, \tilde{\infty}\}$$

08.01.04.0005.01

$$\mathcal{R}_z(E(z), 1) = \log$$

08.01.04.0006.01

$$\mathcal{R}_z(E(z), \tilde{\infty}) = \log$$

Branch cuts

The function $E(z)$ is a single-valued function on the z-plane cut along the interval $(1, \infty)$.
The function $E(z)$ is continuous from below on the interval $(1, \infty)$.

08.01.04.0007.01

$$\mathcal{BC}_z(E(z)) = \{\{(1, \infty), i\}\}$$

08.01.04.0008.01

$$\lim_{\epsilon \to +0} E(x - i\,\epsilon) = E(x) \;/;\; x > 1$$

08.01.04.0009.01

$$\lim_{\epsilon \to +0} E(x + i\,\epsilon) = 2\,i\,(K(1-x) - E(1-x)) + E(x) \;/;\; x > 1$$

Series representations

Generalized power series

Expansions at generic point $z = z_0$

For the function itself

08.01.06.0015.01

$$E(z) \propto E(z_0) - 2\,i\,(-1)^{\left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor (E(1-z_0) - K(1-z_0)) + \frac{1}{2\,z_0}\left(E(z_0) - 2\,i\,(-1)^{\left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor E(1-z_0) - K(z_0)\right)(z-z_0) + \frac{1}{4\pi}\left(\frac{\pi\,i}{(z_0-1)\,z_0^2}(-1)^{\left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor ((E(1-z_0) + K(1-z_0))\,z_0 - 2\,E(1-z_0)) - G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -\frac{1}{2}, -\frac{3}{2} \\ 0, -1 \end{matrix}\right)\right)(z-z_0)^2 + \ldots \;/;\; (z \to z_0)$$

08.01.06.0016.01

$$E(z) \propto E(z_0) - 2\,i\,(-1)^{\left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor (E(1-z_0) - K(1-z_0)) + \frac{1}{2\,z_0}\left(E(z_0) - 2\,i\,(-1)^{\left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor E(1-z_0) - K(z_0)\right)(z-z_0) + \frac{1}{4\pi}\left(\frac{\pi\,i}{(z_0-1)\,z_0^2}(-1)^{\left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor} \left\lfloor \frac{\arg(1-z_0)+\pi}{2\pi} \right\rfloor \left\lfloor \frac{\arg(z_0-z)}{2\pi} \right\rfloor ((E(1-z_0) + K(1-z_0))\,z_0 - 2\,E(1-z_0)) - G_{2,2}^{2,2}\!\left(1-z_0 \,\middle|\, \begin{matrix} -\frac{1}{2}, -\frac{3}{2} \\ 0, -1 \end{matrix}\right)\right)(z-z_0)^2 + O\!\left((z-z_0)^3\right)$$

08.01.06.0017.01

$$E(z) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\pi\, i\,(-1)^k e^{i\pi\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(1-z_0)+\pi}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor (2k-1)\,\Gamma\left(k-\frac{1}{2}\right)^2 {}_2\tilde{F}_1\left(k-\frac{1}{2},k+\frac{1}{2};k;1-z_0\right) - G_{2,2}^{2,2}\left(1-z_0 \middle| \begin{matrix}\frac{3}{2}-k, \frac{1}{2}-k\\ 0, 1-k\end{matrix}\right)\right)(z-z_0)^k$$

08.01.06.0018.01

$$E(z) = \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{(k!)^2}\,{}_2F_1\left(k-\frac{1}{2},k+\frac{1}{2};k+1;z_0\right)(z-z_0)^k$$

08.01.06.0019.01

$$E(z) \propto E(z_0) - 2\,i\,(-1)^{\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(1-z_0)+\pi}{2\pi}\right\rfloor\left\lfloor\frac{\arg(z_0-z)}{2\pi}\right\rfloor(E(1-z_0)-K(1-z_0)) + O(z-z_0)$$

Expansions on branch cuts

For the function itself

08.01.06.0020.01

$$E(z) \propto E(x) - 2(-1)^{\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor} i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor(E(1-x)-K(1-x)) + \frac{1}{2x}\left(E(x) - 2\,i\,(-1)^{\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor E(1-x) - K(x)\right)(z-x) + \frac{1}{4\pi}\left(\frac{\pi\, i}{(x-1)x^2}(-1)^{\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor((E(1-x)+K(1-x))x - 2E(1-x)) - G_{2,2}^{2,2}\left(1-x \middle| \begin{matrix}-\frac{1}{2}, -\frac{3}{2}\\ 0, -1\end{matrix}\right)\right)(z-x)^2 + \ldots /; (z \to x) \land x \in \mathbb{R} \land x > 1$$

08.01.06.0021.01

$$E(z) \propto E(x) - 2(-1)^{\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor} i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor(E(1-x)-K(1-x)) + \frac{1}{2x}\left(E(x) - 2\,i\,(-1)^{\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor E(1-x) - K(x)\right)(z-x) + \frac{1}{4\pi}\left(\frac{\pi\, i}{(x-1)x^2}(-1)^{\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor((E(1-x)+K(1-x))x - 2E(1-x)) - G_{2,2}^{2,2}\left(1-x \middle| \begin{matrix}-\frac{1}{2}, -\frac{3}{2}\\ 0, -1\end{matrix}\right)\right)(z-x)^2 + O\left((z-x)^3\right) /; x \in \mathbb{R} \land x > 1$$

08.01.06.0022.01

$$E(z) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\pi\, i\,(-1)^k e^{i\pi\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor(2k-1)\,\Gamma\left(k-\frac{1}{2}\right)^2 {}_2\tilde{F}_1\left(k-\frac{1}{2},k+\frac{1}{2};k;1-x\right) - G_{2,2}^{2,2}\left(1-x \middle| \begin{matrix}\frac{3}{2}-k, \frac{1}{2}-k\\ 0, 1-k\end{matrix}\right)\right)(z-x)^k /; x \in \mathbb{R} \land x > 1$$

08.01.06.0023.01

$$E(z) \propto E(x) - 2\,i\,(-1)^{\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor(E(1-x)-K(1-x)) + O(z-x) /; x \in \mathbb{R} \land x > 1$$

Expansions at $z = 0$

For the function itself

08.01.06.0001.02

$$E(z) \propto \frac{\pi}{2}\left(1 - \frac{z}{4} - \frac{3z^2}{64} - \ldots\right) /; (z \to 0)$$

08.01.06.0024.01

$$E(z) \propto \frac{\pi}{2}\left(1 - \frac{z}{4} - \frac{3z^2}{64} - O\left(z^3\right)\right)$$

08.01.06.0002.01

$$E(z) = \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k z^k}{k!^2} /; |z| < 1$$

08.01.06.0003.01

$$E(z) = \frac{\pi}{2}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; z\right) /; |z| < 1$$

08.01.06.0004.02

$$E(z) \propto \frac{\pi}{2} + O(z)$$

08.01.06.0025.01

$$E(z) = F_\infty(z) /; \left(\left(F_n(z) = \frac{\pi}{2}\sum_{k=0}^{n}\frac{\left(-\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k z^k}{k!^2} = E(z) + \frac{z^{n+1}\,\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{3}{2}\right)}{4\,(n+1)!^2}\, {}_3F_2\left(1, n+\frac{1}{2}, n+\frac{3}{2}; n+2, n+2; z\right)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Expansions at $z = 1$

For the function itself

08.01.06.0005.02

$$E(z) \propto 1 + \frac{z-1}{4}\left(-2\log(4) + 1 + \frac{24\log(2) - 13}{16}(z-1) - \frac{3(5\log(2) - 3)}{16}(z-1)^2 + \ldots\right) + \log(1-z)\,\frac{z-1}{4}\left(1 + \frac{3(1-z)}{8} + \frac{15}{64}(1-z)^2 + \ldots\right) /; (z \to 1)$$

