EllipticPi3

Notations

Traditional name

Incomplete elliptic integral of the third kind

Traditional notation

$\Pi(n;\, z \mid m)$

Mathematica StandardForm notation

`EllipticPi[n, z, m]`

Primary definition

08.06.02.0001.01

$$\Pi(n;\, z \mid m) = \int_0^z \frac{1}{\left(1 - n\sin^2(t)\right)\sqrt{1 - m\sin^2(t)}}\, dt$$

Specific values

Specialized values

For fixed *n*, *z*

08.06.03.0001.01

$$\Pi(n;\, z \mid 0) = \frac{\tanh^{-1}\left(\sqrt{n-1}\, \tan(z)\right)}{\sqrt{n-1}}$$

08.06.03.0013.01

$$\Pi(n;\, z \mid 1) = \frac{\sqrt{n}\, \tanh^{-1}\left(\sqrt{n}\, \sin(z)\right) - \log(\sec(z) + \tan(z))}{n-1} \;/;\; |\mathrm{Re}(z)| \leq \frac{\pi}{2}$$

08.06.03.0014.01

$$\Pi(n;\, z \mid 1) = \frac{\sqrt{n}\, \tanh^{-1}\left(\sqrt{n}\, \sin(z)\right) - \tanh^{-1}(\sin(z))}{n-1} \;/;\; |\mathrm{Re}(z)| < \frac{\pi}{2}$$

08.06.03.0002.01

$$\Pi(n;\, z \mid 1) = \frac{1}{2\,(n-1)} \left(\sqrt{n}\, \log\left(\frac{1 + \sqrt{n}\, \sin(z)}{1 - \sqrt{n}\, \sin(z)}\right) - 2\log(\sec(z) + \tan(z))\right) \;/;\; |\mathrm{Re}(z)| \leq \frac{\pi}{2}$$

08.06.03.0015.01

$$\Pi(n; z \mid 1) = \tilde{\infty} \;/;\; |\mathrm{Re}(z)| > \frac{\pi}{2}$$

08.06.03.0003.01

$$\Pi(n; z \mid n) == \frac{1}{1-n}\left(E(z \mid n) - \frac{n \sin(2 z)}{2 \sqrt{1 - n \sin^2(z)}}\right)$$

For fixed *n*, *m*

08.06.03.0004.01

$$\Pi\left(n; \frac{\pi}{2} \,\middle|\, m\right) == \Pi(n \mid m)$$

08.06.03.0005.01

$$\Pi\left(n; \frac{k \pi}{2} \,\middle|\, m\right) == k\, \Pi(n \mid m) \;/;\; k \in \mathbb{Z}$$

08.06.03.0006.01

$$\Pi\left(n; \csc^{-1}\left(\sqrt{m}\right) \,\middle|\, m\right) == \frac{1}{\sqrt{m}}\, \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right)$$

08.06.03.0016.01

$$\Pi\left(n; \pi k + \csc^{-1}\left(\sqrt{m}\right) \,\middle|\, m\right) = \frac{1}{\sqrt{m}}\, \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + 2 k\, \Pi(n \mid m) \;/;\; k \in \mathbb{Z}$$

For fixed *z*, *m*

08.06.03.0007.01

$$\Pi(0; z \mid m) == F(z \mid m)$$

08.06.03.0008.01

$$\Pi(1; z \mid m) == \frac{\sqrt{1 - m \sin^2(z)}\, \tan(z) - E(z \mid m)}{1-m} + F(z \mid m)$$

Values at infinities

08.06.03.0009.01

$$\Pi(\infty; z \mid m) == 0$$

08.06.03.0010.01

$$\Pi(-\infty; z \mid m) == 0$$

08.06.03.0011.01

$$\Pi(n; z \mid \infty) == 0$$

08.06.03.0012.01

$$\Pi(n; z \mid -\infty) == 0$$

General characteristics

Domain and analyticity

$\Pi(n; z \mid m)$ is an analytical function of n, z, and m which is defined over $\mathbb{C}^3$.

08.06.04.0001.01

$$(n * z * m) \longrightarrow \Pi(n; z \mid m) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\Pi(n; z \mid m)$ is an odd function with respect to z.

08.06.04.0002.01

$$\Pi(n; -z \mid m) = -\Pi(n; z \mid m)$$

Mirror symmetry

08.06.04.0004.01

$$\Pi(\overline{n}; \overline{z} \mid \overline{m}) = \overline{\Pi(n; z \mid m)} \;/;\; \neg (m \in \mathbb{R} \wedge m > 1 \wedge n \in \mathbb{R} \wedge n > 1)$$

Periodicity

$\Pi(n; z \mid m)$ is a quasi-periodic function with respect to z.

08.06.04.0003.01

$$\Pi(n; z + \pi k \mid m) = 2 k \Pi(n \mid m) + \Pi(n; z \mid m) \;/;\; k \in \mathbb{Z} \bigwedge -1 \leq n \leq 1$$

08.06.04.0016.01

$$\Pi(n; z \mid m) = \Pi\left(n; z - \pi \left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \middle| m\right) + 2 \left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \Pi(n \mid m) \;/;\; -1 \leq n \leq 1$$

Quasi-symmetry

08.06.04.0017.01

$$\Pi(n; x + i y \mid m) = \Pi\left(n; \pi \operatorname{frac}\left(\frac{x}{\pi}\right) + i y \middle| m\right) + 2 \operatorname{sgn}(x) \left\lfloor \frac{|x|}{\pi} \right\rfloor \Pi(n \mid m) \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$$

Poles and essential singularities

With respect to *n*

The function $\Pi(n; z \mid m)$ does not have poles and essential singularities with respect to n.

08.06.04.0007.01

$$\mathcal{S}ing_n(\Pi(n; z \mid m)) = \{\}$$

With respect to *z*

The function $\Pi(n; z \mid m)$ does not have poles and essential singularities with respect to z.

08.06.04.0006.01

$$\mathcal{S}ing_z(\Pi(n; z \mid m)) = \{\}$$

With respect to *m*

The function $\Pi(n; z \mid m)$ does not have poles and essential singularities with respect to m.

08.06.04.0005.01

$$\mathcal{S}ing_m(\Pi(n; z \mid m)) = \{\}$$

Branch points

With respect to *n*

For fixed z, m, the function $\Pi(n; z \mid m)$ does not have branch points.

08.06.04.0014.01

$$\mathcal{BP}_n(\Pi(n; z \mid m)) == \{\}$$

With respect to *z*

For fixed n, m, the function $\Pi(n; z \mid m)$ has an infinite number of branch points at $z == \pm\sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + \pi k /; k \in \mathbb{Z}$, $z == \pm\sin^{-1}\left(\frac{1}{\sqrt{n}}\right) + \pi k /; k \in \mathbb{Z}$, $z == \frac{\pi}{2} + \pi k /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)$ and $z == \tilde{\infty}$.

08.06.04.0009.01

$$\mathcal{BP}_z(\Pi(n; z \mid m)) == \left\{\left\{\sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + \pi k /; k \in \mathbb{Z}\right\}, \left\{-\sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + \pi k /; k \in \mathbb{Z}\right\}, \left\{\sin^{-1}\left(\frac{1}{\sqrt{n}}\right) + \pi k /; k \in \mathbb{Z}\right\}, \left\{-\sin^{-1}\left(\frac{1}{\sqrt{n}}\right) + \pi k /; k \in \mathbb{Z}\right\}, \left\{\frac{\pi}{2} + \pi k /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)\right\}, \tilde{\infty}\right\}$$

08.06.04.0010.01

$$\mathcal{R}_z\left(\Pi(n; z \mid m), \sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + \pi k\right) == 2 /; k \in \mathbb{Z}$$

08.06.04.0011.01

$$\mathcal{R}_z\left(\Pi(n; z \mid m), -\sin^{-1}\left(\frac{1}{\sqrt{m}}\right) + \pi k\right) == 2 /; k \in \mathbb{Z}$$

08.06.04.0012.01

$$\mathcal{R}_z\left(\Pi(n; z \mid m), \sin^{-1}\left(\frac{1}{\sqrt{n}}\right) + \pi k\right) == \log /; k \in \mathbb{Z}$$

08.06.04.0013.01

$$\mathcal{R}_z\left(\Pi(n; z \mid m), -\sin^{-1}\left(\frac{1}{\sqrt{n}}\right) + \pi k\right) == \log /; k \in \mathbb{Z}$$

08.06.04.0018.01

$$\mathcal{R}_z\left(\Pi(n; z \mid m), \frac{\pi}{2} + \pi k\right) == 2 /; k \in \mathbb{Z} \bigwedge m \notin (0, 1)$$

With respect to *m*

For fixed n, z, the function $\Pi(n; z \mid m)$ has two branch points at $m == \csc^2(z)$and $m == \tilde{\infty}$.

08.06.04.0008.01

$$\mathcal{BP}_m(\Pi(n; z \mid m)) == \left\{\csc^2(z), \tilde{\infty}\right\}$$

Branch cuts

With respect to *n*

For fixed z, m the function $\Pi(n; z \mid m)$ does not have branch cuts.

08.06.04.0015.01

$$\mathcal{BC}_n(\Pi(n; z \mid m)) == \{\}$$

With respect to *z*

General description

For fixed m, n, the function $\Pi(n; z \mid m)$ can have up to eight infinite sets of branch cuts (it has at least six), which form very complicated curves in the case of generic m, n.

For fixed real m, n /; $\max(m, n) < 1$, the function $\Pi(n; z \mid m)$ does not have branch cuts on the real axis and on the vertical intervals $\left\{\csc^{-1}\left(\sqrt{m}\right) + \pi k, \pi - \csc^{-1}\left(\sqrt{m}\right) + \pi k\right\}$ /; $k \in \mathbb{Z} \wedge m \in (-\infty, 1)$.

For fixed real $m < 1$, the function $\Pi(n; z \mid m)$ has six infinite sets of branch cuts, four of them are located on vertical intervals starting at the points $z == \pi k \pm \csc^{-1}\left(\sqrt{m}\right)$ /; $k \in \mathbb{Z}$ and extending to imaginary infinity.

1) real intervals $\{\pi k + \csc^{-1}(\sqrt{m}), \pi k + \frac{\pi}{2}\} /; k \in \mathbb{Z} \land m > 1 \land n \notin (m, \infty)$, where $\Pi(n; z \mid m)$ is continuous from below; if $1 < n < m$ these intervals include two sets $\{\pi k + \csc^{-1}(\sqrt{m}), \pi k + \csc^{-1}(\sqrt{n})\}$ and $\{\pi k + \csc^{-1}(\sqrt{n}), \pi k + \frac{\pi}{2}\}$, where $\Pi(n; z \mid m)$ has different values of jumps (for generic complex m these branch cuts deform into a complicated curves; for generic complex n the intervals $\{\pi k + \csc^{-1}(\sqrt{n}), \pi k + \frac{\pi}{2}\}$ also deform into a complicated curves); in the case $\max(m, n) < 1$ these real intervals vanish

2) real intervals $\{\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}(\sqrt{m})\} /; k \in \mathbb{Z} \land m > 1 \land n \notin (m, \infty)$, where $\Pi(n; z \mid m)$ is continuous from above; if $1 < n < m$ these intervals include two sets $\{\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}(\sqrt{n})\}$ and $\{\pi (k+1) - \csc^{-1}(\sqrt{n}), \pi (k+1) - \csc^{-1}(\sqrt{m})\}$, where $\Pi(n; z \mid m)$ has different values of jumps (for generic complex m these branch cuts deform into a complicated curves; for generic complex n the intervals $\{\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}(\sqrt{n})\}$ also deform into a complicated curves); in the case $\max(m, n) < 1$ these real intervals vanish

3) real intervals $\{\pi k + \csc^{-1}(\sqrt{n}), \pi k + \frac{\pi}{2}\} /; k \in \mathbb{Z} \land m \notin (n, \infty) \land n > 1$, where $\Pi(n; z \mid m)$ is continuous from below; if $1 < m < n$ these intervals include two sets $\{\pi k + \csc^{-1}(\sqrt{n}), \pi k + \csc^{-1}(\sqrt{m})\}$ and $\{\pi k + \csc^{-1}(\sqrt{m}), \pi k + \frac{\pi}{2}\}$, where $\Pi(n; z \mid m)$ has different values of jumps (for generic complex n these branch cuts deform into a complicated curves; for generic complex m the intervals $\{\pi k + \csc^{-1}(\sqrt{m}), \pi k + \frac{\pi}{2}\}$ also deform into a complicated curves); in the case $\max(m, n) < 1$ these real intervals vanish

4) real intervals $\{\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}(\sqrt{n})\} /; k \in \mathbb{Z} \land m \notin (n, \infty) \land n > 1$, where $\Pi(n; z \mid m)$ is continuous from above; if $1 < m < n$ these intervals include two sets $\{\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}(\sqrt{m})\}$ and $\{\pi (k+1) - \csc^{-1}(\sqrt{m}), \pi (k+1) - \csc^{-1}(\sqrt{n})\}$, where $\Pi(n; z \mid m)$ has different values of jumps (for generic complex n these branch cuts deform into a complicated curves; for generic complex m the intervals $\{\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}(\sqrt{m})\}$ also deform into a complicated curves); in the case $\max(m, n) < 1$ these real intervals vanish

5) vertical intervals $\{\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k + i\infty\} /; k \in \mathbb{Z} \land m \notin (0, 1)$,or $\{\pi - \csc^{-1}(\sqrt{m}) + 2\pi k, \frac{\pi}{2} + 2\pi k + i\infty\} /; k \in \mathbb{Z} \land m \in (0, 1)$, where $\Pi(n; z \mid m)$ is continuous from the left;

6) vertical intervals $\{\frac{3\pi}{2} + 2\pi k, \frac{3\pi}{2} + 2\pi k + i\infty\} /; k \in \mathbb{Z} \land m \notin (0, 1)$, or $\{2\pi - \csc^{-1}(\sqrt{m}) + 2\pi k, \frac{3\pi}{2} + 2\pi k + i\infty\} /; k \in \mathbb{Z} \land m \in (0, 1)$, where $\Pi(n; z \mid m)$ is continuous from the right;

7) vertical intervals $\{\frac{\pi}{2} + 2\pi k - i\infty, \frac{\pi}{2} + 2\pi k\} /; k \in \mathbb{Z} \land m \notin (0, 1)$, or $\{\frac{\pi}{2} + 2\pi k - i\infty, 2\pi k + \csc^{-1}(\sqrt{m})\} /; k \in \mathbb{Z} \land m \in (0, 1)$, where $\Pi(n; z \mid m)$ is continuous from the left;

8) vertical intervals $\{\frac{3\pi}{2} + 2\pi k - i\infty, \frac{3\pi}{2} + 2\pi k\} /; k \in \mathbb{Z} \land m \notin (0, 1)$, or $\{\frac{3\pi}{2} + 2\pi k - i\infty, 2\pi k + \pi + \csc^{-1}(\sqrt{m})\} /; k \in \mathbb{Z} \land m \in (0, 1)$, where $\Pi(n; z \mid m)$ is continuous from the right.