08.01.06.0026.01

$$E(z) \propto 1 + \frac{z-1}{4}\left(-2\log(4) + 1 + \frac{24\log(2) - 13}{16}(z-1) - \frac{3(5\log(2) - 3)}{16}(z-1)^2 + O\left((z-1)^3\right)\right) + \log(1-z)\,\frac{z-1}{4}\left(1 + \frac{3(1-z)}{8} + \frac{15}{64}(1-z)^2 + O\left((z-1)^3\right)\right)$$

08.01.06.0027.01

$$E(z) = 1 + \frac{1}{4}(z-1)\log(1-z)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!\,(k+1)!}(1-z)^k + \frac{z-1}{4}\left(\frac{29-13z}{16} - 4\log(2)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!\,(k+1)!}(1-z)^k\right) + \frac{45}{64}(z-1)\sum_{k=0}^{\infty}\frac{\left(\frac{5}{2}\right)_k\left(\frac{7}{2}\right)_k}{(k+3)\,((k+2)!)^2}\left(4\sum_{i=k+1}^{2k+1}\frac{1}{i} - \frac{2}{k+2} - \frac{2}{k+1} - \frac{1}{k+3} + \frac{4}{2k+3} + \frac{2}{2k+5}\right)(1-z)^{k+2} /; |z-1| < 1$$

08.01.06.0006.02

$$E(z) = 1 + \frac{1}{4}(z-1)\log(1-z)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!\,(k+1)!}(1-z)^k + \frac{z-1}{4}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k(1-z)^k}{k!\,(k+1)!}\left(-2\,\psi(k+1) + 2\,\psi\left(k+\frac{1}{2}\right) + \frac{1}{2k^2+3k+1}\right) /;$$

$|z-1| < 1$

08.01.06.0028.01

$$E(z) = 2\sqrt{-z} - \frac{\sqrt{-z}\,(\log(-z)+4)}{\pi}E\left(\frac{1}{z}\right) + \frac{(1-z)(\log(-z)+2)}{\pi\sqrt{-z}}K\left(\frac{1}{z}\right) + \frac{2}{\sqrt{-z}}\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_{k+1}^2}{k!\,(k+1)!}\left(\psi(k+1) - \psi\left(\frac{1}{2}-k\right)\right)z^{-k} /;$$

$|z| > 1$

08.01.06.0007.01

$$E(z) = 2 - {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1-z\right) + \frac{z-1}{4}\log(1-z)\,{}_2F_1\left(\frac{3}{2}, \frac{1}{2}; 2; 1-z\right) + \frac{z-1}{2}\sum_{k=0}^{\infty}\frac{(2k+1)\left(\frac{1}{2}\right)_k^2(1-z)^k}{k!\,(k+1)!}\left(\psi\left(k+\frac{1}{2}\right) - \psi(k+1)\right) /; |z-1| < 1$$

08.01.06.0008.02

$$E(z) \propto 1 + (z-1)\left(\frac{1}{4} - \log(2)\right)(1 + O(z-1)) + \frac{z-1}{4}\log(1-z)(1 + O(z-1))$$

08.01.06.0029.01

$$E(z) = F_\infty(z) /; \left(\left(F_n(z) = \frac{1-z}{4}\sum_{k=0}^{n}\frac{\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!\,(k+1)!}\left(-\log(1-z) + 2\,\psi(k+1) - \psi\left(k+\frac{1}{2}\right) - \psi\left(k+\frac{3}{2}\right) + \frac{1}{k+1}\right)(1-z)^k + 1 = E(z) - \frac{1}{2\pi}G_{4,4}^{2,4}\left(1-z\,\middle|\,\begin{matrix} n+2, n+2, \frac{1}{2}, \frac{3}{2} \\ n+2, n+2, 0, 1\end{matrix}\right)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Expansions at $z = \infty$

For the function itself

08.01.06.0009.02

$$E(z) \propto \sqrt{-z} + \frac{1}{\sqrt{-z}}\left(\frac{1}{4} + \log(2) + \frac{8\log(2)-3}{64\,z} + \frac{6\log(2)-3}{128\,z^2} + \ldots\right) + \frac{\log(-z)}{4\sqrt{-z}}\left(1 + \frac{1}{8\,z} + \frac{3}{64\,z^2} + \ldots\right) /; (|z| \to \infty)$$

08.01.06.0030.01

$$E(z) \propto \sqrt{-z} + \frac{1}{\sqrt{-z}}\left(\frac{1}{4} + \log(2) + \frac{8\log(2)-3}{64\,z} + \frac{6\log(2)-3}{128\,z^2} + O\left(\frac{1}{z^3}\right)\right) + \frac{\log(-z)}{4\sqrt{-z}}\left(1 + \frac{1}{8\,z} + \frac{3}{64\,z^2} + O\left(\frac{1}{z^3}\right)\right)$$

08.01.06.0031.01

$$E(z) = \sqrt{-z} + \frac{\log(-z)}{\sqrt{-z}} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{k+1}^{2} z^{-k}}{k!\,(k+1)!} + \frac{1}{\sqrt{-z}} \left(\log(2) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{k}^{2} z^{-k}}{k!\,(k+1)!} + \frac{1}{4} - \frac{3}{64\,z} \sum_{i=0}^{\infty} \delta_{i-1} + \frac{9}{64} \sum_{k=0}^{\infty} \frac{\left(\frac{5}{2}\right)_{k}^{2}}{((k+2)!)^2\,(k+3)} \left(\frac{2}{k+1} + \frac{2}{k+2} - \frac{4}{2\,k+3} + \frac{1}{k+3} - 4 \sum_{i=k+1}^{2\,k+1} \frac{1}{i} \right) z^{-k-2} \right) /;\ |z| > 1$$

08.01.06.0010.02

$$E(z) = \sqrt{-z} + \frac{\log(-z)}{\sqrt{-z}} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{k+1}^{2} z^{-k}}{k!\,(k+1)!} + \frac{1}{\sqrt{-z}} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{k+1}^{2} z^{-k}}{k!\,(k+1)!} \left(2\,\psi(k+1) - 2\,\psi\!\left(\frac{1}{2}+k\right) + \frac{1}{k+1} \right) /;\ |z| > 1$$

08.01.06.0032.01

$$E(z) = 2\sqrt{-z} - \frac{\sqrt{-z}\,(\log(-z)+4)}{\pi} E\!\left(\frac{1}{z}\right) + \frac{(1-z)\,(\log(-z)+2)}{\pi\sqrt{-z}} K\!\left(\frac{1}{z}\right) + \frac{2}{\sqrt{-z}} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{k+1}^{2}}{k!\,(k+1)!} \left(\psi(k+1) - \psi\!\left(\frac{1}{2}+k\right) \right) z^{-k} /;\ |z| > 1$$

08.01.06.0011.02

$$E(z) = 2\sqrt{-z} - \sqrt{-z}\;{}_2F_1\!\left(-\frac{1}{2}, -\frac{1}{2}; 1; \frac{1}{z}\right) + \frac{\log(-z)}{4\sqrt{-z}}\;{}_2F_1\!\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{1}{z}\right) + \frac{2}{\sqrt{-z}} \sum_{k=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{k+1}^{2}}{k!\,(k+1)!} \left(\psi(k+1) - \psi\!\left(\frac{1}{2}+k\right) \right) z^{-k} /;\ |z| > 1$$

08.01.06.0012.02

$$E(z) \propto \sqrt{-z} + \frac{4\log(2)+1}{4\sqrt{-z}} \left(1 + O\!\left(\frac{1}{z}\right)\right) + \frac{\log(-z)}{4\sqrt{-z}} \left(1 + O\!\left(\frac{1}{z}\right)\right)$$

08.01.06.0033.01

$$E(z) \propto \left\{ \begin{cases} i\sqrt{z} & \arg(z) \le 0 \\ -i\sqrt{z} & \text{True} \end{cases} \right. /;\ (|z| \to \infty)$$

08.01.06.0034.01

$$E(z) = F_\infty(z) /; \left(\left(F_n(z) = \frac{1}{\sqrt{-z}} \sum_{k=0}^{m} \frac{1}{k!\,(k+1)!} \left(-\frac{1}{2}\right)_{k+1}^{2} \left(\log(-z) + 2\,\psi(k+1) - \psi\!\left(\frac{1}{2}-k\right) - \psi\!\left(k+\frac{1}{2}\right) + \frac{1}{k+1} \right) z^{-k} + \sqrt{-z} = E(z) - \frac{1}{4} G_{4,4}^{3,2}\!\left(-z \,\middle|\, \begin{matrix} -m-\frac{1}{2}, -m-\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \\ 0, -m-\frac{1}{2}, -m-\frac{1}{2}, 0 \end{matrix} \right) \right) \bigwedge m \in \mathbb{N} \right)$$

Summed form of the truncated series expansion.

Residue representations

08.01.06.0013.01

$$E(z) = -\frac{1}{4} \sum_{j=0}^{\infty} \operatorname{res}_s \left(\frac{\Gamma\!\left(-s-\frac{1}{2}\right) \Gamma\!\left(\frac{1}{2}-s\right) (-z)^{-s}}{\Gamma(1-s)}\, \Gamma(s) \right) (-j) /;\ |z| < 1$$

08.01.06.0014.02

$$E(z) == \frac{1}{4}\left(\operatorname{res}_s\left(\frac{\Gamma(s)\,\Gamma\left(-s-\frac{1}{2}\right)(-z)^{-s}}{\Gamma(1-s)}\,\Gamma\left(\frac{1}{2}-s\right)\right)\left(-\frac{1}{2}\right)+\sum_{j=0}^{\infty}\operatorname{res}_s\left(\frac{\Gamma(s)\,(-z)^{-s}}{\Gamma(1-s)}\,\Gamma\left(-s-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-s\right)\right)\left(j+\frac{1}{2}\right)\right) /;\ |z|>1$$

Integral representations

On the real axis

Of the direct function

08.01.07.0001.01

$$E(z) == \int_0^{\frac{\pi}{2}} \sqrt{1-z\sin^2(t)}\,dt /;\ |\arg(1-z)|<\pi$$

08.01.07.0002.01

$$E(z) == \int_0^1 \frac{\sqrt{1-z\,t^2}}{\sqrt{1-t^2}}\,dt /;\ |\arg(1-z)|<\pi$$

Contour integral representations

08.01.07.0003.01

$$E(z) == -\frac{1}{8\pi i}\int_{\mathcal{L}} \frac{\Gamma(s)\,\Gamma\left(-s-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-s\right)(-z)^{-s}}{\Gamma(1-s)}\,ds$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

08.01.13.0001.01

$$(1-z)\,z\,w''(z)+(1-z)\,w'(z)+\frac{1}{4}\,w(z)==0 /;\ w(z)==c_1\,E(z)\;+c_2(K(1-z)-E(1-z))$$

08.01.13.0002.02

$$W_z(E(z),\,K(1-z)-E(1-z)) == -\frac{\pi}{4\,z}$$

08.01.13.0003.01

$$w''(z)+\left(\frac{g'(z)}{g(z)}-\frac{g''(z)}{g'(z)}\right)w'(z)-\frac{g'(z)^2}{4\,(g(z)-1)\,g(z)}\,w(z)=0 /;\ w(z)=c_1\,E(g(z))+c_2(K(1-g(z))-E(1-g(z)))$$

08.01.13.0004.01

$$W_z(E(g(z)),\,K(1-g(z))-E(1-g\,(z))) == -\frac{\pi\,g'(z)}{4\,g(z)}$$

08.01.13.0005.01

$$w''(z)+\left(\frac{g'(z)}{g(z)}-\frac{2\,h'(z)}{h(z)}-\frac{g''(z)}{g'(z)}\right)w'(z)-\left(\frac{g'(z)^2}{4\,(g(z)-1)\,g(z)}+\frac{h'(z)\,g'(z)}{g(z)\,h(z)}+\frac{h''(z)}{h(z)}-\frac{2\,h'(z)^2}{h(z)^2}-\frac{h'(z)\,g''(z)}{h(z)\,g'(z)}\right)w(z)=0\,/;\; w(z)=c_1\,h(z)\,E(g(z))+c_2\,h(z)\,(K(1-g(z))-E(1-g(z)))$$

08.01.13.0006.01

$$W_z(h(z)\,E(g(z)),\,h(z)\,(K(1-g(z))-E\,(1-g(z))))=-\frac{\pi\,h(z)^2\,g'(z)}{4\,g(z)}$$

08.01.13.0007.01

$$w''(z)+\frac{1}{4\,z^2}\left(\left(\frac{1}{1-a\,z^r}-1\right)r^2+4\,s^2\right)w(z)+\frac{1-2\,s}{z}\,w'(z)=0\,/;\; w(z)=c_1\,z^s\,E(a\,z^r)\,+c_2\,z^s(K(1-a\,z^r)-E(1-a\,z^r))$$

08.01.13.0008.01

$$W_z(z^s\,E(a\,z^r),\,z^s\,(K(1-a\,z^r)-E\,(1-a\,z^r)))=-\frac{\pi\,r}{4}\,z^{2\,s-1}$$

08.01.13.0009.01

$$w''(z)-2\log(s)\,w'(z)+\left(\frac{a\log^2(r)\,r^z}{4-4\,a\,r^z}+\log^2(s)\right)w(z)=0\,/;\; w(z)=c_1\,s^z\,E(a\,r^z)\,+c_2\,s^z(K(1-a\,r^z)-E(1-a\,r^z))$$

08.01.13.0010.01

$$W_z(s^z\,E(a\,r^z),\,s^z\,(K(1-a\,r^z)-E\,(1-a\,r^z)))=-\frac{1}{4}\,\pi\,s^{2\,z}\log(r)$$

Identities

Functional identities

08.01.17.0001.01

$$E\left(1-\frac{1}{z}\right)=\frac{E(1-z)}{\sqrt{z}}\,/;\;|\arg(z)|<\pi$$

08.01.17.0002.01

$$E(z)=\sqrt{1-z}\;E\left(\frac{z}{z-1}\right)/;\;|\arg(1-z)|<\pi$$

08.01.17.0003.01

$$E(z)=\sqrt{1-z}\;\overline{E\left(\frac{z}{z-1}\right)}\,/;\;z\in\mathbb{R}$$

08.01.17.0004.01

$$E(z)=\sqrt{1-z}\;E\left(\frac{z}{z-1}\right)+i\left(1-\sqrt{1-z}\;\sqrt{\frac{1}{1-z}}\right)\sqrt{1-z}\;\left(K\left(\frac{1}{1-z}\right)-E\left(\frac{1}{1-z}\right)\right)$$

08.01.17.0005.01

$$E(z)=\sqrt{z}\;E\left(\frac{1}{z}\right)-\sqrt{-\frac{1}{z}}\;\sqrt{\frac{1}{1-z}}\;\sqrt{1-z}\;\sqrt{z}\;E(1-z)-\sqrt{z}\;K\left(\frac{1}{z}\right)+K(z)$$