08.06.04.0019.01

$$\mathcal{BC}_z(\Pi(n; z \mid m)) == \Big\{\Big\{\Big\{\Big(\pi k + \csc^{-1}\big(\sqrt{m}\big), \pi k + \frac{\pi}{2}\Big), i\Big\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land m > 1 \land n \notin (1, \infty)\Big\},$$
$$\Big\{\Big\{\Big(\pi k + \csc^{-1}\big(\sqrt{m}\big), \pi k + \csc^{-1}\big(\sqrt{n}\big)\Big), i\Big\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land m > 1 \land n \in \mathbb{R} \land 1 < n < m\Big\},$$
$$\Big\{\Big\{\Big(\pi k + \csc^{-1}\big(\sqrt{n}\big), \pi k + \frac{\pi}{2}\Big), i\Big\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land n \in \mathbb{R} \land 1 < n < m\Big\},$$
$$\Big\{\Big\{\Big(\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}\big(\sqrt{n}\big)\Big), -i\Big\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land m > 1 \land n \notin (1, \infty)\Big\},$$
$$\Big\{\Big\{\Big(\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}\big(\sqrt{m}\big)\Big), -i\Big\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land n \in \mathbb{R} \land 1 < n < m\Big\},$$
$$\Big\{\Big\{\Big(\pi (k+1) - \csc^{-1}\big(\sqrt{n}\big), \pi (k+1) - \csc^{-1}\big(\sqrt{m}\big)\Big), -i\Big\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land n \in \mathbb{R} \land 1 < n < m\Big\},$$
$$\Big\{\Big\{\Big(\pi k + \csc^{-1}\big(\sqrt{n}\big), \pi k + \frac{\pi}{2}\Big), i\Big\} /; k \in \mathbb{Z} \land m \notin (1, \infty) \land n \in \mathbb{R} \land n > 1\Big\},$$
$$\Big\{\Big\{\Big(\pi k + \csc^{-1}\big(\sqrt{n}\big), \pi k + \csc^{-1}\big(\sqrt{m}\big)\Big), i\Big\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land n \in \mathbb{R} \land 1 < m < n\Big\},$$
$$\Big\{\Big\{\Big(\pi k + \csc^{-1}\big(\sqrt{m}\big), \pi k + \frac{\pi}{2}\Big), i\Big\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land n \in \mathbb{R} \land 1 < m < n\Big\},$$
$$\Big\{\Big\{\Big(\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}\big(\sqrt{n}\big)\Big), -i\Big\} /; k \in \mathbb{Z} \land m \notin (1, \infty) \land n \in \mathbb{R} \land n > 1\Big\},$$
$$\Big\{\Big\{\Big(\pi k + \frac{\pi}{2}, \pi (k+1) - \csc^{-1}\big(\sqrt{m}\big)\Big), -i\Big\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land n \in \mathbb{R} \land 1 < m < n\Big\},$$
$$\Big\{\Big\{\Big(\pi (k+1) - \csc^{-1}\big(\sqrt{m}\big), \pi (k+1) - \csc^{-1}\big(\sqrt{n}\big)\Big), -i\Big\} /; k \in \mathbb{Z} \land m \in \mathbb{R} \land n \in \mathbb{R} \land 1 < m < n\Big\},$$
$$\Big\{\Big\{\Big(2 \pi k + \frac{\pi}{2}, 2 k \pi + \frac{\pi}{2} + i \infty\Big), 1\Big\} /; k \in \mathbb{Z}\Big\}, \Big\{\Big\{\Big(2 \pi k + \frac{3 \pi}{2}, 2 k \pi + \frac{3 \pi}{2} + i \infty\Big), -1\Big\} /; k \in \mathbb{Z}\Big\},$$
$$\Big\{\Big\{\Big(2 \pi k + \frac{\pi}{2} - i \infty, 2 k \pi + \frac{\pi}{2}\Big), 1\Big\} /; k \in \mathbb{Z}\Big\}, \Big\{\Big\{\Big(2 \pi k + \frac{\pi}{2} - i \infty, 2 k \pi + \frac{\pi}{2}\Big), -1\Big\} /; k \in \mathbb{Z}\Big\}\Big\}$$

Formulas on real axis for real *m*, *n*

For max(*m*, *n*) < 1

For fixed real m, n /; $\max(m, n) < 1$, the function $\Pi(n; z \mid m)$ does not have branch cuts on the real axis.

For *m* < 1 < *n*

08.06.04.0020.01

$$\lim_{\epsilon \to +0} \Pi(n; x + i \epsilon \mid m) == \Pi(n; x \mid m) + \frac{\pi i \sqrt{n}}{\sqrt{n-1} \sqrt{n-m}} /;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge m < 1 < n \bigwedge \pi k + \csc^{-1}\big(\sqrt{n}\big) < x < \pi k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

08.06.04.0021.01

$$\lim_{\epsilon \to +0} \Pi(n; x - i \epsilon \mid m) == \Pi(n; x \mid m) /; x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge m < 1 < n \bigwedge \pi k + \csc^{-1}\big(\sqrt{n}\big) < x < \pi k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

08.06.04.0022.01

$$\lim_{\epsilon \to +0} \Pi(n; x + i \epsilon \mid m) == \Pi(n; x \mid m) /; x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge m < 1 < n \bigwedge \pi k + \frac{\pi}{2} < x < \pi (k+1) - \csc^{-1}\big(\sqrt{n}\big) \bigwedge k \in \mathbb{Z}$$

08.06.04.0023.01

$$\lim_{\epsilon\to+0} \Pi(n; x - i\,\epsilon \mid m) = \Pi(n; x \mid m) - \frac{\pi\, i \sqrt{n}}{\sqrt{n-1}\,\sqrt{n-m}} \,/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge m < 1 < n \bigwedge \pi\, k + \frac{\pi}{2} < x < \pi\,(k+1) - \csc^{-1}\!\left(\sqrt{n}\right) \bigwedge k \in \mathbb{Z}$$

For $n < 1 < m$

08.06.04.0024.01

$$\lim_{\epsilon\to+0} \Pi(n; x + i\,\epsilon \mid m) = \frac{2}{\sqrt{m}}\, \Pi\!\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + 4\left(\left\lfloor \frac{x}{\pi} - \frac{1}{2} \right\rfloor + 1\right) \Pi(n \mid m) - \Pi(n; x \mid m) \,/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge n < 1 < m \bigwedge \pi\, k + \csc^{-1}\!\left(\sqrt{m}\right) < x < \pi\, k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

08.06.04.0025.01

$$\lim_{\epsilon\to+0} \Pi(n; x - i\,\epsilon \mid m) = \Pi(n; x \mid m) \,/;\, x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge n < 1 < m \bigwedge \pi\, k + \csc^{-1}\!\left(\sqrt{m}\right) < x < \pi\, k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

08.06.04.0026.01

$$\lim_{\epsilon\to+0} \Pi(n; x + i\,\epsilon \mid m) = \Pi(n; x \mid m) \,/;\, x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge n < 1 < m \bigwedge \pi\, k + \frac{\pi}{2} < x < \pi\,(k+1) - \csc^{-1}\!\left(\sqrt{m}\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0027.01

$$\lim_{\epsilon\to+0} \Pi(n; x - i\,\epsilon \mid m) = -\frac{2}{\sqrt{m}}\, \Pi\!\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + 4\left(\left\lfloor \frac{x}{\pi} - \frac{1}{2} \right\rfloor + 1\right) \Pi(n \mid m) - \Pi(n; x \mid m) \,/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge n < 1 < m \bigwedge \pi\, k + \frac{\pi}{2} < x < \pi\,(k+1) - \csc^{-1}\!\left(\sqrt{m}\right) \bigwedge k \in \mathbb{Z}$$

For $1 < n < m$

08.06.04.0028.01

$$\lim_{\epsilon\to+0} \Pi(n; x + i\,\epsilon \mid m) = \frac{2}{\sqrt{m}}\, \Pi\!\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) - \Pi(n; x \mid m) \,/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi\, k + \csc^{-1}\!\left(\sqrt{m}\right) < x < \pi\, k + \csc^{-1}\!\left(\sqrt{n}\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0029.01

$$\lim_{\epsilon\to+0} \Pi(n; x + i\,\epsilon \mid m) = -\Pi(n; x \mid m) + \frac{2}{\sqrt{m}}\, \Pi\!\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) - \frac{\pi\, i \sqrt{n}}{\sqrt{n-1}\,\sqrt{n-m}} \,/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi\, k + \csc^{-1}\!\left(\sqrt{n}\right) < x < \pi\, k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

08.06.04.0030.01

$$\lim_{\epsilon\to+0} \Pi(n; x - i\,\epsilon \mid m) = \Pi(n; x \mid m) \,/;\, x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi\, k + \csc^{-1}\!\left(\sqrt{m}\right) < x < \pi\, k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

08.06.04.0031.01

$$\lim_{\epsilon\to+0} \Pi(n; x + i\,\epsilon \mid m) = \Pi(n; x \mid m) \,/;\, x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi\, k + \frac{\pi}{2} < x < \pi\,(k+1) - \csc^{-1}\!\left(\sqrt{m}\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0032.01

$$\lim_{\epsilon \to +0} \Pi(n;\, x - i\,\epsilon \mid m) = -\Pi(n;\, x \mid m) - \frac{2}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + \frac{\pi\, i\, \sqrt{n}}{\sqrt{n-1}\,\sqrt{n-m}} \;/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi\, k + \frac{\pi}{2} < x < \pi\,(k+1) - \csc^{-1}\!\left(\sqrt{n}\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0033.01

$$\lim_{\epsilon \to +0} \Pi(n;\, x - i\,\epsilon \mid m) = -\frac{2}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) - \Pi(n;\, x \mid m) \;/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi\, k - \csc^{-1}\!\left(\sqrt{n}\right) < x < \pi\, k - \csc^{-1}\!\left(\sqrt{m}\right) \bigwedge k \in \mathbb{Z}$$

For $1 < m < n$

08.06.04.0034.01

$$\lim_{\epsilon \to +0} \Pi(n;\, x + i\,\epsilon \mid m) = \Pi(n;\, x \mid m) - \frac{\pi\, i\, \sqrt{n}}{\sqrt{n-1}\,\sqrt{n-m}} \;/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi\, k + \csc^{-1}\!\left(\sqrt{n}\right) < x < \pi\, k + \csc^{-1}\!\left(\sqrt{m}\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0035.01

$$\lim_{\epsilon \to +0} \Pi(n;\, x + i\,\epsilon \mid m) = -\Pi(n;\, x \mid m) + \frac{2}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + \frac{\pi\, i\, \sqrt{n}}{\sqrt{n-1}\,\sqrt{n-m}} \;/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi\, k + \csc^{-1}\!\left(\sqrt{m}\right) < x < \pi\, k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

08.06.04.0036.01

$$\lim_{\epsilon \to +0} \Pi(n;\, x - i\,\epsilon \mid m) = \Pi(n;\, x \mid m) \;/;\; x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi\, k + \csc^{-1}\!\left(\sqrt{n}\right) < x < \pi\, k + \frac{\pi}{2} \bigwedge k \in \mathbb{Z}$$

08.06.04.0037.01

$$\lim_{\epsilon \to +0} \Pi(n;\, x + i\,\epsilon \mid m) = \Pi(n;\, x \mid m) \;/;\; x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi\, k + \frac{\pi}{2} < x < \pi\,(k+1) - \csc^{-1}\!\left(\sqrt{n}\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0038.01

$$\lim_{\epsilon \to +0} \Pi(n;\, x - i\,\epsilon \mid m) = -\Pi(n;\, x \mid m) - \frac{2}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) - \frac{\pi\, i\, \sqrt{n}}{\sqrt{n-1}\,\sqrt{n-m}} \;/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi\, k + \frac{\pi}{2} < x < \pi\,(k+1) - \csc^{-1}\!\left(\sqrt{m}\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0039.01

$$\lim_{\epsilon \to +0} \Pi(n;\, x - i\,\epsilon \mid m) = \Pi(n;\, x \mid m) + \frac{\pi\, i\, \sqrt{n}}{\sqrt{n-1}\,\sqrt{n-m}} \;/;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi\, k - \csc^{-1}\!\left(\sqrt{m}\right) < x < \pi\, k - \csc^{-1}\!\left(\sqrt{n}\right) \bigwedge k \in \mathbb{Z}$$

Formulas for vertical intervals

For $m < 1$

For fixed real $m < 1$, the function $\Pi(n; z \mid m)$ has branch points $\csc^{-1}(\sqrt{m}) + \pi k /; k \in \mathbb{Z}$ and $\pi - \csc^{-1}(\sqrt{m}) + \pi k /; k \in \mathbb{Z}$. In this case branch cuts lay at the vertical lines beginning from these points and going to imaginary infinity. By this reason for fixed real $m < 1$, the function $\Pi(n; z \mid m)$ does not have branch cuts on the vertical intervals $\{\csc^{-1}(\sqrt{m}) + \pi k, \pi - \csc^{-1}(\sqrt{m}) + \pi k\} /; k \in \mathbb{Z} \bigwedge m \in (-\infty, 1)$.

For $m > 0$

08.06.04.0040.01

$$\lim_{\epsilon \to +0} \Pi\left(n; 2\pi k + i x - \epsilon + \frac{\pi}{2} \,\middle|\, m\right) = \Pi\left(n; 2\pi k + i x + \frac{\pi}{2} \,\middle|\, m\right) /; x \in \mathbb{R} \bigwedge k \in \mathbb{Z}$$

08.06.04.0041.01

$$\lim_{\epsilon \to +0} \Pi\left(n; 2\pi k + i x + \epsilon + \frac{\pi}{2} \,\middle|\, m\right) = -\frac{2}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + 4(k+1)\Pi(n \mid m) - \Pi\left(n; i x + \frac{\pi}{2} \,\middle|\, m\right) /;$$
$$m \in \mathbb{R} \bigwedge x \in \mathbb{R} \bigwedge \left(0 < m < 1 \bigwedge x > -\mathrm{Im}\left(\csc^{-1}(\sqrt{m})\right) \bigvee m > 1 \wedge x < 0\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0042.01

$$\lim_{\epsilon \to +0} \Pi\left(n; 2\pi k + i x + \epsilon + \frac{\pi}{2} \,\middle|\, m\right) = \frac{2}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + 4k\,\Pi(n \mid m) - \Pi\left(n; i x + \frac{\pi}{2} \,\middle|\, m\right) /;$$
$$m \in \mathbb{R} \bigwedge x \in \mathbb{R} \bigwedge \left(0 < m < 1 \bigwedge x < \mathrm{Im}\left(\csc^{-1}(\sqrt{m})\right) \bigvee m > 1 \wedge x > 0\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0043.01

$$\lim_{\epsilon \to +0} \Pi\left(n; 2\pi k + i x - \epsilon + \frac{3\pi}{2} \,\middle|\, m\right) = -\frac{2}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + 4(k+2)\Pi(n \mid m) - \Pi\left(n; i x + \frac{3\pi}{2} \,\middle|\, m\right) /;$$
$$m \in \mathbb{R} \bigwedge x \in \mathbb{R} \bigwedge \left(0 < m < 1 \bigwedge x > -\mathrm{Im}\left(\csc^{-1}(\sqrt{m})\right) \bigvee m > 1 \wedge x < 0\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0044.01

$$\lim_{\epsilon \to +0} \Pi\left(n; 2\pi k + i x - \epsilon + \frac{3\pi}{2} \,\middle|\, m\right) = \frac{2}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + 4(k+1)\Pi(n \mid m) - \Pi\left(n; i x + \frac{3\pi}{2} \,\middle|\, m\right) /;$$
$$m \in \mathbb{R} \bigwedge x \in \mathbb{R} \bigwedge \left(0 < m < 1 \bigwedge x < \mathrm{Im}\left(\csc^{-1}(\sqrt{m})\right) \bigvee m > 1 \wedge x > 0\right) \bigwedge k \in \mathbb{Z}$$

08.06.04.0045.01

$$\lim_{\epsilon \to +0} \Pi\left(n; 2\pi k + i x + \epsilon + \frac{3\pi}{2} \,\middle|\, m\right) = \Pi\left(n; 2\pi k + i x + \frac{3\pi}{2} \,\middle|\, m\right) /; x \in \mathbb{R} \bigwedge k \in \mathbb{Z}$$

With respect to *m*

Branch cut locations: complicated.