08.01.17.0006.01

$$E(z)=\sqrt{z}\;E\left(\frac{1}{z}\right)-\sqrt{-\frac{1}{z}}\;\sqrt{\frac{1}{1-z}}\;\sqrt{1-z}\;\sqrt{z}\;E(1-z)+\sqrt{\frac{1}{1-z}}\;\sqrt{z\,(1-z)}\;\sqrt{-\frac{1}{z}}\;z\,K(1-z)+(1-z)\,K(z)$$

08.01.17.0007.01

$$E(z) = \sqrt{z-1}\,\sqrt{\frac{z-1}{z}}\,\sqrt{\frac{z}{z-1}}\left(E\left(\frac{1}{1-z}\right) - K\left(\frac{1}{1-z}\right) + \frac{K(1-z)-E(1-z)}{\sqrt{1-z}}\right)$$

Complex characteristics

Real part

08.01.19.0001.01

$$\operatorname{Re}(E(x+i\,y)) = \frac{\pi}{2}\,F_{2\times1\times1}^{4\times0\times0}\left(\begin{matrix}\frac{3}{4},\frac{5}{4},\frac{1}{4},\frac{3}{4};;;\\ 1,\frac{3}{2};\frac{1}{2};\frac{3}{2};\end{matrix}\;-y^2, x^2\right) - \frac{\pi\,x}{8}\,F_{2\times1\times1}^{4\times0\times0}\left(\begin{matrix}\frac{1}{4},\frac{3}{4},-\frac{1}{4},\frac{1}{4};;;\\ 1,\frac{1}{2};\frac{1}{2};\frac{1}{2};\end{matrix}\;-y^2, x^2\right) /; x\in\mathbb{R}\bigwedge y\in\mathbb{R}$$

Imaginary part

08.01.19.0002.01

$$\operatorname{Im}(E(x+i\,y)) = -\frac{\pi\,y}{8}\,F_{2\times1\times1}^{4\times0\times0}\left(\begin{matrix}\frac{3}{4},\frac{5}{4},\frac{1}{4},\frac{3}{4};;;\\ 1,\frac{3}{2};\frac{3}{2};\frac{1}{2};\end{matrix}\;-y^2, x^2\right) - \frac{3\,\pi\,x\,y}{64}\,F_{2\times1\times1}^{4\times0\times0}\left(\begin{matrix}\frac{5}{4},\frac{7}{4},\frac{3}{4},\frac{5}{4};;;\\ 2,\frac{3}{2};\frac{3}{2};\frac{3}{2};\end{matrix}\;-y^2, x^2\right) /; x\in\mathbb{R}\bigwedge y\in\mathbb{R}$$

Differentiation

Low-order differentiation

08.01.20.0001.01

$$\frac{\partial E(z)}{\partial z} = \frac{E(z)-K(z)}{2\,z}$$

08.01.20.0002.01

$$\frac{\partial^2 E(z)}{\partial z^2} = \frac{(2-z)\,E(z) + 2\,(z-1)\,K(z)}{4\,(z-1)\,z^2}$$

Symbolic differentiation

08.01.20.0003.02

$$\frac{\partial^n E(z)}{\partial z^n} = \frac{\pi\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{2\,n!}\,{}_2F_1\left(n-\frac{1}{2}, n+\frac{1}{2}; n+1; z\right) /; n\in\mathbb{N}$$

08.01.20.0004.02

$$\frac{\partial^n E(z)}{\partial z^n} = \frac{\pi\,z^{-n}}{2}\,{}_2\tilde{F}_1\left(-\frac{1}{2}, \frac{1}{2}; 1-n; z\right) /; n\in\mathbb{N}$$

Fractional integro-differentiation

08.01.20.0005.01

$$\frac{\partial^\alpha E(z)}{\partial z^\alpha} = \frac{\pi\,z^{-\alpha}}{2}\,{}_2\tilde{F}_1\left(-\frac{1}{2}, \frac{1}{2}; 1-\alpha; z\right)$$

Integration

Indefinite integration

Involving only one direct function

08.01.21.0001.01

$$\int E(a\,z)\,dz = \frac{2\,((a\,z+1)\,E(a\,z)+(a\,z-1)\,K(a\,z))}{3\,a}$$

08.01.21.0002.01

$$\int E(z)\,dz = \frac{2}{3}\,((z+1)\,E(z)+(z-1)\,K(z))$$

Involving one direct function and elementary functions

Involving power function

Involving power

Linear argument

08.01.21.0003.01

$$\int z^{\alpha-1}\,E(a\,z)\,dz = \frac{1}{2}\,\pi\,z^{\alpha}\,\Gamma(\alpha)\,{}_3\tilde{F}_2\left(-\frac{1}{2},\frac{1}{2},\alpha;1,\alpha+1;a\,z\right)$$

08.01.21.0004.01

$$\int z^{\alpha-1}\,E(z)\,dz = \frac{\pi\,z^{\alpha}}{2\,\alpha}\,{}_3F_2\left(-\frac{1}{2},\frac{1}{2},\alpha;1,\alpha+1;z\right)$$

08.01.21.0005.01

$$\int z\,E(a\,z)\,dz = \frac{2\left(\left(9\,a^2\,z^2+a\,z+4\right)E(a\,z)+\left(3\,a^2\,z^2+a\,z-4\right)K(a\,z)\right)}{45\,a^2}$$

08.01.21.0006.01

$$\int \frac{E(a\,z)}{\sqrt{z}}\,dz = \pi\,\sqrt{z}\,{}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};a\,z\right)$$

08.01.21.0007.01

$$\int \frac{E(a\,z)}{z}\,dz = -\frac{1}{8}\,\pi\left(a\,z\,{}_4F_3\left(\frac{1}{2},1,1,\frac{3}{2};2,2,2;a\,z\right)+4\left(\log(16)+\gamma^2-\log(-a\,z)-3\right)\right)$$

08.01.21.0008.01

$$\int \frac{E(a\,z)}{z^2}\,dz = -\frac{1}{128\,z}\left(\pi\left(3\,a^2\,{}_4F_3\left(1,1,\frac{3}{2},\frac{5}{2};2,3,3;a\,z\right)z^2+16\,(-a\,z\log(16)+a\,z\log(-a\,z)+4)\right)\right)$$

Power arguments

08.01.21.0009.01

$$\int z^{\alpha-1}\,E(a\,z^r)\,dz = \frac{\pi}{2\,\alpha}\,z^{\alpha}\,{}_3F_2\left(-\frac{1}{2},\frac{1}{2},\frac{\alpha}{r};1,\frac{\alpha}{r}+1;a\,z^r\right)$$

08.01.21.0010.01

$$\int z\, E(a z^2)\, dz = \frac{(a z^2+1)\, E(a z^2) + (a z^2-1)\, K(a z^2)}{3 a}$$

08.01.21.0011.01

$$\int z^3\, E(a z^2)\, dz = \frac{(9 a^2 z^4 + a z^2 + 4)\, E(a z^2) + (3 a^2 z^4 + a z^2 - 4)\, K(a z^2)}{45 a^2}$$

08.01.21.0012.01

$$\int z^5\, E(a z^2)\, dz = \frac{1}{1575 a^3} \left((225 a^3 z^6 + 9 a^2 z^4 + 16 a z^2 + 64)\, E(a z^2) + (45 a^3 z^6 + 3 a^2 z^4 + 16 a z^2 - 64)\, K(a z^2)\right)$$

08.01.21.0013.01

$$\int E(a z^2)\, dz = \frac{1}{2} \pi z\, {}_3F_2\!\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; a z^2\right)$$

08.01.21.0014.01

$$\int \frac{E(a z^2)}{z}\, dz = -\frac{1}{16} \pi \left(a\, {}_4F_3\!\left(\frac{1}{2}, 1, 1, \frac{3}{2}; 2, 2, 2; a z^2\right) z^2 + 4 \left(\log(16) + \gamma^2 - \log(-a z^2) - 3\right)\right)$$

08.01.21.0015.01

$$\int \frac{E(a z^2)}{z^2}\, dz = \frac{(1 - a z^2)\, K(a z^2) - 2 E(a z^2)}{z}$$

08.01.21.0016.01

$$\int \frac{E(a z^2)}{z^3}\, dz = -\frac{1}{256 z^2} \left(\pi \left(3 a^2\, {}_4F_3\!\left(1, 1, \frac{3}{2}, \frac{5}{2}; 2, 3, 3; a z^2\right) z^4 - 2 a \log(4\,294\,967\,296)\, z^2 + 16 a \log(-a z^2)\, z^2 + 64\right)\right)$$

08.01.21.0017.01

$$\int \frac{E(a z^2)}{z^4}\, dz = \frac{2 (a z^2 - 2)\, E(a z^2) + (1 - a z^2)\, K(a z^2)}{9 z^3}$$

Involving rational functions

08.01.21.0018.01

$$\int \frac{E(z^2)}{1 - z^2}\, dz = z\, K(z^2)$$

08.01.21.0019.01

$$\int \frac{z\, E(z^2)}{1 - z^2}\, dz = K(z^2) - E(z^2)$$

08.01.21.0020.01

$$\int \frac{z^3\, E(z^2)}{1 - z^2}\, dz = \frac{1}{3} \left(-(z^2 + 4)\, E(z^2) - (z^2 - 4)\, K(z^2)\right)$$

08.01.21.0021.01

$$\int \frac{E(z^2)}{z^2 (z^2 - 1)}\, dz = \frac{2 E(z^2) - K(z^2)}{z}$$

Involving algebraic functions

08.01.21.0022.01

$$\int \frac{z\,E(z^2)}{(1-z^2)^{3/2}}\,dz = \frac{E(z^2)+(z^2-1)\,K(z^2)}{\sqrt{1-z^2}}$$

08.01.21.0023.01

$$\int \frac{z\,E(z^2)}{(1-z^2)^{5/2}}\,dz = -\frac{(1-2\,z^2)\,E(z^2)+(z^2-1)\,K(z^2)}{3\,(1-z^2)^{3/2}}$$

Definite integration

For the direct function itself

08.01.21.0024.01

$$\int_0^1 t^{\alpha-1}\,E(t)\,dt = \frac{\pi}{2\,\alpha}\,{}_3F_2\!\left(-\frac{1}{2},\frac{1}{2},\alpha;1,\alpha+1;1\right)\ /;\ \mathrm{Re}(\alpha)>0$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2F_1$

08.01.26.0001.01

$$E(z) = \frac{\pi}{2}\,{}_2F_1\!\left(-\frac{1}{2},\frac{1}{2};1;z\right)$$

Through Meijer *G*

Classical cases for the direct function itself

08.01.26.0002.01

$$E(z) = -\frac{1}{4}\,G_{2,2}^{1,2}\!\left(-z\,\middle|\,\begin{matrix}\frac{1}{2},\frac{3}{2}\\ 0,0\end{matrix}\right)$$

08.01.26.0003.01

$$E(z) = \frac{1}{4}\,G_{3,3}^{1,3}\!\left(-z\,\middle|\,\begin{matrix}\frac{1}{2},1,\frac{3}{2}\\ 1,0,0\end{matrix}\right)+\frac{\pi}{2}$$

Classical cases involving algebraic functions

08.01.26.0004.01

$$\frac{1}{1-z}\,E(z) = G_{2,2}^{1,2}\!\left(-z\,\middle|\,\begin{matrix}-\frac{1}{2},\frac{1}{2}\\ 0,0\end{matrix}\right)$$

08.01.26.0005.01

$$\frac{1}{1-z}\,E\!\left(\frac{1}{z}\right) = G_{2,2}^{2,1}\!\left(-z\,\middle|\,\begin{matrix}0,0\\ -\frac{1}{2},\frac{1}{2}\end{matrix}\right)$$

08.01.26.0006.01

$$\sqrt{z+1}\,E\!\left(\frac{1}{z+1}\right) = -\frac{1}{4}\,G_{2,2}^{2,1}\!\left(z\,\middle|\,\begin{matrix}\frac{3}{2},\frac{3}{2}\\ 0,1\end{matrix}\right)\ /;\ z\notin(-1,0)$$

08.01.26.0007.01

$$\frac{1}{\sqrt{z+1}}\,E\left(\frac{1}{z+1}\right) = G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 0, 1 \end{matrix}\right) /;\, z \notin (-1, 0)$$

08.01.26.0008.01

$$\frac{1}{\sqrt{z+1}}\,E\left(\frac{z}{z+1}\right) = G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} -\frac{1}{2}, \frac{1}{2} \\ 0, 0 \end{matrix}\right) /;\, z \notin (-\infty, -1)$$

08.01.26.0009.01

$$\sqrt{z+1}\,E\left(\frac{z}{z+1}\right) = -\frac{1}{4}\,G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{3}{2} \\ 0, 0 \end{matrix}\right)$$

08.01.26.0010.01

$$\frac{1}{\sqrt{z+1}\,\sqrt{\sqrt{z+1}-\sqrt{z}}}\,E\left(2\sqrt{z}\,\sqrt{z+1}-2z\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} -\frac{1}{4}, \frac{3}{4} \\ 0, 0 \end{matrix}\right) /;\, z \notin (-\infty, -1)$$

08.01.26.0011.01

$$\frac{\sqrt{\sqrt{z}+\sqrt{z+1}}}{\sqrt{z+1}}\,E\left(2\sqrt{z}\,\sqrt{z+1}-2z\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} -\frac{1}{4}, \frac{3}{4} \\ 0, 0 \end{matrix}\right) /;\, z \notin (-\infty, -1)$$

08.01.26.0012.01

$$\frac{1}{\sqrt{z+1}\,\sqrt{\sqrt{z+1}-1}}\,E\left(\frac{2}{\sqrt{z+1}+1}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{1}{4} \\ -\frac{1}{2}, \frac{1}{2} \end{matrix}\right) /;\, z \notin (-1, 0)$$

08.01.26.0013.01

$$\frac{\sqrt{\sqrt{z+1}+1}}{\sqrt{z+1}}\,E\left(\frac{2}{\sqrt{z+1}+1}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{3}{4}, \frac{3}{4} \\ 0, 1 \end{matrix}\right) /;\, z \notin (-1, 0)$$

08.01.26.0014.01

$$\frac{\sqrt{\sqrt{z+1}-1}}{\sqrt{z+1}}\,E\left(\frac{2}{1-\sqrt{z+1}}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{3}{4}, \frac{3}{4} \\ 0, 1 \end{matrix}\right) /;\, z \notin (-1, 0)$$

08.01.26.0015.01

$$\frac{1}{\sqrt{z+1}\,\sqrt{\sqrt{z+1}+1}}\,E\left(\frac{2}{1-\sqrt{z+1}}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{1}{4} \\ -\frac{1}{2}, \frac{1}{2} \end{matrix}\right) /;\, z \notin (-1, 0)$$

08.01.26.0016.01

$$\frac{1}{\sqrt{z+1}\,\sqrt{\sqrt{z+1}-\sqrt{z}}}\,E\left(\frac{2\sqrt{z}}{\sqrt{z}+\sqrt{z+1}}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} -\frac{1}{4}, \frac{3}{4} \\ 0, 0 \end{matrix}\right) /;\, z \notin (-\infty, -1)$$