Series representations

Generalized power series

Expansions at generic point $n == n_0$

For the function itself

08.06.06.0008.01

$$\Pi(n; z \mid m) \propto \Pi(n_0; z \mid m) + \frac{1}{2(m-n_0)(n_0-1)} \left(E(z \mid m) - \frac{(m-n_0^2)\,\Pi(n_0; z \mid m)}{n_0} + F(z \mid m)\left(\frac{m}{n_0}-1\right) + \frac{\sqrt{1-m\sin^2(z)}\,\sin(2z)\,n_0}{2(n_0\sin^2(z)-1)} \right)(n-n_0) + \frac{1}{4(m-n_0)^2(n_0-1)^2} \left(\frac{1}{2n_0^2}\Big(E(z \mid m)(-n_0)(2n_0 m + m + n_0(2-5n_0)) + F(z \mid m)\big((1-4n_0)m^2 + 3n_0(3n_0-1)m + n_0^2(2-5n_0)\big) + \Pi(n_0; z \mid m)\big(3n_0^4 + 2m(2-5n_0)n_0 + m^2(4n_0-1)\big)\Big) + \frac{\sin(2z)}{8(n_0\sin^2(z)-1)^2}\big(5n_0^3 - 2(m+4)n_0^2 - m n_0 + \cos(2z)(2n_0 m + m + n_0(2-5n_0))n_0 + 6m\big)\sqrt{1-m\sin^2(z)} \right)(n-n_0)^2 + \ldots /; (n \to n_0)$$

08.06.06.0009.01

$$\Pi(n; z \mid m) \propto \Pi(n_0; z \mid m) + \frac{1}{2(m-n_0)(n_0-1)} \left(E(z \mid m) - \frac{(m-n_0^2)\,\Pi(n_0; z \mid m)}{n_0} + F(z \mid m)\left(\frac{m}{n_0}-1\right) + \frac{\sqrt{1-m\sin^2(z)}\,\sin(2z)\,n_0}{2(n_0\sin^2(z)-1)} \right)(n-n_0) + \frac{1}{4(m-n_0)^2(n_0-1)^2} \left(\frac{1}{2n_0^2}\Big(E(z \mid m)(-n_0)(2n_0 m + m + n_0(2-5n_0)) + F(z \mid m)\big((1-4n_0)m^2 + 3n_0(3n_0-1)m + n_0^2(2-5n_0)\big) + \Pi(n_0; z \mid m)\big(3n_0^4 + 2m(2-5n_0)n_0 + m^2(4n_0-1)\big)\Big) + \frac{\sin(2z)}{8(n_0\sin^2(z)-1)^2}\big(5n_0^3 - 2(m+4)n_0^2 - m n_0 + \cos(2z)(2n_0 m + m + n_0(2-5n_0))n_0 + 6m\big)\sqrt{1-m\sin^2(z)} \right)(n-n_0)^2 + O\big((n-n_0)^3\big)$$

08.06.06.0010.01

$$\Pi(n; z \mid m) = \sum_{k=0}^{\infty} \frac{1}{k!}\,\Pi^{(k,0,0)}(n_0; z \mid m)\,(n-n_0)^k$$

08.06.06.0011.01

$$\Pi(n; z \mid m) \propto \Pi(n_0; z \mid m)\,(1 + O(n-n_0))$$

Expansions at $n == 0$

08.06.06.0001.02

$$\Pi(n; z \mid m) \propto F(z \mid m) + \frac{n}{3} (-1)^{\left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil} \left(\sin^3(z) F_1\left(\frac{3}{2}; \frac{1}{2}, \frac{1}{2}; \frac{5}{2}; \sin^2(z), m \sin^2(z)\right) + \frac{3 \sin^5(z)}{5} F_1\left(\frac{5}{2}; \frac{1}{2}, \frac{1}{2}; \frac{7}{2}; \sin^2(z), m \sin^2(z)\right) n + \frac{3 \sin^7(z)}{7} F_1\left(\frac{7}{2}; \frac{1}{2}, \frac{1}{2}; \frac{9}{2}; \sin^2(z), m \sin^2(z)\right) n^2 + \ldots \right) + \frac{\pi n}{2} \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \left({}_2F_1\left(\frac{3}{2}, \frac{1}{2}; 2; m\right) + \frac{3}{4} {}_2F_1\left(\frac{5}{2}, \frac{1}{2}; 3; m\right) n + \frac{5}{8} {}_2F_1\left(\frac{7}{2}, \frac{1}{2}; 4; m\right) n^2 + \ldots \right) /; (n \to 0)$$

08.06.06.0012.01

$$\Pi(n; z \mid m) \propto \left(F(z \mid m) + \frac{F(z \mid m) - E(z \mid m)}{m} n - \frac{n^2 \left(4 (m+1) E(z \mid m) - 2 \left((m+2) F(z \mid m) + m \cos(z) \sin(z) \sqrt{1 - m \sin^2(z)}\right)\right)}{6 m^2} n^2 + O\left(n^3\right) \right) + 2 \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \left(K(m) - \frac{E(m) - K(m)}{m} n - \frac{2 (m+1) E(m) - (m+2) K(m)}{3 m^2} n^2 + O\left(n^3\right) \right) /; (n \to 0)$$

08.06.06.0002.01

$$\Pi(n; z \mid m) = (-1)^{\left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil} \sum_{k=0}^{\infty} \frac{\sin^{2k+1}(z)}{2k+1} F_1\left(k + \frac{1}{2}; \frac{1}{2}, \frac{1}{2}; k + \frac{3}{2}; \sin^2(z), m \sin^2(z)\right) n^k + 2 \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \Pi(n \mid m)$$

08.06.06.0013.01

$$\Pi(n; z \mid m) \propto F(z \mid m) (1 + O(n))$$

Expansions at $n = 1$

08.06.06.0014.01

$$\Pi(n; z \mid m) \propto (-1)^{\left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil} \left(\sin(z) F_1\left(\frac{1}{2}; \frac{3}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2(z), m \sin^2(z)\right) + \frac{\sin^3(z)}{3} F_1\left(\frac{3}{2}; \frac{5}{2}, \frac{1}{2}; \frac{5}{2}; \sin^2(z), m \sin^2(z)\right) (n-1) + \frac{\sin^5(z)}{5} F_1\left(\frac{5}{2}; \frac{7}{2}, \frac{1}{2}; \frac{7}{2}; \sin^2(z), m \sin^2(z)\right) (n-1)^2 + \ldots \right) + \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \left(\frac{\pi}{\sqrt{1 - \frac{m}{n}} \sqrt{1-n}} + \frac{2 E(m)}{m-1} + 2 K(m) + 2 \frac{(m+1) E(m) + (m-1) K(m)}{3 (m-1)^2} (n-1) - 2 \frac{\left(2 m^2 - 7 m - 3\right) E(m) - \left(m^2 + 2 m - 3\right) K(m)}{15 (m-1)^3} (n-1)^2 + \ldots \right) /; (n \to 1) \bigwedge \neg \frac{2 \text{Re}(z) + \pi}{4 \pi} \in \mathbb{Z}$$

08.06.06.0015.01

$$\Pi(n; z \mid m) = 2\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \Pi(n \mid m) + (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil} \sum_{k=0}^{\infty} \frac{\sin^{2k+1}(z)}{2k+1} F_1\left(k+\frac{1}{2}; k+\frac{3}{2}, \frac{1}{2}; k+\frac{3}{2}; \sin^2(z), m\sin^2(z)\right)(n-1)^k \;/;\; \neg \frac{2\,\mathrm{Re}(z)+\pi}{4\pi} \in \mathbb{Z}$$

08.06.06.0016.01

$$\Pi(n; z \mid m) \propto \left(\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \left(\frac{2E(m)}{m-1} + 2K(m)\right) + (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil} \sin(z)\, F_1\left(\frac{1}{2}; \frac{3}{2}, \frac{1}{2}; \frac{3}{2}; \sin^2(z), m\sin^2(z)\right)\right)(1+O(n-1)) + \frac{\pi \left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil}{\sqrt{1-\frac{m}{n}}\,\sqrt{1-n}} \;/;\; \neg \frac{2\,\mathrm{Re}(z)+\pi}{4\pi} \in \mathbb{Z}$$

Expansions at $n == \infty$

08.06.06.0017.01

$$\Pi(n; z \mid m) \propto (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil} \csc(z)\left(\left(\frac{1}{n} F_1\left(-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{1}{2}; m\sin^2(z), \sin^2(z)\right) + \frac{1}{3n^2 z^2} F_1\left(-\frac{3}{2}; \frac{1}{2}, \frac{1}{2}; -\frac{1}{2}; m\sin^2(z), \sin^2(z)\right) + \frac{1}{5n^3 z^4} F_1\left(-\frac{5}{2}; \frac{1}{2}, \frac{1}{2}; -\frac{3}{2}; m\sin^2(z), \sin^2(z)\right) + \ldots\right) - \frac{\pi\sqrt{-n\sin^2(z)}}{2n}\left(1 + \frac{m+1}{2n} + \frac{3m^2+2m+3}{8n^2} + \ldots\right)\right) + \left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \left(\frac{\pi}{\sqrt{-n}}\left(1 + \frac{m+1}{2n} + \frac{3m^2+2m+3}{8n^2} + \ldots\right) + \frac{1}{2n}\left(4(E(m)-K(m)) - \frac{4(m+2)K(m) - 8(m+1)E(m)}{3n} + \frac{8\left(\left(8m^2+7m+8\right)E(m) - \left(4m^2+3m+8\right)K(m)\right)}{15n^2} + \ldots\right)\right) \;/;\; (|n| \to \infty)$$

08.06.06.0018.01

$$\Pi(n; z \mid m) = (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil} \left(\sum_{k=0}^{\infty} n^{-k-1}\left(\frac{\sin^{-2k-3}(z)\left(m\sin^2(z)\right)^{k+\frac{3}{2}}\sqrt{1-m\sin^2(z)}}{m\sqrt{\cos^2(z)}}\sqrt{\frac{m\cos^2(z)}{m-1}}\, F_1\left(\frac{1}{2}; k+\frac{3}{2}, \frac{1}{2}; \frac{3}{2}; 1-m\sin^2(z), \frac{1-m\sin^2(z)}{1-m}\right) + \frac{\pi\sqrt{m\sin^2(z)}\; m^k}{\sin(z)\left(\frac{3}{2}\right)_k}\sqrt{\frac{m}{m-1}}\; {}_2\tilde{F}_1\left(\frac{1}{2}, \frac{1}{2}; -k; \frac{1}{1-m}\right)\right) + \frac{\pi\, i\sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}}\right) + 2\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \Pi(n \mid m)$$

08.06.06.0019.01

$$\Pi(n; z \mid m) = 2\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \Pi(n \mid m) + \frac{(-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil} \csc(z)}{n} \left(\sum_{k=0}^{\infty} \frac{n^{-k} z^{-2k}}{2k+1} F_1\left(-k-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{1}{2}-k; m\sin^2(z), \sin^2(z)\right) - \frac{1}{2}\left(\pi\sqrt{-n\sin^2(z)}\right)\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!}\, {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2}-k; \frac{1}{m}\right)\left(\frac{m}{n}\right)^k\right)$$

08.06.06.0020.01

$$\Pi(n; z \mid m) \propto \left(\frac{\pi \left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil}{\sqrt{-n}} - \frac{(-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil} \pi \csc(z) \sqrt{-n \sin^2(z)}}{2\,n} \right) \left(1 + O\left(\frac{1}{\sqrt{n}} \right) \right) /; (|n| \to \infty)$$

Expansions at generic point $z == z_0$

For the function itself

08.06.06.0021.01

$$\Pi(n; z \mid m) \propto \Pi(n; z_0 \mid m) + \frac{1}{\sqrt{1 - m \sin^2(z_0)}\, \left(1 - n \sin^2(z_0)\right)} (z - z_0) - \frac{\sin(2\,z_0) \left(3\,m\,n \sin^2(z_0) - m - 2\,n\right)}{4 \left(1 - m \sin^2(z_0)\right)^{3/2} \left(n \sin^2(z_0) - 1\right)^2} (z - z_0)^2 + \ldots /;$$
$$(z \to z_0)$$

08.06.06.0022.01

$$\Pi(n; z \mid m) \propto$$
$$\Pi(n; z_0 \mid m) + \frac{1}{\sqrt{1 - m \sin^2(z_0)}\, \left(1 - n \sin^2(z_0)\right)} (z - z_0) - \frac{\sin(2\,z_0) \left(3\,m\,n \sin^2(z_0) - m - 2\,n\right)}{4 \left(1 - m \sin^2(z_0)\right)^{3/2} \left(n \sin^2(z_0) - 1\right)^2} (z - z_0)^2 + O\left((z - z_0)^3\right)$$

08.06.06.0023.01

$$\Pi(n; z \mid m) ==$$
$$\Pi(n; z_0 \mid m) + \sum_{p=1}^{\infty} \frac{1}{p!} \left(\sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1}\, 2^{k_1 - j} \sin^{k_1}(z_0)\, (k_1 + 2\,k_2 - j)^p\, e^{-\frac{1}{2} i \left(\pi (j + p - k_1 - 2\,k_2) + 2\,(k_1 + 2\,k_2 - j)\,z_0\right)} \binom{j - k_1}{k_2} \right.$$
$$\sum_{i=0}^{j-1} \frac{(1 - j)_{2\,(j-i)-2}}{(j - i - 1)!\, (2 \sin(z_0))^{j - 2\,i - 1}} \sum_{i_1=0}^{i} \sum_{i_2=0}^{i} \sum_{i_3=0}^{i} (-1)^{i_1}\, \delta_{i_1 + i_2 + i_3 - i}\, (i_1 + i_2 + i_3;\, i_1,\, i_2,\, i_3)\, n^{i_1}\, m^{i_3}$$
$$\left. (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1 - n \sin^2(z_0)\right)^{-i_1 - 1} \cos^{-2\,i_2 - 1}(z_0) \left(1 - m \sin^2(z_0)\right)^{-i_3 - \frac{1}{2}} (z - z_0)^p \right)$$

08.06.06.0024.01

$$\Pi(n; z \mid m) = \sum_{k=0}^{\infty} \frac{1}{k!} \Pi^{(0,k,0)}(n; z_0 \mid m)\, (z - z_0)^k$$

08.06.06.0025.01

$$\Pi(n; z \mid m) \propto \Pi(n; z_0 \mid m)\, (1 + O[z - z_0])$$

Expansions on branch cuts

Formulas on real axis for real m, n

For $m < 1 < n$, $\csc^{-1}\left(\sqrt{n}\right) + \pi\,u < x < \pi\left(u + \frac{1}{2}\right) /; u \in \mathbb{Z}$

08.06.06.0026.01

$$\Pi(n; z \mid m) \propto \Pi(n; x \mid m) + \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\sqrt{n-m}} \left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \left(\frac{z-x}{\sqrt{1-m\sin^2(x)}\,(1-n\sin^2(x))} - \frac{\sin(2x)\left(3mn\sin^2(x)-m-2n\right)}{4\left(1-m\sin^2(x)\right)^{3/2}\left(n\sin^2(x)-1\right)^2}(z-x)^2 + \ldots \right) /;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge m < 1 < n \bigwedge \pi u + \csc^{-1}\left(\sqrt{n}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

08.06.06.0027.01

$$\Pi(n; z \mid m) = \Pi(n; x \mid m) + \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\sqrt{n-m}} \left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \sum_{p=1}^{\infty} \frac{1}{p!} \left(\sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(x) (k_1+2k_2-j)^p e^{-\frac{1}{2} i (\pi (j+p-k_1-2k_2)+2(k_1+2k_2-j)x)} \binom{j-k_1}{k_2} \sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}} \sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i} (-1)^{i_1} \delta_{i_1+i_2+i_3-i}(i_1+i_2+i_3;\, i_1,\, i_2,\, i_3)\, n^{i_1} m^{i_3} (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1-n\sin^2(x)\right)^{-i_1-1} \cos^{-2i_2-1}(x) \left(1-m\sin^2(x)\right)^{-i_3-\frac{1}{2}} (z-x)^p \right) /;$$

$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge m < 1 < n \bigwedge \pi u + \csc^{-1}\left(\sqrt{n}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

08.06.06.0028.01

$$\Pi(n; z \mid m) \propto \left(\Pi(n; x \mid m) + \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\sqrt{n-m}} \left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right)\right)(1 + O(z-x)) /;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge m < 1 < n \bigwedge \pi u + \csc^{-1}\left(\sqrt{n}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

For $m < 1 < n$, $\pi\left(u + \frac{1}{2}\right) < x < \pi(u+1) - \csc^{-1}\left(\sqrt{n}\right) /; u \in \mathbb{Z}$

08.06.06.0029.01

$$\Pi(n; z \mid m) \propto \Pi(n; x \mid m) - \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\sqrt{n-m}} \left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) + e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(\frac{z-x}{\sqrt{1-m\sin^2(x)}\,(1-n\sin^2(x))} - \frac{\sin(2x)\left(3mn\sin^2(x)-m-2n\right)}{4\left(1-m\sin^2(x)\right)^{3/2}\left(n\sin^2(x)-1\right)^2}(z-x)^2 + \ldots \right) /;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge m < 1 < n \bigwedge \pi u + \frac{\pi}{2} < x < \pi(u+1) - \csc^{-1}\left(\sqrt{n}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0030.01

$$\Pi(n;z\mid m)=\Pi(n;x\mid m)-\frac{\pi\,i\sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}}\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)+ e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\sum_{p=1}^{\infty}\frac{1}{p!}\left(\sum_{j=1}^{p}\frac{1}{j!}\sum_{k_1=0}^{j-1}\binom{j}{k_1}\sum_{k_2=0}^{j-k_1}(-1)^{k_1}2^{k_1-j}\sin^{k_1}(x)(k_1+2k_2-j)^p e^{-\frac{1}{2}i(\pi(j+p-k_1-2k_2)+2(k_1+2k_2-j)x)}\binom{j-k_1}{k_2}\sum_{i=0}^{j-1}\frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}}\sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i}(-1)^{i_1}\delta_{i_1+i_2+i_3-i}(i_1+i_2+i_3;i_1,i_2,i_3)\,n^{i_1}m^{i_3}(-i_1)_{i_1}\left(\frac{1}{2}\right)_{i_2}\left(\frac{1}{2}\right)_{i_3}\left(1-n\sin^2(x)\right)^{-i_1-1}\cos^{-2i_2-1}(x)\left(1-m\sin^2(x)\right)^{-i_3-\frac{1}{2}}(z-x)^p\right)\;/;$$

$$x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge m<1<n\bigwedge\pi u+\frac{\pi}{2}<x<\pi(u+1)-\csc^{-1}\left(\sqrt{n}\right)\bigwedge u\in\mathbb{Z}$$