08.01.26.0017.01

$$\frac{\sqrt{\sqrt{z+1}-\sqrt{z}}}{\sqrt{z+1}}\,E\left(\frac{2\sqrt{z}}{\sqrt{z}-\sqrt{z+1}}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} -\frac{1}{4}, \frac{3}{4} \\ 0, 0 \end{matrix}\right)$$

08.01.26.0018.01

$$\frac{\sqrt{\sqrt{z}+\sqrt{z+1}}}{\sqrt{z+1}}\,E\left(\frac{2\sqrt{z}}{\sqrt{z}+\sqrt{z+1}}\right)=\frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{1,2}\left(z\,\middle|\,\begin{matrix}-\frac{1}{4},\frac{3}{4}\\0,0\end{matrix}\right)\,/;\,z\notin(-\infty,-1)$$

08.01.26.0019.01

$$\frac{1}{\sqrt{z+1}\,\sqrt{\sqrt{z}+\sqrt{z+1}}}\,E\left(\frac{2\sqrt{z}}{\sqrt{z}-\sqrt{z+1}}\right)=\frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{1,2}\left(z\,\middle|\,\begin{matrix}-\frac{1}{4},\frac{3}{4}\\0,0\end{matrix}\right)$$

08.01.26.0020.01

$$\frac{1}{\sqrt{z+1}\,\sqrt{\sqrt{z+1}-1}}\,E\left(\frac{2\left(\sqrt{z+1}-1\right)}{z}\right)=\frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{2,1}\left(z\,\middle|\,\begin{matrix}\frac{1}{4},\frac{1}{4}\\-\frac{1}{2},\frac{1}{2}\end{matrix}\right)\,/;\,z\notin(-1,0)$$

08.01.26.0021.01

$$\frac{\sqrt{\sqrt{z+1}+1}}{\sqrt{z+1}}\,E\left(\frac{2\left(\sqrt{z+1}-1\right)}{z}\right)=\frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\,G_{2,2}^{2,1}\left(z\,\middle|\,\begin{matrix}\frac{3}{4},\frac{3}{4}\\0,1\end{matrix}\right)\,/;\,z\notin(-1,0)$$

Classical cases involving unit step θ

08.01.26.0022.01

$$\theta(1-|z|)\,E(1-z)=\frac{\pi}{2}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}\frac{1}{2},\frac{3}{2}\\0,1\end{matrix}\right)\,/;\;\notin(-1,0)$$

08.01.26.0023.01

$$\theta(|z|-1)\,E(1-z)=\frac{\pi}{2}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}\frac{1}{2},\frac{3}{2}\\0,1\end{matrix}\right)$$

08.01.26.0024.01

$$\theta(1-|z|)\,E\left(1-\frac{1}{z}\right)=\frac{\pi}{2}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}0,1\\-\frac{1}{2},\frac{1}{2}\end{matrix}\right)\,/;\,z\notin(-1,0)$$

08.01.26.0025.01

$$\theta(|z|-1)\,E\left(1-\frac{1}{z}\right)=\frac{\pi}{2}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}0,1\\-\frac{1}{2},\frac{1}{2}\end{matrix}\right)\,/;\;\notin(-\infty,-1)$$

08.01.26.0026.01

$$\theta(1-|z|)\,\sqrt{\sqrt{1-z}+1}\;E\left(\frac{2\sqrt{1-z}}{\sqrt{1-z}+1}\right)=\frac{\pi}{2}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}\frac{3}{4},\frac{5}{4}\\0,1\end{matrix}\right)\,/;\;\notin(-1,0)$$

08.01.26.0027.01

$$\theta(1-|z|)\,\sqrt{1-\sqrt{1-z}}\;E\left(\frac{2\sqrt{1-z}}{\sqrt{1-z}-1}\right)=\frac{\pi}{2}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{matrix}\frac{3}{4},\frac{5}{4}\\0,1\end{matrix}\right)\,/;\;\notin(-1,0)$$

08.01.26.0028.01

$$\theta(|z|-1)\,\sqrt{1+\sqrt{1-z}}\;E\left(\frac{2\sqrt{1-z}}{\sqrt{1-z}+1}\right)=\frac{\pi}{2}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{matrix}\frac{3}{4},\frac{5}{4}\\0,1\end{matrix}\right)$$

08.01.26.0029.01

$$\theta(|z|-1)\sqrt{1-\sqrt{1-z}}\;E\left(\frac{2\sqrt{1-z}}{\sqrt{1-z}-1}\right)=\frac{\pi}{2}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\frac{3}{4},\frac{5}{4}\\0,1\end{array}\right)/;\,z\notin(-\infty,-1)$$

08.01.26.0030.01

$$\frac{\theta(1-|z|)}{\sqrt{1-\sqrt{1-z}}}\,E\left(\frac{2\sqrt{1-z}}{\sqrt{1-z}+1}\right)=\frac{\pi}{2}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{3}{4}\\-\frac{1}{2},\frac{1}{2}\end{array}\right)/;\,\notin(-1,0)$$

08.01.26.0031.01

$$\frac{\theta(1-|z|)}{\sqrt{\sqrt{1-z}+1}}\,E\left(\frac{2\sqrt{1-z}}{\sqrt{1-z}-1}\right)=\frac{\pi}{2}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{3}{4}\\-\frac{1}{2},\frac{1}{2}\end{array}\right)/;\,\notin(-1,0)$$

08.01.26.0032.01

$$\frac{\theta(|z|-1)}{\sqrt{1-\sqrt{1-z}}}\,E\left(\frac{2\sqrt{1-z}}{\sqrt{1-z}+1}\right)=\frac{\pi}{2}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{3}{4}\\-\frac{1}{2},\frac{1}{2}\end{array}\right)$$

08.01.26.0033.01

$$\frac{\theta(|z|-1)}{\sqrt{\sqrt{1-z}+1}}\,E\left(\frac{2\sqrt{1-z}}{\sqrt{1-z}-1}\right)=\frac{\pi}{2}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{3}{4}\\-\frac{1}{2},\frac{1}{2}\end{array}\right)/;\,z\notin(-\infty,-1)$$

08.01.26.0034.01

$$\theta(1-|z|)\sqrt{\sqrt{z-1}+\sqrt{z}}\;E\left(\frac{2\sqrt{z-1}}{\sqrt{z-1}+\sqrt{z}}\right)=\frac{\pi}{2}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{5}{4}\\0,\frac{1}{2}\end{array}\right)/;\,z\notin(-1,0)$$

08.01.26.0035.01

$$\theta(1-|z|)\sqrt{\sqrt{z}-\sqrt{z-1}}\;E\left(\frac{2\sqrt{z-1}}{\sqrt{z-1}-\sqrt{z}}\right)=\frac{\pi}{2}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{5}{4}\\0,\frac{1}{2}\end{array}\right)/;\,z\notin(-1,0)$$

08.01.26.0036.01

$$\theta(|z|-1)\sqrt{\sqrt{z-1}+\sqrt{z}}\;E\left(\frac{2\sqrt{z-1}}{\sqrt{z-1}+\sqrt{z}}\right)=\frac{\pi}{2}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{5}{4}\\0,\frac{1}{2}\end{array}\right)/;\,z\notin(-\infty,-1)$$

08.01.26.0037.01

$$\theta(|z|-1)\sqrt{\sqrt{z}-\sqrt{z-1}}\;E\left(\frac{2\sqrt{z-1}}{\sqrt{z-1}-\sqrt{z}}\right)=\frac{\pi}{2}\,G_{2,2}^{0,2}\left(z\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{5}{4}\\0,\frac{1}{2}\end{array}\right)$$