08.06.06.0031.01

$$\Pi(n;z\mid m)\propto\left(\Pi(n;x\mid m)-\frac{\pi\,i\sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}}\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)\right)(1+O(z-x))\;/;$$

$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge m<1<n\bigwedge\pi u+\frac{\pi}{2}<x<\pi(u+1)-\csc^{-1}\left(\sqrt{n}\right)\bigwedge u\in\mathbb{Z}$$

For $n<1<m$, $\csc^{-1}\left(\sqrt{m}\right)+\pi u<x<\pi\left(u+\frac{1}{2}\right)\;/;\;u\in\mathbb{Z}$

08.06.06.0032.01

$$\Pi(n;z\mid m)\propto\left(\frac{1}{\sqrt{m}}\Pi\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)+2\left(\left\lfloor\frac{x}{\pi}-\frac{1}{2}\right\rfloor+1\right)\Pi(n\mid m)\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\right)+\Pi(n;x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}+ e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\left(\frac{z-x}{\sqrt{1-m\sin^2(x)}\,\left(1-n\sin^2(x)\right)}-\frac{\sin(2x)\left(3mn\sin^2(x)-m-2n\right)}{4\left(1-m\sin^2(x)\right)^{3/2}\left(n\sin^2(x)-1\right)^2}(z-x)^2+\ldots\right)\;/;$$

$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge n<1<m\bigwedge\pi u+\csc^{-1}\left(\sqrt{m}\right)<x<\pi u+\frac{\pi}{2}\bigwedge u\in\mathbb{Z}$$

08.06.06.0033.01

$$\Pi(n;\,z\mid m) = \left(\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right) + 2\left(\left\lfloor\frac{x}{\pi}-\frac{1}{2}\right\rfloor+1\right)\Pi(n\mid m)\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\right) + \Pi(n;\,x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor} + e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\sum_{p=1}^{\infty}\frac{1}{p!}\left(\sum_{j=1}^{p}\frac{1}{j!}\sum_{k_1=0}^{j-1}\binom{j}{k_1}\sum_{k_2=0}^{j-k_1}(-1)^{k_1}\,2^{k_1-j}\sin^{k_1}(x)\,(k_1+2k_2-j)^p\,e^{-\frac{1}{2}i(\pi(j+p-k_1-2k_2)+2(k_1+2k_2-j)x)}\binom{j-k_1}{k_2}\sum_{i=0}^{j-1}\frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}}\sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i}(-1)^{i_1}\,\delta_{i_1+i_2+i_3-i}\,(i_1+i_2+i_3;\,i_1,\,i_2,\,i_3)\,n^{i_1}\,m^{i_3}\,(-i_1)_{i_1}\left(\frac{1}{2}\right)_{i_2}\left(\frac{1}{2}\right)_{i_3}\left(1-n\sin^2(x)\right)^{-i_1-1}\cos^{-2i_2-1}(x)\left(1-m\sin^2(x)\right)^{-i_3-\frac{1}{2}}(z-x)^p\right)\;/;$$

$$x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge n<1<m\bigwedge \pi u+\csc^{-1}\!\left(\sqrt{m}\right)<x<\pi u+\frac{\pi}{2}\bigwedge u\in\mathbb{Z}$$

08.06.06.0034.01

$$\Pi(n;\,z\mid m)\propto\left(\left(\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)+2\left(\left\lfloor\frac{x}{\pi}-\frac{1}{2}\right\rfloor+1\right)\Pi(n\mid m)\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\right)+\Pi(n;\,x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\right)(1+O(z-x))\;/;$$

$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge n<1<m\bigwedge \pi u+\csc^{-1}\!\left(\sqrt{m}\right)<x<\pi u+\frac{\pi}{2}\bigwedge u\in\mathbb{Z}$$

For $n<1<m,\ \pi\left(u+\frac{1}{2}\right)<x<\pi(u+1)-\csc^{-1}\!\left(\sqrt{m}\right)\ /;\ u\in\mathbb{Z}$

08.06.06.0035.01

$$\Pi(n;\,z\mid m)\propto\left(2\left(\left\lfloor\frac{x}{\pi}-\frac{1}{2}\right\rfloor+1\right)\Pi(n\mid m)-\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)+\Pi(n;\,x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}+e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left(\frac{z-x}{\sqrt{1-m\sin^2(x)}\,\left(1-n\sin^2(x)\right)}-\frac{\sin(2x)\left(3mn\sin^2(x)-m-2n\right)}{4\left(1-m\sin^2(x)\right)^{3/2}\left(n\sin^2(x)-1\right)^2}(z-x)^2+\ldots\right)\;/;$$

$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge n<1<m\bigwedge \pi u+\frac{\pi}{2}<x<\pi(u+1)-\csc^{-1}\!\left(\sqrt{m}\right)\bigwedge u\in\mathbb{Z}$$

08.06.06.0036.01

$$\Pi(n;z\mid m)=\left(2\left(\left\lfloor\frac{x}{\pi}-\frac{1}{2}\right\rfloor+1\right)\Pi(n\mid m)-\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)+\Pi(n;x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}+$$
$$e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\sum_{p=1}^{\infty}\frac{1}{p!}\left(\sum_{j=1}^{p}\frac{1}{j!}\sum_{k_1=0}^{j-1}\binom{j}{k_1}\sum_{k_2=0}^{j-k_1}(-1)^{k_1}\,2^{k_1-j}\sin^{k_1}(x)\,(k_1+2k_2-j)^p\,e^{-\frac{1}{2}i(\pi(j+p-k_1-2k_2)+2(k_1+2k_2-j)x)}\binom{j-k_1}{k_2}\right.$$
$$\sum_{i=0}^{j-1}\frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}}\sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i}(-1)^{i_1}\,\delta_{i_1+i_2+i_3-i}\,(i_1+i_2+i_3;\,i_1,\,i_2,\,i_3)\,n^{i_1}\,m^{i_3}$$
$$\left.(-i_1)_{i_1}\left(\frac{1}{2}\right)_{i_2}\left(\frac{1}{2}\right)_{i_3}\left(1-n\sin^2(x)\right)^{-i_1-1}\cos^{-2i_2-1}(x)\left(1-m\sin^2(x)\right)^{-i_3-\frac{1}{2}}(z-x)^p\right)\,/;$$
$$x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge n<1<m\bigwedge \pi u+\frac{\pi}{2}<x<\pi(u+1)-\csc^{-1}\left(\sqrt{m}\right)\bigwedge u\in\mathbb{Z}$$

08.06.06.0037.01

$$\Pi(n;z\mid m)\propto\left(\left(2\left(\left\lfloor\frac{x}{\pi}-\frac{1}{2}\right\rfloor+1\right)\Pi(n\mid m)-\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)+\Pi(n;x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)(1+O(z-x))\,/;$$
$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge n<1<m\bigwedge \pi u+\frac{\pi}{2}<x<\pi(u+1)-\csc^{-1}\left(\sqrt{m}\right)\bigwedge u\in\mathbb{Z}$$

For $1<n<\ m,\ \csc^{-1}\left(\sqrt{m}\right)+\pi u<x<\pi\left(u+\frac{1}{2}\right)/;\ u\in\mathbb{Z}$

08.06.06.0038.01

$$\Pi(n;z\mid m)\propto\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\right)+\Pi(n;x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}+$$
$$e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\left(\frac{z-x}{\sqrt{1-m\sin^2(x)}\,\left(1-n\sin^2(x)\right)}-\frac{\sin(2x)\left(3mn\sin^2(x)-m-2n\right)}{4\left(1-m\sin^2(x)\right)^{3/2}\left(n\sin^2(x)-1\right)^2}(z-x)^2+\ldots\right)/;$$
$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge 1<n<m\bigwedge \pi u+\csc^{-1}\left(\sqrt{m}\right)<x<\pi u+\csc^{-1}\left(\sqrt{n}\right)\bigwedge u\in\mathbb{Z}$$

08.06.06.0039.01

$$\Pi(n;z\mid m)\propto\left(\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)-\frac{\pi i\sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}}\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\right)+\Pi(n;x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}+$$
$$e^{-\pi i\left\lfloor\frac{\arg(x-z)}{2\pi}\right\rfloor}\left(\frac{z-x}{\sqrt{1-m\sin^2(x)}\,\left(1-n\sin^2(x)\right)}-\frac{\sin(2x)\left(3mn\sin^2(x)-m-2n\right)}{4\left(1-m\sin^2(x)\right)^{3/2}\left(n\sin^2(x)-1\right)^2}(z-x)^2+\ldots\right)/;$$
$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge 1<n<m\bigwedge \pi u+\csc^{-1}\left(\sqrt{n}\right)<x<\pi u+\frac{\pi}{2}\bigwedge u\in\mathbb{Z}$$

08.06.06.0040.01

$$\Pi(n; z \mid m) = \frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} + e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \sum_{p=1}^{\infty} \frac{1}{p!} \left(\sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(x) (k_1 + 2k_2 - j)^p e^{-\frac{1}{2} i (\pi (j+p-k_1-2k_2)+2(k_1+2k_2-j)x)} \binom{j-k_1}{k_2} \sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}} \sum_{i_1=0}^{i} \sum_{i_2=0}^{i} \sum_{i_3=0}^{i} (-1)^{i_1} \delta_{i_1+i_2+i_3-i} (i_1+i_2+i_3; i_1, i_2, i_3)\, n^{i_1} m^{i_3} (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1 - n\sin^2(x)\right)^{-i_1-1} \cos^{-2i_2-1}(x) \left(1 - m\sin^2(x)\right)^{-i_3-\frac{1}{2}} (z-x)^p \right) /;$$

$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \csc^{-1}\left(\sqrt{n}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0041.01

$$\Pi(n; z \mid m) = \left(\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) - \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} + e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \sum_{p=1}^{\infty} \frac{1}{p!} \left(\sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(x) (k_1 + 2k_2 - j)^p e^{-\frac{1}{2} i (\pi (j+p-k_1-2k_2)+2(k_1+2k_2-j)x)} \binom{j-k_1}{k_2} \sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}} \sum_{i_1=0}^{i} \sum_{i_2=0}^{i} \sum_{i_3=0}^{i} (-1)^{i_1} \delta_{i_1+i_2+i_3-i} (i_1+i_2+i_3; i_1, i_2, i_3)\, n^{i_1} m^{i_3} (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1 - n\sin^2(x)\right)^{-i_1-1} \cos^{-2i_2-1}(x) \left(1 - m\sin^2(x)\right)^{-i_3-\frac{1}{2}} (z-x)^p \right) /;$$

$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi u + \csc^{-1}\left(\sqrt{n}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

08.06.06.0042.01

$$\Pi(n; z \mid m) \propto \left(\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right)(1 + O(z-x)) /;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \csc^{-1}\left(\sqrt{n}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0043.01

$$\Pi(n; z \mid m) \propto \left(\left(\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) - \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right)(1 + O(z-x)) /;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi u + \csc^{-1}\left(\sqrt{n}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

For $1 < n < m$, $\pi\left(u + \frac{1}{2}\right) < x < \pi(u+1) - \csc^{-1}\left(\sqrt{m}\right) /; u \in \mathbb{Z}$

08.06.06.0044.01

$$\Pi(n; z \mid m) \propto \left(-\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \middle| \frac{1}{m}\right) + \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\sqrt{n-m}}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} + e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\left(\frac{z-x}{\sqrt{1-m\sin^2(x)}\,\left(1-n\sin^2(x)\right)} - \frac{\sin(2x)\left(3mn\sin^2(x)-m-2n\right)}{4\left(1-m\sin^2(x)\right)^{3/2}\left(n\sin^2(x)-1\right)^2}(z-x)^2 + \ldots\right) /;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi u + \frac{\pi}{2} < x < \pi(u+1) - \csc^{-1}\left(\sqrt{n}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0045.01

$$\Pi(n; z \mid m) \propto -\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \middle| \frac{1}{m}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} + e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\left(\frac{z-x}{\sqrt{1-m\sin^2(x)}\,\left(1-n\sin^2(x)\right)} - \frac{\sin(2x)\left(3mn\sin^2(x)-m-2n\right)}{4\left(1-m\sin^2(x)\right)^{3/2}\left(n\sin^2(x)-1\right)^2}(z-x)^2 + \ldots\right) /;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi u - \csc^{-1}\left(\sqrt{n}\right) < x < \pi u - \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0046.01

$$\Pi(n; z \mid m) = \left(-\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \middle| \frac{1}{m}\right) + \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\sqrt{n-m}}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} + e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sum_{p=1}^{\infty} \frac{1}{p!} \left(\sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(x)\,(k_1+2k_2-j)^p\, e^{-\frac{1}{2} i (\pi(j+p-k_1-2k_2)+2(k_1+2k_2-j)x)} \binom{j-k_1}{k_2} \sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}} \sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i} (-1)^{i_1} \delta_{i_1+i_2+i_3-i}\,(i_1+i_2+i_3; i_1, i_2, i_3)\, n^{i_1} m^{i_3} (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1-n\sin^2(x)\right)^{-i_1-1} \cos^{-2i_2-1}(x) \left(1-m\sin^2(x)\right)^{-i_3-\frac{1}{2}} (z-x)^p\right) /;$$

$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi u + \frac{\pi}{2} < x < \pi(u+1) - \csc^{-1}\left(\sqrt{n}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0047.01

$$\Pi(n; z \mid m) == -\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) \left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} +$$
$$e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sum_{p=1}^{\infty} \frac{1}{p!} \left(\sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(x) (k_1 + 2k_2 - j)^p e^{-\frac{1}{2} i (\pi (j+p-k_1-2k_2) + 2(k_1+2k_2-j)x)} \binom{j-k_1}{k_2} \right.$$
$$\sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}} \sum_{i_1=0}^{i} \sum_{i_2=0}^{i} \sum_{i_3=0}^{i} (-1)^{i_1} \delta_{i_1+i_2+i_3-i} (i_1 + i_2 + i_3; i_1, i_2, i_3)\, n^{i_1} m^{i_3}$$
$$\left. (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1 - n\sin^2(x)\right)^{-i_1-1} \cos^{-2i_2-1}(x) \left(1 - m\sin^2(x)\right)^{-i_3-\frac{1}{2}} (z-x)^p \right) /;$$
$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi u - \csc^{-1}\left(\sqrt{n}\right) < x < \pi u - \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0048.01

$$\Pi(n; z \mid m) \propto \left(\left(-\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}} \right) \left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \right) (1 + O(z-x)) /;$$
$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi u + \frac{\pi}{2} < x < \pi (u+1) - \csc^{-1}\left(\sqrt{n}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0049.01

$$\Pi(n; z \mid m) \propto \left(e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \Pi(n; x \mid m) - \frac{\Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right)}{\sqrt{m}} \left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) \right) (1 + O(z-x)) /;$$
$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < n < m \bigwedge \pi u - \csc^{-1}\left(\sqrt{n}\right) < x < \pi u - \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

For $1 < m < n$, $\csc^{-1}\left(\sqrt{n}\right) + \pi u < x < \pi\left(u + \frac{1}{2}\right) /; u \in \mathbb{Z}$