08.01.26.0038.01

$$\frac{\theta(1-|z|)}{\sqrt{\sqrt{z}-\sqrt{z-1}}}\,E\left(\frac{2\sqrt{z-1}}{\sqrt{z-1}+\sqrt{z}}\right)=\frac{\pi}{2}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{5}{4}\\0,\frac{1}{2}\end{array}\right)/;\,z\notin(-1,0)$$

08.01.26.0039.01

$$\frac{\theta(1-|z|)}{\sqrt{\sqrt{z-1}+\sqrt{z}}}\,E\left(\frac{2\sqrt{z-1}}{\sqrt{z-1}-\sqrt{z}}\right)=\frac{\pi}{2}\,G_{2,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\frac{1}{4},\frac{5}{4}\\0,\frac{1}{2}\end{array}\right)/;\,z\notin(-1,0)$$

08.01.26.0040.01

$$\frac{\theta(|z|-1)}{\sqrt{\sqrt{z}-\sqrt{z-1}}}\, E\left(\frac{2\sqrt{z-1}}{\sqrt{z-1}+\sqrt{z}}\right) = \frac{\pi}{2}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{5}{4} \\ 0, \frac{1}{2} \end{matrix}\right) /;\ \notin (-\infty, -1)$$

08.01.26.0041.01

$$\frac{\theta(|z|-1)}{\sqrt{\sqrt{z-1}+\sqrt{z}}}\, E\left(\frac{2\sqrt{z-1}}{\sqrt{z-1}-\sqrt{z}}\right) = \frac{\pi}{2}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{5}{4} \\ 0, \frac{1}{2} \end{matrix}\right)$$

08.01.26.0042.01

$$\theta(1-|z|)\sqrt{\sqrt{1-z}+1}\; E\left(\frac{2\sqrt{1-z}\left(1-\sqrt{1-z}\right)}{z}\right) = \frac{\pi}{2}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{3}{4}, \frac{5}{4} \\ 0, 1 \end{matrix}\right) /;\ \notin (-1, 0)$$

08.01.26.0043.01

$$\theta(1-|z|)\sqrt{\sqrt{1-z}+1}\; E\left(\frac{2\sqrt{1-z}\left(1-\sqrt{1-z}\right)}{z}\right) = \frac{\pi}{2}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{3}{4}, \frac{5}{4} \\ 0, 1 \end{matrix}\right) /;\ \notin (-1, 0)$$

08.01.26.0044.01

$$\theta(|z|-1)\sqrt{\sqrt{1-z}+1}\; E\left(\frac{2\sqrt{1-z}\left(1-\sqrt{1-z}\right)}{z}\right) = \frac{\pi}{2}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{3}{4}, \frac{5}{4} \\ 0, 1 \end{matrix}\right)$$

08.01.26.0045.01

$$\theta(|z|-1)\sqrt{1-\sqrt{1-z}}\; E\left(-\frac{2\sqrt{1-z}\left(\sqrt{1-z}+1\right)}{z}\right) = \frac{\pi}{2}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{3}{4}, \frac{5}{4} \\ 0, 1 \end{matrix}\right) /;\ z \notin (-\infty, -1)$$

08.01.26.0046.01

$$\frac{\theta(1-|z|)}{\sqrt{1-\sqrt{1-z}}}\, E\left(\frac{2\sqrt{1-z}\left(1-\sqrt{1-z}\right)}{z}\right) = \frac{\pi}{2}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ -\frac{1}{2}, \frac{1}{2} \end{matrix}\right) /;\ z \notin (-1, 0)$$

08.01.26.0047.01

$$\frac{\theta(|z|-1)}{\sqrt{1-\sqrt{1-z}}}\, E\left(\frac{2\sqrt{1-z}\left(1-\sqrt{1-z}\right)}{z}\right) = \frac{\pi}{2}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ -\frac{1}{2}, \frac{1}{2} \end{matrix}\right)$$

08.01.26.0048.01

$$\frac{\theta(|z|-1)}{\sqrt{\sqrt{1-z}+1}}\, E\left(-\frac{2\sqrt{1-z}\left(\sqrt{1-z}+1\right)}{z}\right) = \frac{\pi}{2}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{3}{4} \\ -\frac{1}{2}, \frac{1}{2} \end{matrix}\right) /;\ z \notin (-\infty, -1)$$

08.01.26.0049.01

$$\theta(1-|z|)\left(\sqrt{1-z}+1\right) E\left(\frac{4\sqrt{1-z}}{\left(\sqrt{1-z}+1\right)^2}\right) = \frac{\pi}{2}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{3}{2}, \frac{3}{2} \\ 0, 2 \end{matrix}\right) /;\ |z| > 1 \lor \operatorname{Re}(z) > 0$$

08.01.26.0050.01

$$\theta(1-|z|)\left(1-\sqrt{1-z}\right) E\left(-\frac{4\sqrt{1-z}}{\left(1-\sqrt{1-z}\right)^2}\right) = \frac{\pi}{2}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{3}{2}, \frac{3}{2} \\ 0, 2 \end{matrix}\right) /;\ |z| > 1 \lor \operatorname{Re}(z) > 0$$

08.01.26.0051.01

$$\frac{\theta(1-|z|)}{1-\sqrt{1-z}}\, E\left(\frac{4\sqrt{1-z}}{\left(\sqrt{1-z}+1\right)^2}\right) = \frac{\pi}{2}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ -1, 1 \end{matrix}\right) /;\ |z| > 1 \lor \mathrm{Re}(z) > 0$$

08.01.26.0052.01

$$\frac{\theta(1-|z|)}{\sqrt{1-z}+1}\, E\left(-\frac{4\sqrt{1-z}}{\left(1-\sqrt{1-z}\right)^2}\right) = \frac{\pi}{2}\, G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ -1, 1 \end{matrix}\right) /;\ |z| > 1 \lor \mathrm{Re}(z) > 0$$

08.01.26.0053.01

$$\theta(|z|-1)\left(\sqrt{z-1}+\sqrt{z}\right) E\left(\frac{4\sqrt{z-1}\,\sqrt{z}}{\left(\sqrt{z-1}+\sqrt{z}\right)^2}\right) = \frac{\pi}{2}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ 0, 0 \end{matrix}\right) /;\ |z| < 1 \lor \mathrm{Re}(z) > 0$$

08.01.26.0054.01

$$\theta(|z|-1)\left(\sqrt{z}-\sqrt{z-1}\right) E\left(-\frac{4\sqrt{z-1}\,\sqrt{z}}{\left(\sqrt{z}-\sqrt{z-1}\right)^2}\right) = \frac{\pi}{2}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ 0, 0 \end{matrix}\right) /;\ |z| < 1 \lor \mathrm{Re}(z) > 0$$

08.01.26.0055.01

$$\frac{\theta(|z|-1)}{\sqrt{z}-\sqrt{z-1}}\, E\left(\frac{4\sqrt{z-1}\,\sqrt{z}}{\left(\sqrt{z-1}+\sqrt{z}\right)^2}\right) = \frac{\pi}{2}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ 0, 0 \end{matrix}\right) /;\ |z| < 1 \lor \mathrm{Re}(z) > 0$$

08.01.26.0056.01

$$\frac{\theta(|z|-1)}{\sqrt{z-1}+\sqrt{z}}\, E\left(-\frac{4\sqrt{z-1}\,\sqrt{z}}{\left(\sqrt{z}-\sqrt{z-1}\right)^2}\right) = \frac{\pi}{2}\, G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} -\frac{1}{2}, \frac{3}{2} \\ 0, 0 \end{matrix}\right) /;\ |z| < 1 \lor \mathrm{Re}(z) > 0$$