08.06.06.0050.01

$$\Pi(n; z \mid m) \propto \Pi(n; x \mid m) - \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}} \left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) +$$
$$e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \left(\frac{z-x}{\sqrt{1 - m\sin^2(x)}\,\left(1 - n\sin^2(x)\right)} - \frac{\sin(2x)\left(3mn\sin^2(x) - m - 2n\right)}{4\left(1 - m\sin^2(x)\right)^{3/2}\left(n\sin^2(x) - 1\right)^2} (z-x)^2 + \ldots \right) /;$$
$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi u + \csc^{-1}\left(\sqrt{n}\right) < x < \pi u + \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0051.01

$$\Pi(n; z \mid m) \propto \left(\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\sqrt{n-m}}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} + e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\left(\frac{z-x}{\sqrt{1-m\sin^2(x)}\,\left(1-n\sin^2(x)\right)} - \frac{\sin(2x)\left(3mn\sin^2(x)-m-2n\right)}{4\left(1-m\sin^2(x)\right)^{3/2}\left(n\sin^2(x)-1\right)^2}(z-x)^2 + \ldots\right) /;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

08.06.06.0052.01

$$\Pi(n; z \mid m) \propto \Pi(n; x \mid m) - \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\sqrt{n-m}}\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \sum_{p=1}^{\infty} \frac{1}{p!}\left(\sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(x)(k_1+2k_2-j)^p e^{-\frac{1}{2} i (\pi(j+p-k_1-2k_2)+2(k_1+2k_2-j)x)} \binom{j-k_1}{k_2} \sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}} \sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i} (-1)^{i_1} \delta_{i_1+i_2+i_3-i}(i_1+i_2+i_3; i_1, i_2, i_3)\, n^{i_1} m^{i_3} (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1-n\sin^2(x)\right)^{-i_1-1} \cos^{-2i_2-1}(x)\left(1-m\sin^2(x)\right)^{-i_3-\frac{1}{2}} (z-x)^p\right) /;$$

$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi u + \csc^{-1}\left(\sqrt{n}\right) < x < \pi u + \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0053.01

$$\Pi(n; z \mid m) \propto \left(\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + \frac{\pi i \sqrt{n}}{2\sqrt{n-1}\sqrt{n-m}}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} + e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \sum_{p=1}^{\infty} \frac{1}{p!}\left(\sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(x)(k_1+2k_2-j)^p e^{-\frac{1}{2} i (\pi(j+p-k_1-2k_2)+2(k_1+2k_2-j)x)} \binom{j-k_1}{k_2} \sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}} \sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i} (-1)^{i_1} \delta_{i_1+i_2+i_3-i}(i_1+i_2+i_3; i_1, i_2, i_3)\, n^{i_1} m^{i_3} (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1-n\sin^2(x)\right)^{-i_1-1} \cos^{-2i_2-1}(x)\left(1-m\sin^2(x)\right)^{-i_3-\frac{1}{2}} (z-x)^p\right) /;$$

$$x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

08.06.06.0054.01

$$\Pi(n; z \mid m) \propto \left(\Pi(n; x \mid m) - \frac{\pi i \sqrt{n}}{2 \sqrt{n-1} \sqrt{n-m}} \left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right)\right)(1 + O(z-x)) \,/;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi u + \csc^{-1}\left(\sqrt{n}\right) < x < \pi u + \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0055.01

$$\Pi(n; z \mid m) \propto \left(\left(\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + \frac{\pi i \sqrt{n}}{2 \sqrt{n-1} \sqrt{n-m}}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right) + \Pi(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor}\right)(1 + O(z-x)) \,/;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi u + \csc^{-1}\left(\sqrt{m}\right) < x < \pi u + \frac{\pi}{2} \bigwedge u \in \mathbb{Z}$$

For $1 < m < n,\ \pi\left(u + \frac{1}{2}\right) < x < \pi(u+1) - \csc^{-1}\left(\sqrt{n}\right) /;\ u \in \mathbb{Z}$

08.06.06.0056.01

$$\Pi(n; z \mid m) \propto \Pi\,(n; x \mid m)\, e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} - \left(\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + \frac{\pi i \sqrt{n}}{2 \sqrt{n-1} \sqrt{n-m}}\right)\left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) +$$

$$e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\left(\frac{z-x}{\sqrt{1 - m \sin^2(x)}\, \left(1 - n \sin^2(x)\right)} - \frac{\sin(2x)\left(3 m n \sin^2(x) - m - 2n\right)}{4\left(1 - m \sin^2(x)\right)^{3/2} \left(n \sin^2(x) - 1\right)^2} (z-x)^2 + \ldots\right) /;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi u + \frac{\pi}{2} < x < \pi\,(u+1) - \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0057.01

$$\Pi(n; z \mid m) \propto \Pi(n; x \mid m) + \frac{\pi i \sqrt{n}}{2 \sqrt{n-1} \sqrt{n-m}} \left(1 - e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\right) +$$

$$e^{-\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor}\left(\frac{z-x}{\sqrt{1 - m \sin^2(x)}\, \left(1 - n \sin^2(x)\right)} - \frac{\sin(2x)\left(3 m n \sin^2(x) - m - 2n\right)}{4\left(1 - m \sin^2(x)\right)^{3/2} \left(n \sin^2(x) - 1\right)^2} (z-x)^2 + \ldots\right) /;$$

$$(z \to x) \bigwedge x \in \mathbb{R} \bigwedge m \in \mathbb{R} \bigwedge n \in \mathbb{R} \bigwedge 1 < m < n \bigwedge \pi u - \csc^{-1}\left(\sqrt{m}\right) < x < \pi u - \csc^{-1}\left(\sqrt{n}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0058.01

$$\Pi(n;z\mid m)\propto -\left(\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)+\frac{\pi\,i\,\sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}}\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)+\Pi(n;x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}+ e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\sum_{p=1}^{\infty}\frac{1}{p!}\left(\sum_{j=1}^{p}\frac{1}{j!}\sum_{k_1=0}^{j-1}\binom{j}{k_1}\sum_{k_2=0}^{j-k_1}(-1)^{k_1}\,2^{k_1-j}\sin^{k_1}(x)\,(k_1+2k_2-j)^p\,e^{-\frac{1}{2}i(\pi(j+p-k_1-2k_2)+2(k_1+2k_2-j)x)}\binom{j-k_1}{k_2}\right.$$
$$\sum_{i=0}^{j-1}\frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}}\sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i}(-1)^{i_1}\,\delta_{i_1+i_2+i_3-i}\,(i_1+i_2+i_3;\,i_1,\,i_2,\,i_3)\,n^{i_1}\,m^{i_3}$$
$$\left.(-i_1)_{i_1}\left(\frac{1}{2}\right)_{i_2}\left(\frac{1}{2}\right)_{i_3}\left(1-n\sin^2(x)\right)^{-i_1-1}\cos^{-2i_2-1}(x)\left(1-m\sin^2(x)\right)^{-i_3-\frac{1}{2}}(z-x)^p\right)\,/;$$
$$x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge 1<m<n\bigwedge \pi u+\frac{\pi}{2}<x<\pi(u+1)-\csc^{-1}\!\left(\sqrt{m}\right)\bigwedge u\in\mathbb{Z}$$

08.06.06.0059.01

$$\Pi(n;z\mid m)\propto \Pi(n;x\mid m)+\frac{\pi\,i\,\sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}}\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)+ e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\sum_{p=1}^{\infty}\frac{1}{p!}\left(\sum_{j=1}^{p}\frac{1}{j!}\sum_{k_1=0}^{j-1}\binom{j}{k_1}\sum_{k_2=0}^{j-k_1}(-1)^{k_1}\,2^{k_1-j}\sin^{k_1}(x)\,(k_1+2k_2-j)^p\,e^{-\frac{1}{2}i(\pi(j+p-k_1-2k_2)+2(k_1+2k_2-j)x)}\binom{j-k_1}{k_2}\right.$$
$$\sum_{i=0}^{j-1}\frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(x))^{j-2i-1}}\sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i}(-1)^{i_1}\,\delta_{i_1+i_2+i_3-i}\,(i_1+i_2+i_3;\,i_1,\,i_2,\,i_3)\,n^{i_1}\,m^{i_3}$$
$$\left.(-i_1)_{i_1}\left(\frac{1}{2}\right)_{i_2}\left(\frac{1}{2}\right)_{i_3}\left(1-n\sin^2(x)\right)^{-i_1-1}\cos^{-2i_2-1}(x)\left(1-m\sin^2(x)\right)^{-i_3-\frac{1}{2}}(z-x)^p\right)\,/;$$
$$x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge 1<m<n\bigwedge \pi u-\csc^{-1}\!\left(\sqrt{m}\right)<x<\pi u-\csc^{-1}\!\left(\sqrt{n}\right)\bigwedge u\in\mathbb{Z}$$

08.06.06.0060.01

$$\Pi(n;z\mid m)\propto\left(\Pi(n;x\mid m)\,e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}-\left(\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)+\frac{\pi\,i\,\sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}}\right)\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)\right)(1+O(z-x))\,/;$$
$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge 1<m<n\bigwedge \pi u+\frac{\pi}{2}<x<\pi(u+1)-\csc^{-1}\!\left(\sqrt{m}\right)\bigwedge u\in\mathbb{Z}$$

08.06.06.0061.01

$$\Pi(n;z\mid m)\propto\left(\Pi(n;x\mid m)+\frac{\pi\,i\,\sqrt{n}}{2\sqrt{n-1}\,\sqrt{n-m}}\left(1-e^{-\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\right)\right)(1+O(z-x))\,/;$$
$$(z\to x)\bigwedge x\in\mathbb{R}\bigwedge m\in\mathbb{R}\bigwedge n\in\mathbb{R}\bigwedge 1<m<n\bigwedge \pi u-\csc^{-1}\!\left(\sqrt{m}\right)<x<\pi u-\csc^{-1}\!\left(\sqrt{n}\right)\bigwedge u\in\mathbb{Z}$$

Formulas for vertical intervals

For $\mathrm{Re}\left(\frac{z_0}{2\pi} - \frac{1}{4}\right) \in \mathbb{Z}$

08.06.06.0062.01

$$\Pi(n; z \mid m) \propto -e^{-\pi i \left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} + \frac{3}{4} \right\rfloor + \left\lfloor \frac{3}{4} - \frac{\arg(z-z_0)}{2\pi} \right\rfloor\right)} \Pi(n; z_0 \mid m) + \left(2\left(2\,\mathrm{Re}\left(\frac{z_0}{2\pi} - \frac{1}{4}\right) + 1\right)\Pi(n \mid m) - \frac{1}{\sqrt{m}}\,\Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right)\right)\left(e^{-\pi i \left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} + \frac{3}{4} \right\rfloor + \left\lfloor \frac{3}{4} - \frac{\arg(z-z_0)}{2\pi} \right\rfloor\right)} + 1\right) + \frac{z - z_0}{\sqrt{1 - m\sin^2(z_0)}\,\left(1 - n\sin^2(z_0)\right)} - \frac{\sin(2 z_0)\left(3 m n \sin^2(z_0) - m - 2n\right)}{4\left(1 - m\sin^2(z_0)\right)^{3/2}\left(n\sin^2(z_0) - 1\right)^2}(z - z_0)^2 + \ldots \;/;\; (z \to z_0) \bigwedge \mathrm{Re}\left(\frac{z_0}{2\pi} - \frac{1}{4}\right) \in \mathbb{Z}$$

08.06.06.0063.01

$$\Pi(n; z \mid m) = -\Pi(n; z_0 \mid m)\, e^{-\pi i \left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} + \frac{3}{4} \right\rfloor + \left\lfloor \frac{3}{4} - \frac{\arg(z-z_0)}{2\pi} \right\rfloor\right)} + \left(2\left(2\,\mathrm{Re}\left(\frac{z_0}{2\pi} - \frac{1}{4}\right) + 1\right)\Pi(n \mid m) - \frac{1}{\sqrt{m}}\,\Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right)\right)\left(e^{-\pi i \left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} + \frac{3}{4} \right\rfloor + \left\lfloor \frac{3}{4} - \frac{\arg(z-z_0)}{2\pi} \right\rfloor\right)} + 1\right) + \sum_{p=1}^{\infty} \frac{1}{p!}\left(\sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(z_0)\,(k_1 + 2k_2 - j)^p\, e^{-\frac{1}{2} i \left(\pi (j+p-k_1-2k_2) + 2(k_1+2k_2-j) z_0\right)} \binom{j-k_1}{k_2} \sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(z_0))^{j-2i-1}} \sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i} (-1)^{i_1}\, \delta_{i_1+i_2+i_3-i}\, (i_1+i_2+i_3; i_1, i_2, i_3)\, n^{i_1} m^{i_3} (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1 - n\sin^2(z_0)\right)^{-i_1-1} \cos^{-2 i_2 - 1}(z_0) \left(1 - m\sin^2(z_0)\right)^{-i_3-\frac{1}{2}} (z - z_0)^p\right) \;/;\; \mathrm{Re}\left(\frac{z_0}{2\pi} - \frac{1}{4}\right) \in \mathbb{Z}$$

08.06.06.0064.01

$$\Pi(n; z \mid m) \propto \left(-e^{-\pi i \left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} + \frac{3}{4} \right\rfloor + \left\lfloor \frac{3}{4} - \frac{\arg(z-z_0)}{2\pi} \right\rfloor\right)} \Pi(n; z_0 \mid m) + \left(2\left(2\,\mathrm{Re}\left(\frac{z_0}{2\pi} - \frac{1}{4}\right) + 1\right)\Pi(n \mid m) - \frac{1}{\sqrt{m}}\,\Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right)\right)\left(e^{-\pi i \left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} + \frac{3}{4} \right\rfloor + \left\lfloor \frac{3}{4} - \frac{\arg(z-z_0)}{2\pi} \right\rfloor\right)} + 1\right)\right)(1 + O(z - z_0)) \;/;\; (z \to z_0) \bigwedge \mathrm{Re}\left(\frac{z_0}{2\pi} - \frac{1}{4}\right) \in \mathbb{Z}$$

For $\mathrm{Re}\left(\frac{z_0}{2\pi} - \frac{3}{4}\right) \in \mathbb{Z}$

08.06.06.0065.01

$$\Pi(n;z\mid m)\propto\left(\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)+2\left(2\operatorname{Re}\!\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)+1\right)\Pi(n\mid m)\right)\left(1-e^{-\pi i\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor\frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}\right)+\Pi(n;z_0\mid m)\,e^{-\pi i\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor\frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}+\frac{z-z_0}{\sqrt{1-m\sin^2(z_0)}\,\left(1-n\sin^2(z_0)\right)}-\frac{\sin(2z_0)\left(3mn\sin^2(z_0)-m-2n\right)}{4\left(1-m\sin^2(z_0)\right)^{3/2}\left(n\sin^2(z_0)-1\right)^2}(z-z_0)^2+\ldots\;/;\;(z\to z_0)\bigwedge\operatorname{Re}\!\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)\in\mathbb{Z}$$

08.06.06.0066.01

$$\Pi(n;z\mid m)\propto\left(\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)+2\left(2\operatorname{Re}\!\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)+1\right)\Pi(n\mid m)\right)\left(1-e^{-\pi i\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor\frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}\right)+e^{-\pi i\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor\frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}\Pi(n;z_0\mid m)+\sum_{p=1}^{\infty}\frac{1}{p!}\left(\sum_{j=1}^{p}\frac{1}{j!}\sum_{k_1=0}^{j-1}\binom{j}{k_1}\sum_{k_2=0}^{j-k_1}(-1)^{k_1}2^{k_1-j}\sin^{k_1}(z_0)\,(k_1+2k_2-j)^p\,e^{-\frac{1}{2}i\left(\pi(j+p-k_1-2k_2)+2(k_1+2k_2-j)z_0\right)}\binom{j-k_1}{k_2}\sum_{i=0}^{j-1}\frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(z_0))^{j-2i-1}}\sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i}(-1)^{i_1}\delta_{i_1+i_2+i_3-i}\,(i_1+i_2+i_3;i_1,i_2,i_3)\,n^{i_1}m^{i_3}(-i_1)_{i_1}\left(\frac{1}{2}\right)_{i_2}\left(\frac{1}{2}\right)_{i_3}\left(1-n\sin^2(z_0)\right)^{-i_1-1}\cos^{-2i_2-1}(z_0)\left(1-m\sin^2(z_0)\right)^{-i_3-\frac{1}{2}}(z-z_0)^p\right)\;/;\;\operatorname{Re}\!\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)\in\mathbb{Z}$$