Generalized cases involving algebraic functions

08.01.26.0057.01

$$\frac{1}{\sqrt{z^2+1}\,\sqrt{\sqrt{z^2+1}-z}}\, E\left(2z\sqrt{z^2+1}-2z^2\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\, G_{2,2}^{1,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} -\frac{1}{4}, \frac{3}{4} \\ 0, 0 \end{matrix}\right)$$

08.01.26.0058.01

$$\frac{\sqrt{z+\sqrt{z^2+1}}}{\sqrt{z^2+1}}\, E\left(2z\sqrt{z^2+1}-2z^2\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\, G_{2,2}^{1,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} -\frac{1}{4}, \frac{3}{4} \\ 0, 0 \end{matrix}\right)$$

08.01.26.0059.01

$$\frac{1}{\sqrt{z^2+1}\,\sqrt{\sqrt{z^2+1}-z}}\, E\left(\frac{2z}{z+\sqrt{z^2+1}}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2}\, G_{2,2}^{1,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} -\frac{1}{4}, \frac{3}{4} \\ 0, 0 \end{matrix}\right)$$

08.01.26.0060.01

$$\frac{\sqrt{\sqrt{z^2+1}-z}}{\sqrt{z^2+1}} E\left(\frac{2z}{z-\sqrt{z^2+1}}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2} G_{2,2}^{1,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} -\frac{1}{4}, \frac{3}{4} \\ 0, 0 \end{matrix}\right)$$

08.01.26.0061.01

$$\frac{\sqrt{z+\sqrt{z^2+1}}}{\sqrt{z^2+1}} E\left(\frac{2z}{z+\sqrt{z^2+1}}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2} G_{2,2}^{1,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} -\frac{1}{4}, \frac{3}{4} \\ 0, 0 \end{matrix}\right)$$

08.01.26.0062.01

$$\frac{1}{\sqrt{z^2+1}\sqrt{z+\sqrt{z^2+1}}} E\left(\frac{2z}{z-\sqrt{z^2+1}}\right) = \frac{2\pi}{\Gamma\left(\frac{1}{4}\right)^2} G_{2,2}^{1,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} -\frac{1}{4}, \frac{3}{4} \\ 0, 0 \end{matrix}\right)$$

Generalized cases involving unit step θ

08.01.26.0063.01

$$\theta(1-|z|)\sqrt{z+\sqrt{z^2-1}}\; E\left(\frac{2\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right) = \frac{\pi}{2} G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{5}{4} \\ 0, \frac{1}{2} \end{matrix}\right) /;\ \mathrm{Re}(z) > 0$$

08.01.26.0064.01

$$\theta(1-|z|)\sqrt{z-\sqrt{z^2-1}}\; E\left(\frac{2\sqrt{z^2-1}}{\sqrt{z^2-1}-z}\right) = \frac{\pi}{2} G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{5}{4} \\ 0, \frac{1}{2} \end{matrix}\right) /;\ \mathrm{Re}(z) > 0$$

08.01.26.0065.01

$$\theta(|z|-1)\sqrt{z+\sqrt{z^2-1}}\; E\left(\frac{2\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right) = \frac{\pi}{2} G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{5}{4} \\ 0, \frac{1}{2} \end{matrix}\right) /;\ \mathrm{Re}(z) > 0$$

08.01.26.0066.01

$$\theta(|z|-1)\sqrt{z-\sqrt{z^2-1}}\; E\left(\frac{2\sqrt{z^2-1}}{\sqrt{z^2-1}-z}\right) = \frac{\pi}{2} G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{5}{4} \\ 0, \frac{1}{2} \end{matrix}\right) /;\ \mathrm{Re}(z) > 0$$

08.01.26.0067.01

$$\frac{\theta(1-|z|)}{\sqrt{z-\sqrt{z^2-1}}} E\left(\frac{2\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right) = \frac{\pi}{2} G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{5}{4} \\ 0, \frac{1}{2} \end{matrix}\right) /;\ \mathrm{Re}(z) > 0$$

08.01.26.0068.01

$$\frac{\theta(1-|z|)}{\sqrt{z+\sqrt{z^2-1}}} E\left(\frac{2\sqrt{z^2-1}}{\sqrt{z^2-1}-z}\right) = \frac{\pi}{2} G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{5}{4} \\ 0, \frac{1}{2} \end{matrix}\right) /;\ \mathrm{Re}(z) > 0$$

08.01.26.0069.01

$$\frac{\theta(|z|-1)}{\sqrt{z-\sqrt{z^2-1}}} E\left(\frac{2\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right) = \frac{\pi}{2} G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{5}{4} \\ 0, \frac{1}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

08.01.26.0070.01

$$\frac{\theta(|z|-1)}{\sqrt{z+\sqrt{z^2-1}}} E\left(\frac{2\sqrt{z^2-1}}{\sqrt{z^2-1}-z}\right) = \frac{\pi}{2} G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{4}, \frac{5}{4} \\ 0, \frac{1}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

Through other functions

Involving incomplete elliptic integrals

08.01.26.0071.01

$$E(z) = (1-z)\, \Pi\left(z; \frac{\pi}{2} \,\middle|\, z\right)$$

08.01.26.0072.01

$$E(z) = (1-z)\, \Pi(z \mid z)$$

08.01.26.0073.01

$$E(z) = E\left(\frac{\pi}{2} \,\middle|\, z\right)$$

Involving Legendre functions

08.01.26.0074.01

$$E(z) = \frac{\pi}{4}\left(P_{\frac{1}{2}}(1-2z) + P_{-\frac{1}{2}}(1-2z)\right)$$

08.01.26.0075.01

$$E(z) = \frac{1}{2}\left(Q_{-\frac{1}{2}}(2z-1) - Q_{\frac{1}{2}}(2z-1)\right)$$

Involving some hypergeometric-type functions

08.01.26.0076.01

$$E(z) = 2\, F_1\left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; 1, z\right)$$

Representations through equivalent functions

With related functions

08.01.27.0001.01

$$E(z)\, K(1-z) - K(z)\, K(1-z) + E(1-z)\, K(z) = \frac{\pi}{2}$$

08.01.27.0002.01

$$E(2) = \sqrt{\frac{2}{\pi}}\, \Gamma\left(\frac{3}{4}\right)^2 - \frac{2}{\pi^2}\, \Gamma\left(\frac{3}{4}\right)^4 K(2)$$

Theorems

The perimeter of an ellipse

The perimeter s of an ellipse with semi-axes a and b is $s = 4\,b\,E\left(1 - \frac{a^2}{b^2}\right)$.

The gravitational or electrostatic potential of a uniform circular ring

The gravitational or electrostatic potential $V(r)$ of a uniform circular ring of radius r_0 is given by

$$V(r) \propto \int_0^{2\pi} \frac{\cos(\phi)}{\sqrt{r^2 + r_0{}^2 + z^2 - 2\,r\,r_0 \cos(\phi)}}\,d\phi ==$$

$$\frac{2}{r\,r_0\sqrt{z^2 + (r + r_0)^2}}\left(\left(r^2 + z^2 + r_0^2\right) K\left(-\frac{4\,r\,r_0}{z^2 + (r + r_0)^2}\right) - \left(z^2 + (r + r_0)^2\right) E\left(-\frac{4\,r\,r_0}{z^2 + (r + r_0)^2}\right)\right)$$

History

– J. Wallis (1655)
– A. M. Legendre (1811)
– C. G. J. Jacobi (1829)

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