08.06.06.0067.01

$$\Pi(n;z\mid m)\propto\left(\left(\frac{1}{\sqrt{m}}\,\Pi\!\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)+2\left(2\operatorname{Re}\!\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)+1\right)\Pi(n\mid m)\right)\left(1-e^{-\pi i\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor\frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}\right)+\Pi(n;z_0\mid m)\,e^{-\pi i\left(\left\lfloor\frac{\arg(z-z_0)}{2\pi}+\frac{1}{4}\right\rfloor+\left\lfloor\frac{1}{4}-\frac{\arg(z-z_0)}{2\pi}\right\rfloor\right)}\right)(1+O(z-z_0))\;/;\;(z\to z_0)\bigwedge\operatorname{Re}\!\left(\frac{z_0}{2\pi}-\frac{3}{4}\right)\in\mathbb{Z}$$

Expansions at $z == 0$

08.06.06.0003.01

$$\Pi(n;z\mid m)\propto z+\frac{m+2n}{6}z^3+\frac{1}{120}\left(9m^2+4(m+2n)(3n-1)\right)z^5+\frac{1}{5040}\left(225m^3+90m^2(3n-2)+8(m+2n)\left(2-30n+45n^2\right)\right)z^7+\frac{1}{362\,880}\left(11\,025m^4+12\,600m^3(n-1)+3024m^2\left(1-5n+5n^2\right)+64(m+2n)\left(-1+63n-315n^2+315n^3\right)\right)z^9+O\!\left(z^{11}\right)\;/;\;(z\to 0)$$

08.06.06.0068.01

$$\Pi(n;z\mid m)\propto-\frac{2^{-\frac{5}{2}}\sqrt{\pi}\,n}{\left(\frac{m-2\left(\sqrt{1-m}+1\right)}{m}\right)^{3/2}\sqrt{1-n}}\sqrt{1+\frac{1}{\sqrt{1-m}}}$$

$$\sum_{q=0}^{\infty}\frac{(-1)^q\,2^{2q+1}}{(2q+1)!}\sum_{k=0}^{2q}\mathcal{S}_{2q}^{(k)}\sum_{j=0}^{k}\frac{2^{-k}\,(-1)^j\,j!\,n^j}{\Gamma\left(j-k+\frac{1}{2}\right)}\binom{k}{k-j}\left(\frac{m}{1-\sqrt{1-m}}\right)^{k-j}\left(\left(\sqrt{1-n}+1\right)^{-j-1}-\left(1-\sqrt{1-n}\right)^{-j-1}\right)$$

$$F_1\left(\frac{1}{2};\frac{1}{2},-\frac{3}{2};j-k+\frac{1}{2};\frac{1}{2}-\frac{1}{2\sqrt{1-m}},\frac{2\left(\sqrt{1-m}+1\right)}{m}\right)z^{2q+1}\;/;\;|z|<1$$

08.06.06.0069.01

$$\Pi(n;z\mid m)\propto z+O\left(z^3\right)\;/;\;(z\to 0)$$

Expansions at $z==\csc^{-1}\left(\sqrt{m}\right)+\pi u\;/;\;u\in\mathbb{Z}$

08.06.06.0070.01

$$\Pi(n;z\mid m)\propto\frac{1}{\sqrt{m}}\,\Pi\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)+2u\,\Pi(n\mid m)-\frac{m\sqrt{2}\sqrt{-\sqrt{m-1}\,(z-z_0)}}{(n-m)\sqrt{m-1}}\left(1-\frac{\left(m^2-(9n+2)m+10n\right)(z-z_0)}{12\sqrt{m-1}\,(m-n)}+\right.$$
$$\left((m(9m-4)+4)m^2+2(m(15m-68)+44)nm+(m(345m-628)+292)n^2\right)\Big/\left(480(m-1)(m-n)^2\right)(z-z_0)^2-$$
$$\left(15m^6-(21n+26)m^5+\left(-1155n^2+214n-12\right)m^4+\left(-1911n^3+4258n^2-284n+8\right)m^3+\right.$$
$$\left.2n\left(2385n^2-2578n+68\right)m^2-4n^2(941n-502)m+920n^3\right)\Big/$$
$$\left.\left(2688(m-1)^{3/2}(m-n)^3\right)(z-z_0)^3+\ldots\right)\;/;\;(z\to z_0)\bigwedge z_0=\csc^{-1}\left(\sqrt{m}\right)+\pi u\bigwedge u\in\mathbb{Z}$$

08.06.06.0071.01

$$\Pi(n;z\mid m)=$$

$$2u\,\Pi(n\mid m)+\frac{1}{\sqrt{m}}\,\Pi\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)-\frac{m\sqrt{2}\sqrt{-\sqrt{m-1}\,(z-z_0)}}{(n-m)\sqrt{m-1}}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sum_{j=0}^{k}(j+1)\,c_{k-j}\sum_{r=0}^{j}\frac{(-1)^r\binom{j}{r}}{r+1}\,q_{r,j}\,(z-z_0)^k\;/;$$

$$z_0=\pi u+\csc^{-1}\left(\sqrt{m}\right)\bigwedge u\in\mathbb{Z}\bigwedge a_0=1\bigwedge a_{2k}=\frac{(-1)^k\,2^{2k}}{(2k+1)!}\bigwedge a_{2k+1}=\frac{(-1)^{k-1}\,2^{2k}\,(2-m)}{\sqrt{m-1}\,(2k+2)!}\bigwedge$$

$$b_0=1\bigwedge b_k=\frac{2n\sqrt{m-1}}{n-m}\,a_{k-1}\bigwedge c_k=\frac{1}{k!}\left(\frac{3}{2}\right)_k\sum_{j=0}^{k}\frac{(-1)^j}{2j+1}\binom{k}{j}p_{j,k}\bigwedge p_{u,0}=1\bigwedge$$

$$p_{u,v}=\frac{1}{v}\sum_{j=1}^{v}(uj+j-v)\,a_j\,p_{u,v-j}\bigwedge q_{u,0}=1\bigwedge q_{j,k}=\frac{1}{k}\sum_{i=1}^{k}(ji+i-k)\,b_i\,q_{j,k-i}\bigwedge k\in\mathbb{N}$$

08.06.06.0072.01

$$\Pi(n;z\mid m)\propto\frac{1}{\sqrt{m}}\,\Pi\left(\frac{n}{m}\,\middle|\,\frac{1}{m}\right)+2u\,\Pi(n\mid m)-\frac{m\sqrt{2}\sqrt{-\sqrt{m-1}\,(z-z_0)}}{(n-m)\sqrt{m-1}}\,(1+O(z-z_0))\;/;$$

$$(z\to z_0)\bigwedge z_0=\csc^{-1}\left(\sqrt{m}\right)+\pi u\bigwedge u\in\mathbb{Z}$$

Expansions at $z == -\csc^{-1}\left(\sqrt{m}\right) + \pi u /; u \in \mathbb{Z}$

08.06.06.0073.01

$$\Pi(n; z \mid m) \propto \frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + 2 u \Pi(n \mid m) - \frac{m \sqrt{2} \sqrt{-\sqrt{m-1}\, (z - z_0)}}{(n-m)\sqrt{m-1}} \left(1 - \frac{\left(m^2 - (9n+2)m + 10n\right)(z - z_0)}{12\sqrt{m-1}\,(m-n)} + \left((m(9m-4)+4)m^2 + 2(m(15m-68)+44)nm + (m(345m-628)+292)n^2\right) \Big/ \left(480(m-1)(m-n)^2\right)(z-z_0)^2 - \left(15m^6 - (21n+26)m^5 + \left(-1155n^2 + 214n - 12\right)m^4 + \left(-1911n^3 + 4258n^2 - 284n + 8\right)m^3 + 2n\left(2385n^2 - 2578n + 68\right)m^2 - 4n^2(941n - 502)m + 920n^3\right) \Big/ \left(2688(m-1)^{3/2}(m-n)^3\right)(z-z_0)^3 + \ldots\right) /; (z \to z_0) \bigwedge z_0 = \csc^{-1}\left(\sqrt{m}\right) + \pi u \bigwedge u \in \mathbb{Z}$$

08.06.06.0074.01

$$\Pi(n; z \mid m) = 2 u \Pi(n \mid m) - \frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) - \frac{m\sqrt{2}\sqrt{\sqrt{m-1}\,(z-z_0)}}{(n-m)\sqrt{m-1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \sum_{j=0}^{k} (j+1) c_{k-j} \sum_{r=0}^{j} \frac{(-1)^r \binom{j}{r}}{r+1} q_{r,j} (z-z_0)^k /;$$

$$z_0 = \pi u - \csc^{-1}\left(\sqrt{m}\right) \bigwedge u \in \mathbb{Z} \bigwedge a_0 = 1 \bigwedge a_{2k} = \frac{(-1)^k 2^{2k}}{(2k+1)!} \bigwedge a_{2k+1} = \frac{(-1)^{k-1} 2^{2k} (2-m)}{\sqrt{m-1}\,(2k+2)!} \bigwedge$$

$$b_0 = 1 \bigwedge b_k = \frac{2n\sqrt{m-1}}{n-m} a_{k-1} \bigwedge c_k = \frac{1}{k!} \left(\frac{3}{2}\right)_k \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \binom{k}{j} p_{j,k} \bigwedge p_{u,0} = 1 \bigwedge$$

$$p_{u,v} = \frac{1}{v} \sum_{j=1}^{v} (u j + j - v) a_j\, p_{u,v-j} \bigwedge q_{u,0} = 1 \bigwedge q_{j,k} = \frac{1}{k} \sum_{i=1}^{k} (j i + i - k) b_i\, q_{j,k-i} \bigwedge k \in \mathbb{N}$$

08.06.06.0075.01

$$\Pi(n; z \mid m) \propto -\frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m} \,\middle|\, \frac{1}{m}\right) + 2 u \Pi(n \mid m) - \frac{m\sqrt{2}\sqrt{\sqrt{m-1}\,(z-z_0)}}{(n-m)\sqrt{m-1}} (1 + O(z - z_0)) /;$$

$$(z \to z_0) \bigwedge z_0 = -\csc^{-1}\left(\sqrt{m}\right) + \pi u \bigwedge u \in \mathbb{Z}$$

Expansions at $z == \csc^{-1}\left(\sqrt{n}\right) + \pi u /; u \in \mathbb{Z}$

08.06.06.0076.01

$$\Pi(n; z \mid m) \propto -\frac{\log(z - z_0)}{2\sqrt{n-1}\sqrt{\frac{n-m}{n}}} - \frac{1}{2\sqrt{n-1}\sqrt{\frac{n-m}{n}}} \left(-\frac{(n-2)n + m(4-3n)}{2\sqrt{n-1}\,(n-m)}(z-z_0) + \left(\left(21n^2 - 40n + 22\right)m^2 - 2n\left(3n^2 - 4n + 4\right)m + n^2\left(3n^2 - 4n + 4\right)\right) \Big/ \left(24(m-n)^2(n-1)\right)(z-z_0)^2 + \left(-(n-2)n^5 + 3m(n-2)n^4 + 3m^2\left(n^3 - 6n^2 + 10n - 4\right)n + m^3\left(15n^3 - 38n^2 + 30n - 8\right)\right) \Big/ \left(24(n-1)^{3/2}(n-m)^3\right)(z-z_0)^3 + \ldots\right) + \frac{\pi i \sqrt{\frac{1}{n}}}{2\sqrt{1-\frac{1}{n}}\sqrt{1-\frac{m}{n}}} /; (z \to z_0) \bigwedge z_0 = \pi u + \csc^{-1}\left(\sqrt{n}\right) \bigwedge u \in \mathbb{Z}$$

08.06.06.0077.01

$$\Pi(n; z \mid m) = -\frac{\log(z-z_0)}{2\sqrt{n-1}\sqrt{\frac{n-m}{n}}} - \frac{1}{2\sqrt{n-1}\sqrt{\frac{n-m}{n}}}\sum_{k=0}^{\infty}\frac{1}{k+1}\sum_{j=0}^{k+1}(j+1)\,c_{-j+k+1}\sum_{r=0}^{j}\frac{(-1)^r}{r+1}\binom{j}{r}q_{r,j}\,(z-z_0)^{k+1} + \frac{\pi\,i\sqrt{\frac{1}{n}}}{2\sqrt{1-\frac{1}{n}}\sqrt{1-\frac{m}{n}}} \;/;$$

$$z_0 = \pi\,u + \csc^{-1}\left(\sqrt{n}\right) \bigwedge u \in \mathbb{Z} \bigwedge a_0 = 1 \bigwedge a_{2k} = \frac{(-1)^k\,2^{2k}}{(2k+1)!} \bigwedge a_{2k+1} = \frac{(-1)^{k-1}\,2^{2k}\,(2-m)}{\sqrt{m-1}\,(2k+2)!} \bigwedge$$

$$b_0 = 1 \bigwedge b_k = \frac{2\,n\sqrt{m-1}}{n-m}\,a_{k-1} \bigwedge c_k = \frac{\left(\frac{3}{2}\right)_k}{k!}\sum_{j=0}^{k}\frac{(-1)^j}{2j+1}\binom{k}{j}p_{j,k} \bigwedge p_{u,0} = 1 \bigwedge$$

$$p_{u,v} = \frac{1}{v}\sum_{j=1}^{v}(u\,j + j - v)\,b_j\,p_{u,v-j} \bigwedge q_{u,0} = 1 \bigwedge q_{j,k} = \frac{1}{k}\sum_{i=1}^{k}(j\,i + i - k)\,a_i\,q_{j,k-i} \bigwedge k \in \mathbb{N}$$

08.06.06.0078.01

$$\Pi(n; z \mid m) \propto -\frac{\log(z-z_0)}{2\sqrt{n-1}\sqrt{\frac{n-m}{n}}} + \frac{\pi\,i\sqrt{\frac{1}{n}}}{2\sqrt{1-\frac{1}{n}}\sqrt{1-\frac{m}{n}}} + \frac{(n-2)\,n + m\,(4-3\,n)\,(z-z_0)}{4\sqrt{n-1}\,(n-m)\sqrt{n-1}\sqrt{\frac{n-m}{n}}}\,(1 + O(z-z_0)) \;/;$$

$$(z \to z_0) \bigwedge z_0 = \pi\,u + \csc^{-1}\left(\sqrt{n}\right) \bigwedge u \in \mathbb{Z}$$

Expansions at *z* == ∞

08.06.06.0079.01

$$\Pi\left(n; \sin^{-1}(z) \,\middle|\, m\right) \propto$$

$$\frac{i\sqrt{-z^2}}{z}\left(\frac{i}{\sqrt{-m}}\left(K\left(\frac{1}{m}\right) + i\left(1 - \sqrt{\frac{m}{m-1}}\sqrt{\frac{m-1}{m}}\right)\left(K\left(1-\frac{1}{m}\right) + \frac{n}{m-n}\,\Pi\left(\frac{m-1}{m-n} \,\middle|\, \frac{m-1}{m}\right)\right) - \Pi\left(\frac{1}{n} \,\middle|\, \frac{1}{m}\right)\right)\right) +$$

$$\Pi(n \mid m) + \frac{\sqrt{-z^2}\sqrt{-m\,z^2}}{n\,m\,z^5}\sum_{k=0}^{\infty}\sum_{u=0}^{k}\sum_{i=0}^{u}\frac{m^{u-k}\,n^{i-u}\left(\frac{1}{2}\right)_i\left(\frac{1}{2}\right)_{k-u}}{i!\,(k-u)!\,(2k+3)}\,z^{-2k} \;/;\; (|z| \to \infty) \bigwedge$$

$$\left(\left(0 < \arg(n) < \frac{\pi}{2} \bigwedge 0 < \arg(m) < \pi\right) \bigvee \left(\frac{\pi}{2} < \arg(n) < \pi \bigwedge 0 < \arg(m) < \frac{\pi}{2} \bigwedge (|m| < 1 \lor (|n| > 1 \land |n| > |m|))\right) \bigvee \left(\frac{\pi}{2} < \arg(n) < \pi \bigwedge \frac{\pi}{2} < \arg(m) < \pi \bigwedge |n| > |m|\right)\right)$$

Expansions at generic point *m* == m_0

For the function itself

08.06.06.0080.01

$$\Pi(n;z\mid m)\propto\Pi(n;z\mid m_0)+\frac{1}{2(n-m_0)}\left(\frac{E(z\mid m_0)}{m_0-1}+\Pi(n;z\mid m_0)-\frac{m_0\sin(2z)}{2(m_0-1)\sqrt{1-m_0\sin^2(z)}}\right)(m-m_0)+\frac{1}{2}\left(\frac{\left(4m_0^2-(n+2)m_0-n\right)E(z\mid m_0)}{4(m_0-1)^2m_0(m_0-n)^2}+\frac{F(z\mid m_0)}{4(m_0-1)m_0(m_0-n)}+\frac{3\Pi(n;z\mid m_0)}{4(m_0-n)^2}+\frac{\left(\left(4m_0^2-(n+2)m_0-n\right)m_0\sin^2(z)-3m_0^2+2n+m_0\right)\sin(2z)}{8(m_0-1)^2(m_0-n)^2\left(1-m_0\sin^2(z)\right)^{3/2}}\right)(m-m_0)^2+\ldots\;/;\;(m\to m_0)$$

08.06.06.0081.01

$$\Pi(n;z\mid m)\propto\Pi(n;z\mid m_0)+\frac{1}{2(n-m_0)}\left(\frac{E(z\mid m_0)}{m_0-1}+\Pi(n;z\mid m_0)-\frac{m_0\sin(2z)}{2(m_0-1)\sqrt{1-m_0\sin^2(z)}}\right)(m-m_0)+\frac{1}{2}\left(\frac{\left(4m_0^2-(n+2)m_0-n\right)E(z\mid m_0)}{4(m_0-1)^2m_0(m_0-n)^2}+\frac{F(z\mid m_0)}{4(m_0-1)m_0(m_0-n)}+\frac{3\Pi(n;z\mid m_0)}{4(m_0-n)^2}+\frac{\left(\left(4m_0^2-(n+2)m_0-n\right)m_0\sin^2(z)-3m_0^2+2n+m_0\right)\sin(2z)}{8(m_0-1)^2(m_0-n)^2\left(1-m_0\sin^2(z)\right)^{3/2}}\right)(m-m_0)^2+O\left((m-m_0)^3\right)$$

08.06.06.0082.01

$$\Pi(n;z\mid m)=\sum_{k=0}^{\infty}\frac{1}{k!}\Pi^{(0,0,k)}(n;z\mid m_0)(m-m_0)^k$$

08.06.06.0083.01

$$\Pi(n;z\mid m)\propto\Pi(n;z\mid m_0)\,(1+O(m-m_0))$$

Expansions at $m == 0$

08.06.06.0084.01

$$\Pi(n;z\mid m)\propto\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil\left(\frac{\pi}{\sqrt{1-n}}-\frac{\pi}{2\sqrt{1-n}\,n}\left(\sqrt{1-n}-1\right)m-\frac{3\pi}{16n^2}\left(n-\frac{2}{\sqrt{1-n}}+2\right)m^2+\ldots\right)+\frac{\tanh^{-1}\left(\sqrt{n-1}\tan(z)\right)}{\sqrt{n-1}}+\frac{m}{2n}\left(\frac{\tanh^{-1}\left(\sqrt{n-1}\tan(z)\right)}{\sqrt{n-1}}-z\right)+\frac{3\left(4\tanh^{-1}\left(\sqrt{n-1}\tan(z)\right)+\sqrt{n-1}\,(n\sin(2z)-2(n+2)z)\right)m^2}{32\sqrt{n-1}\,n^2}+\ldots\;/;\;(m\to 0)$$

08.06.06.0085.01

$$\Pi(n;z\mid m)\propto\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil\left(\frac{\pi}{\sqrt{1-n}}-\frac{\pi}{2\sqrt{1-n}\,n}\left(\sqrt{1-n}-1\right)m-\frac{3\pi}{16n^2}\left(n-\frac{2}{\sqrt{1-n}}+2\right)m^2+O\left(m^3\right)\right)+\frac{\tanh^{-1}\left(\sqrt{n-1}\tan(z)\right)}{\sqrt{n-1}}+\frac{m}{2n}\left(\frac{\tanh^{-1}\left(\sqrt{n-1}\tan(z)\right)}{\sqrt{n-1}}-z\right)+\frac{3\left(4\tanh^{-1}\left(\sqrt{n-1}\tan(z)\right)+\sqrt{n-1}\,(n\sin(2z)-2(n+2)z)\right)m^2}{32\sqrt{n-1}\,n^2}+O\left(m^3\right)$$

08.06.06.0086.01

$$\Pi(n;z\,|\,m)=\frac{\tanh^{-1}\left(\sqrt{n-1}\,\tan(z)\right)}{\sqrt{n-1}\,\sqrt{\frac{n-m}{n}}}+\frac{m}{4n}$$

$$\sum_{k=0}^{\infty}\frac{\left(\frac{3}{2}\right)_k 2^{-k} m^k}{(k+1)!}\sum_{j=0}^{k}(-1)^j\binom{k+1}{j+1}\left(\frac{n-2}{n}\right)^j\sum_{i=0}^{j+1}(-1)^i\binom{j+1}{i}\left(\sum_{q=0}^{\left\lfloor\frac{i}{2}\right\rfloor-1} q!\binom{i-1}{2q+1}\left(\frac{n}{2-n}\right)^{2q+1}\sin^{2q+1}(2z)\sum_{p=0}^{q}\frac{\cot^{2p}(2z)}{\left(\frac{3}{2}\right)_{q-p}\,p!}+\right.$$

$$\left.\sum_{q=0}^{\left\lfloor\frac{i-1}{2}\right\rfloor}\frac{\binom{i-1}{2q}\left(\frac{1}{2}\right)_q\left(\frac{n}{2-n}\right)^{2q}}{2q!}\left(4z+\tan(2z)\sum_{p=1}^{q}\frac{(p-1)!\cos^{2p}(2z)}{\left(\frac{1}{2}\right)_p}\right)\right)+2\left\lfloor\frac{\mathrm{Re}(z)}{\pi}\right\rceil\Pi(n\,|\,m)\;/;\;|m|<1$$

08.06.06.0005.01

$$\Pi(n;z\,|\,m)=(-1)^{\left\lfloor\frac{\mathrm{Re}(z)}{\pi}\right\rceil}\sin(z)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\sin^{2k}(z)}{(2k+1)\,k!}F_1\left(k+\frac{1}{2};\frac{1}{2},1;k+\frac{3}{2};\sin^2(z),n\sin^2(z)\right)m^k+2\left\lfloor\frac{\mathrm{Re}(z)}{\pi}\right\rceil\Pi(n\,|\,m)\;/;\;|m|<1$$

08.06.06.0087.01

$$\Pi(n;z\,|\,m)\propto\left(\frac{\tanh^{-1}\left(\sqrt{n-1}\,\tan(z)\right)}{\sqrt{n-1}}+\frac{\pi}{\sqrt{1-n}}\left\lfloor\frac{\mathrm{Re}(z)}{\pi}\right\rceil\right)(1+O(m))$$

Expansions at $m = 1$

08.06.06.0088.01

$$\Pi(n;z\,|\,m)\propto$$

$$(-1)^{\left\lfloor\frac{\mathrm{Re}(z)}{\pi}\right\rceil}\left(\frac{\sqrt{n}\,\tanh^{-1}\left(\sqrt{n}\,\sin(z)\right)-\tanh^{-1}(\sin(z))}{n-1}-\frac{(n+1)\tanh^{-1}(\sin(z))-2\sqrt{n}\,\tanh^{-1}\left(\sqrt{n}\,\sin(z)\right)+(n-1)\sec(z)\tan(z)}{4(n-1)^2}\right.$$

$$(m-1)+\frac{3}{64(n-1)^3}\left(\frac{1}{2}(n-1)(-3n+(n-5)\cos(2z)-1)\tan(z)\sec^3(z)+\right.$$

$$\left.\left.\left(n^2-6n-3\right)\tanh^{-1}(\sin(z))+8\sqrt{n}\,\tanh^{-1}\left(\sqrt{n}\,\sin(z)\right)\right)(m-1)^2+\ldots\right)-\left\lfloor\frac{\mathrm{Re}(z)}{\pi}\right\rceil$$

$$\left(\frac{\log(1-m)}{1-n}\left(1+\frac{(n+1)(m-1)}{4(n-1)}-\frac{3\left(n^2-6n-3\right)(m-1)^2}{64(n-1)^2}+\ldots\right)-\frac{\sqrt{n}\left(\log\left(\sqrt{n}+1\right)-\log\left(1-\sqrt{n}\right)\right)-4\log(2)}{n-1}+\right.$$

$$\frac{\left(2n\log(2)+2\log(2)+\sqrt{n}\left(\log\left(1-\sqrt{n}\right)-\log\left(\sqrt{n}+1\right)\right)-1\right)(m-1)}{2(n-1)^2}-$$

$$\frac{1}{64(n-1)^3}\left(-5n^2+12n+24\left(\log\left(\sqrt{n}+1\right)-\log\left(1-\sqrt{n}\right)\right)\sqrt{n}+12((n-6)n-3)\log(2)+21\right)(m-1)^2+$$

$$\left.\ldots\right)/;\;(m\to1)\bigwedge\neg\frac{2\,\mathrm{Re}(z)+\pi}{4\pi}\in\mathbb{Z}$$

08.06.06.0089.01

$$\Pi(n;z\mid m)\propto\frac{1}{2(n-1)}\left(\sqrt{n}\,\log\left(\frac{1+\sqrt{n}\,\sin(z)}{1-\sqrt{n}\,\sin(z)}\right)-2\log(\sec(z)+\tan(z))\right)+\frac{1}{6}\sin^3(z)\,F_1\left(\frac{3}{2};2,1;\frac{5}{2};\sin^2(z),n\sin^2(z)\right)(m-1)+\frac{3\sin^5(z)}{40}F_1\left(\frac{5}{2};3,1;\frac{7}{2};\sin^2(z),n\sin^2(z)\right)(m-1)^2+\ldots\;/;\;(m\to1)\bigwedge|\text{Re}(z)|\le\frac{\pi}{2}$$

08.06.06.0090.01

$$\Pi(n;z\mid m)=(-1)^{\left\lceil\frac{\text{Re}(z)}{\pi}\right\rfloor}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{k!}\left(\sqrt{n}\,\tanh^{-1}\left(\sqrt{n}\,\sin(z)\right)(n-1)^{-k-1}+\frac{1}{2\,k!}\sum_{p=0}^{k}\frac{(-1)^{k-p}(p)_{2(k-p)}}{2^{2k-p}(k-p)!}\left((-1)^p\left(\left(1-\sqrt{n}\right)^{-p-1}+\left(\sqrt{n}+1\right)^{-p-1}\right)\tanh^{-1}(\sin(z))\,p!+\tan(z)\sec(z)\sum_{j=0}^{p}\binom{p}{j}(-1)^{p-j}(p-j)!\left(\left(1-\sqrt{n}\right)^{j-p-1}+\left(\sqrt{n}+1\right)^{j-p-1}\right)\sum_{q=0}^{j-1}\frac{\left(2^{-j+2q+1}q!\tan^{2q}(z)\right)(-j+2q+2)_{2(j-q-1)}}{(j-q-1)!}\right)\right)(m-1)^k-\left\lceil\frac{\text{Re}(z)}{\pi}\right\rfloor\left(\log(1-m)\sum_{j=0}^{\infty}\frac{(1-m)^j\left(\frac{1}{2}\right)_j^2}{(j!)^2}\,{}_2F_1\left(1,j+\frac{1}{2};\frac{1}{2};n\right)-\sum_{k=0}^{\infty}\frac{(1-m)^k\left(\frac{1}{2}\right)_k^2}{(k!)^2}\sum_{s=0}^{\infty}\frac{\left(k+\frac{1}{2}\right)_s\left(2\psi(k+1)-\psi\left(k+\frac{1}{2}\right)-\psi\left(k+s+\frac{1}{2}\right)\right)n^s}{\left(\frac{1}{2}\right)_s}\right)\;/;\;\neg\frac{2\,\text{Re}(z)+\pi}{4\pi}\in\mathbb{Z}$$

08.06.06.0091.01

$$\Pi(n;z\mid m)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{k!}\left(\sqrt{n}\,\tanh^{-1}\left(\sqrt{n}\,\sin(z)\right)(n-1)^{-k-1}+\frac{1}{2\,k!}\sum_{p=0}^{k}\frac{(-1)^{k-p}(p)_{2(k-p)}}{2^{2k-p}(k-p)!}\left((-1)^p\left(\left(1-\sqrt{n}\right)^{-p-1}+\left(\sqrt{n}+1\right)^{-p-1}\right)\tanh^{-1}(\sin(z))\,p!+\tan(z)\sec(z)\sum_{j=0}^{p}\binom{p}{j}(-1)^{p-j}(p-j)!\left(\left(1-\sqrt{n}\right)^{j-p-1}+\left(\sqrt{n}+1\right)^{j-p-1}\right)\sum_{q=0}^{j-1}\frac{\left(2^{-j+2q+1}q!\tan^{2q}(z)\right)(-j+2q+2)_{2(j-q-1)}}{(j-q-1)!}\right)\right)(m-1)^k\;/;\;|\text{Re}(z)|\le\frac{\pi}{2}$$

08.06.06.0092.01

$$\Pi(n;z\mid m)=(-1)^{\left\lceil\frac{\text{Re}(z)}{\pi}\right\rfloor}\sin(z)\sum_{k=0}^{\infty}\frac{\sin^{2k}(z)\left(\frac{1}{2}\right)_k}{(2k+1)\,k!}F_1\left(k+\frac{1}{2};k+1,1;k+\frac{3}{2};\sin^2(z),n\sin^2(z)\right)(m-1)^k+2\left\lceil\frac{\text{Re}(z)}{\pi}\right\rfloor\Pi(n\mid m)\;/;\;\neg\frac{2\,\text{Re}(z)+\pi}{4\pi}\in\mathbb{Z}$$

08.06.06.0093.01

$$\Pi(n; z \mid m) \propto \left(\frac{(-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil}}{n-1} \left(\sqrt{n} \tanh^{-1}\left(\sqrt{n} \sin(z)\right) - \tanh^{-1}(\sin(z)) \right) + \frac{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil}{n-1} \left(\sqrt{n} \left(\log\left(\sqrt{n} + 1\right) - \log\left(1 - \sqrt{n}\right) \right) - 4 \log(2) \right) \right) (1 + O(m-1)) - \frac{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \log(1-m)}{1-n} (1 + O(m-1)) \;/;\; \neg \frac{2\,\mathrm{Re}(z) + \pi}{4\pi} \in \mathbb{Z}$$

Expansions at $m == \infty$

08.06.06.0094.01

$$\Pi(n; z \mid m) \propto 2 \left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \left(\frac{\log(-m)}{2\sqrt{-m}} \left(1 + \frac{2n+1}{4m} + \frac{3\left(8n^2 + 4n + 3\right)}{64m^2} + \ldots \right) + \frac{\sqrt{n} \sin^{-1}\left(\sqrt{n}\right)}{\sqrt{1-n}\,\sqrt{-m}} \left(1 + \frac{n}{2m} + \frac{3n^2}{8m^2} + \ldots \right) + \frac{1}{2\sqrt{-m}} \left(4\log(2) + \frac{2\log(2) + n(4\log(2) - 1) - 1}{2m} - \frac{28n^2 + 26n - 12\left(8n^2 + 4n + 3\right)\log(2) + 21}{64m^2} + \ldots \right) \right) +$$
$$(-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil} \frac{\sqrt{-m \sin^2(z)}}{2m\sin(z)} \left(-\log\left(-4m\sin^2(z)\right) + \left(2\log\left(\cos^2\left(\frac{z}{2}\right)\right) - n\sin^2(z) \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j \sin^{2j}(z)}{(j+1)\,j!} \, {}_2F_1\left(1, -j; \frac{1}{2} - j; n\right) \right) + \right.$$
$$\frac{1}{4m} \left(-n\sin^2(z) \sum_{j=0}^{\infty} \frac{\left(\frac{3}{2}\right)_j \sin^{2j}(z)}{(j+1)(j+1)!} \, {}_2F_1\left(1, -j-1; -j - \frac{1}{2}; n\right) + 2\left(n + \cot(z)\csc(z) + \log\left(\cos^2\left(\frac{z}{2}\right)\right) + 1 \right) - (2n+1)\log\left(-4m\sin^2(z)\right) \right) +$$
$$\left. \frac{1}{64m^2} \left(21 + 24n\csc^2(z) + 6\cot(z)\left(2\csc^2(z) + 3\right)\csc(z) + 2n(12n+7) + 18\log\left(\cos^2\left(\frac{z}{2}\right)\right) - 3\left(8n^2 + 4n + 3\right)\log\left(-4m\sin^2(z)\right) - 18n\sin^2(z) \sum_{j=0}^{\infty} \frac{\left(\frac{5}{2}\right)_j \sin^{2j}(z)}{(j+1)(j+2)!} \, {}_2F_1\left(1, -j-2; -j - \frac{3}{2}; n\right) \right) \right) \;/;\; (|m| \to \infty)$$

08.06.06.0095.01

$$\Pi(n; z \mid m) =$$

$$2\left\lfloor\frac{\text{Re}(z)}{\pi}\right\rceil \Pi(n \mid m) + (-1)^{\left\lfloor\frac{\text{Re}(z)}{\pi}\right\rceil} \frac{\sqrt{-m\sin^2(z)}}{2\,m\sin(z)} \left(-n\sin^2(z)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{m^{-k}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{j+k}\sin^{2j}(z)}{(j+1)\,k!\,(j+k)!}\,{}_2F_1\!\left(1, -j-k; -j-k+\frac{1}{2}; n\right) - \right.$$

$$\frac{2\log\left(-4\,m\sin^2(z)\right)}{\pi}K\!\left(\frac{1}{m}\right) + \frac{1}{4\,m}\,{}_4F_3\!\left(1, 1, \frac{3}{2}, \frac{3}{2}; 2, 2, 2; \frac{1}{m}\right) + \log(4) -$$

$$2\log\left(\sqrt{1-\frac{\csc^2(z)}{m}} + 1\right) - \frac{n\log\left(-4\,m\sin^2(z)\right)}{2\,m}\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{(k+1)!\,k!}\,{}_2F_1\!\left(1, -k; \frac{1}{2}-k; n\right) +$$

$$\frac{n}{2\,m}\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{((k+1)!)^2}\,{}_2F_1\!\left(1, -k; \frac{1}{2}-k; n\right) - \frac{\sin^2(z)}{2}\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!\,(k+1)!}\,{}_3F_2\!\left(1, 1, k+\frac{3}{2}; 2, k+2; \sin^2(z)\right) +$$

$$\frac{3\csc^2(z)}{8\,m^2}\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{3}{2}\right)_k\left(\frac{5}{2}\right)_k}{(k+1)!}\,{}_3\tilde{F}_2\!\left(1, 1, k+\frac{5}{2}; k+3, 2; \frac{\csc^2(z)}{m}\right) + \frac{9\,n^2}{8\,m^2}\sum_{k=0}^{\infty}\frac{m^{-k}\,n^k\left(\frac{5}{2}\right)_k}{(k+2)!}\sum_{i=0}^{k}\frac{1}{i+k+3} +$$

$$\frac{9}{8\,m^2}\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{5}{2}\right)_k^2}{((k+2)!)^2}\sum_{i=0}^{k}\frac{1}{i+k+3} + \frac{3\,n}{4\,m^2\sin^2(z)}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{m^{-k}\,n^j\left(\frac{5}{2}\right)_k\left(\frac{1}{2}\right)_{k-j}}{(k-j)!}\,{}_3\tilde{F}_2\!\left(1, 1, k+\frac{5}{2}; k+3, 2; \frac{\csc^2(z)}{m}\right) +$$

$$\left.\frac{3\,n}{4\,m^2}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{m^{-k}\,n^j\left(\frac{3}{2}\right)_{k-j}\left(\frac{5}{2}\right)_k}{(k-j+1)!\,(k+2)!}\sum_{i=0}^{k}\frac{1}{i+k+3}\right)$$

08.06.06.0096.01

$$\Pi(n; z \mid m) =$$

$$2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil \Pi(n \mid m) + (-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil} \frac{\sqrt{-m\sin^2(z)}}{2\, m\sin(z)} \left(-n\sin^2(z) \sum_{i=0}^{\infty} \frac{\left(\frac{1}{2}\right)_i}{i!} \left(\frac{n}{m}\right)^i \sum_{j=0}^{i} \frac{n^{-j}\left(\frac{1}{2}\right)_j}{j!}\, {}_3F_2\left(1, 1, j+\frac{1}{2}; 2, j+1; \sin^2(z)\right) - \right.$$

$$2\log\left(\sqrt{1-\frac{\csc^2(z)}{m}}+1\right) - \log\left(-m\sin^2(z)\right) - \frac{n^2\sin^4(z)}{2} \sum_{i=0}^{\infty} \frac{\left(\frac{1}{2}\right)_i}{i!} F_1\left(2; \frac{1}{2}, 1; 3; \sin^2(z), n\sin^2(z)\right)\left(\frac{n}{m}\right)^i +$$

$$\frac{n}{2\, m} \sum_{k=0}^{\infty} \frac{m^{-k}\left(\frac{1}{2}\right)_k\left(\frac{3}{2}\right)_k}{k!\,(k+1)!} \left(\frac{1}{k+1} - \log\left(-4\, m\sin^2(z)\right)\right) {}_2F_1\left(1, -k; \frac{1}{2}-k; n\right) -$$

$$\frac{\sin^2(z)}{2} \sum_{i=0}^{\infty} \frac{m^{-i}\left(\frac{3}{2}\right)_i\left(\frac{1}{2}\right)_i}{i!\,(i+1)!}\, {}_3F_2\left(1, 1, i+\frac{3}{2}; 2, i+2; \sin^2(z)\right) + \frac{1}{2\, m} \sum_{k=0}^{\infty} \frac{m^{-k}\left(\frac{3}{2}\right)_k}{(k+1)!}$$

$$\left(\frac{3\csc^2(z)\left(\frac{5}{2}\right)_k}{4\, m}\, {}_3\tilde{F}_2\left(1, 1, k+\frac{5}{2}; k+3, 2; \frac{\csc^2(z)}{m}\right) - \frac{\left(\frac{3}{2}\right)_k}{2\,(k+1)!}\left(\log\left(-4\, m\sin^2(z)\right) - \sum_{i=0}^{k-1} \frac{2}{i+k+2} - \frac{1}{k+1}\right)\right) +$$

$$\left.\frac{n}{m} \sum_{k=0}^{\infty} \left(\frac{3\csc^2(z)\left(\frac{5}{2}\right)_k}{4\, m}\, {}_3\tilde{F}_2\left(1, 1, k+\frac{5}{2}; k+3, 2; \frac{\csc^2(z)}{m}\right) + \frac{\left(\frac{3}{2}\right)_k}{(k+1)!} \sum_{i=0}^{k-1} \frac{1}{i+k+2}\right)\left(\frac{n}{m}\right)^k \sum_{j=0}^{k} \frac{n^{-j}}{j!}\left(\frac{1}{2}\right)_j\right)$$

08.06.06.0097.01

$$\Pi(n; z \mid m) =$$

$$2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil \Pi(n \mid m) + (-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil} \frac{\sqrt{-m\sin^2(z)}}{2\, m\sin(z)} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k n^j m^{-j-k}}{k!} \left(2\, m\sin^2(z) \sum_{i=0}^{j+k-1} \frac{\left(-\frac{1}{2}\right)_{j+k-i} m^i \sin^{2i}(z)}{(i+1)\,(j+k-i-1)!} - \frac{\left(\frac{1}{2}\right)_{j+k}}{(j+k)!}\right.$$

$$\left.\left(\log\left(-m\sin^2(z)\right) + \psi(j+k+1) - \psi\left(j+k+\frac{1}{2}\right)\right) + \frac{\left(\frac{3}{2}\right)_{j+k}}{2\, m\sin^2(z)}\, {}_3\tilde{F}_2\left(1, 1, j+k+\frac{3}{2}; j+k+2, 2; \frac{\csc^2(z)}{m}\right)\right)$$

08.06.06.0098.01

$$\Pi(n; z \mid m) = 2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil \Pi(n \mid m) + (-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{n^j\left(\frac{1}{2}\right)_k \sin^{2j+2k+1}(z)}{(2\, j+2\, k+1)\, k!}\, {}_2F_1\left(j+k+\frac{1}{2}, \frac{1}{2}; j+k+\frac{3}{2}; m\sin^2(z)\right)$$

08.06.06.0099.01

$$\Pi(n; z \mid m) = 2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil \Pi(n \mid m) + (-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil} \frac{\sqrt{m\sin^2(z)}}{2\, m\sin(z)} \left(\frac{m\,\pi}{m-n} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2 m^{-k}}{(k!)^2}\, {}_2F_1\left(\frac{1}{2}, 1; k+1; \frac{n}{n-m}\right) - \right.$$

$$\left.\frac{2\, m\sqrt{1-m\sin^2(z)}}{m-n} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k m^{-k}}{k!} F_1\left(\frac{1}{2}; \frac{1}{2}-k, 1; \frac{3}{2}; 1-m\sin^2(z), \frac{n\left(1-m\sin^2(z)\right)}{n-m}\right)\right)$$

08.06.06.0100.01

$$\Pi(n;z\mid m)=2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil\Pi(n\mid m)+(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\sin^{2k+1}(z)}{(2k+1)\,k!}\,F_1\!\left(k+\frac{1}{2};\frac{1}{2},1;k+\frac{3}{2};m\sin^2(z),n\sin^2(z)\right)$$

08.06.06.0101.01

$$\Pi(n;z\mid m)=(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\frac{\sin(z)}{\sqrt{-m\sin^2(z)}}\left(\frac{\log\left(1-n\sin^2(z)\right)}{2}\left(1-\frac{1}{\sqrt{1-\frac{1}{n}}}\right)\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{1}{2}\right)_k n^k}{k!}-\right.$$

$$\frac{1}{4m\sin^2(z)}\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{3}{2}\right)_k}{(k+1)!}\,{}_3F_2\!\left(1,1,k+\frac{3}{2};k+2,2;\frac{1}{m\sin^2(z)}\right)\left(\frac{n^k}{\sqrt{1-\frac{1}{n}}}-\frac{\left(\frac{3}{2}\right)_k}{2n(k+1)!}\,{}_2F_1\!\left(1,k+\frac{3}{2};k+2;\frac{1}{n}\right)\right)+$$

$$\sum_{k=0}^{\infty}\frac{m^{-k}\left(\frac{3}{2}\right)_k\left(\log\left(-m\sin^2(z)\right)+\psi(k+1)-\psi\left(k+\frac{1}{2}\right)\right)}{2(2k+1)k!}\left(\frac{n^k}{\sqrt{1-\frac{1}{n}}}-\frac{\left(\frac{3}{2}\right)_k}{2n(k+1)!}\,{}_2F_1\!\left(1,k+\frac{3}{2};k+2;\frac{1}{n}\right)\right)-$$

$$\left.\frac{3\sin^2(z)}{8n}\sum_{i=0}^{\infty}\frac{m^{-i}\left(\frac{5}{2}\right)_i\left(\frac{1}{2}\right)_i}{i!\,(i+2)!}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{n^{-j}\sin^{2k}(z)\,(1)_k^2\,(1)_j\left(i+\frac{5}{2}\right)_{j+k}}{j!\,k!\,(2)_k\,(i+3)_{j+k}}-\frac{\log\left(1-n\sin^2(z)\right)}{2\sqrt{1-\frac{n}{m}}}\right)+2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil\Pi(n\mid m)$$

08.06.06.0102.01

$$\Pi(n;z\mid m)=F(z\mid m)+2\left(\Pi(n\mid m)-K(m)\right)\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil-(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\frac{n\sqrt{-m\sin^2(z)}}{m\sin(z)}$$

$$\left(\frac{\tanh^{-1}\left(\frac{\cos(z)\sqrt{n}}{\sqrt{n-1}}\right)-\tanh^{-1}\left(\frac{\sqrt{n}}{\sqrt{n-1}}\right)}{\sqrt{n-1}\,\sqrt{n}}+\frac{\left(1-\sqrt{\frac{m-n}{m}}\right)}{\sqrt{\frac{m-n}{m}}\,(n-1)}\left(\tanh^{-1}\left(\frac{\sqrt{n}\,\cos(z)}{\sqrt{n-1}}\right)\cos(z)-\tanh^{-1}\left(\frac{\sqrt{n}}{\sqrt{n-1}}\right)\right)+\right.$$

$$\left.\frac{1}{2nm}\sum_{k=0}^{\infty}\frac{\left(\frac{3}{2}\right)_k m^{-k}}{k!}\left(F_{1\times1\times0}^{1\times2\times1}\left(\begin{matrix}k+2;\frac{1}{2},1;1;\\2;\frac{3}{2};;\end{matrix}\;1,\frac{n-1}{n}\right)-\cos(z)\,F_{1\times1\times0}^{1\times2\times1}\left(\begin{matrix}k+2;\frac{1}{2},1;1;\\2;\frac{3}{2};;\end{matrix}\;\cos^2(z),\frac{n-1}{n}\right)\right)\right)$$

08.06.06.0103.01

$$\Pi(n; z \mid m) \propto (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rfloor} \frac{\sqrt{-m \sin^2(z)}}{2\, m \sin(z)} \left(-\log\left(-4\, m \sin^2(z)\right) + \left(2 \log\left(\cos^2\left(\frac{z}{2}\right)\right) - \frac{2\sqrt{n}}{\sqrt{n-1}} \left(\tanh^{-1}\left(\frac{\cos(z)\sqrt{n}}{\sqrt{n-1}}\right) - \tanh^{-1}\left(\frac{\sqrt{n}}{\sqrt{n-1}}\right)\right)\right)\left(1 + O\left(\frac{1}{m}\right)\right) + 2\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rfloor \left(\frac{4\log(2)}{2\sqrt{-m}}\left(1 + O\left(\frac{1}{m}\right)\right) + \frac{\log(-m)}{2\sqrt{-m}}\left(1 + O\left(\frac{1}{m}\right)\right) + \frac{\sqrt{n}\, \sin^{-1}\left(\sqrt{n}\right)}{\sqrt{1-n}\, \sqrt{-m}}\left(1 + O\left(\frac{1}{m}\right)\right)\right)\right) /; (|m| \to \infty)$$

Other series representations

Expansions $\Pi\left(n; \sin^{-1}(z) \mid m\right)$ at $z == \infty$

08.06.06.0104.01

$$\Pi\left(n; \sin^{-1}(z) \mid m\right) \propto \frac{i\sqrt{-z^2}}{z}\left(\frac{i}{\sqrt{-m}}\left(K\left(\frac{1}{m}\right) + i\left(1 - \sqrt{\frac{m}{m-1}}\sqrt{\frac{m-1}{m}}\right)\left(K\left(1 - \frac{1}{m}\right) + \frac{n}{m-n}\Pi\left(\frac{m-1}{m-n} \,\middle|\, \frac{m-1}{m}\right)\right) - \Pi\left(\frac{1}{n} \,\middle|\, \frac{1}{m}\right)\right)\right) + \Pi(n \mid m) + \frac{\sqrt{-z^2}\,\sqrt{-m z^2}}{n\, m\, z^5} \sum_{k=0}^{\infty} \sum_{u=0}^{k} \sum_{i=0}^{u} \frac{m^{u-k}\, n^{i-u} \left(\frac{1}{2}\right)_i \left(\frac{1}{2}\right)_{k-u}}{i!\, (k-u)!\, (2k+3)} z^{-2k} /; (|z| \to \infty) \bigwedge \left(\left(0 < \arg(n) < \frac{\pi}{2} \bigwedge 0 < \arg(m) < \pi\right) \bigvee \left(\frac{\pi}{2} < \arg(n) < \pi \bigwedge 0 < \arg(m) < \frac{\pi}{2} \bigwedge (|m| < 1 \vee (|n| > 1 \wedge |n| > |m|))\right) \bigvee \left(\frac{\pi}{2} < \arg(n) < \pi \bigwedge \frac{\pi}{2} < \arg(m) < \pi \bigwedge |n| > |m|\right)\right)$$

Other expansions

08.06.06.0006.01

$$\Pi(n; z \mid m) == \sum_{k=0}^{\infty} n^k \left(\sqrt{1 + \frac{m}{n}} - \frac{\sqrt{\pi}}{2\,\Gamma\left(\frac{1}{2} - k\right)(k+1)!}\left(\frac{m}{n}\right)^{k+1} {}_2F_1\left(1, k + \frac{1}{2}; k+2; -\frac{m}{n}\right)\right) \left(\frac{\sqrt{\pi}\, \Gamma\left(k + \frac{1}{2}\right)}{2\, k!} - \cos(z)\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - k; \frac{3}{2}; \cos^2(z)\right)\right) /; 0 \le m < 1 \bigwedge 0 \le n < 1 \bigwedge -\frac{\pi}{2} \le z \le \frac{\pi}{2}$$

08.06.06.0007.01

$$\Pi(n; z \mid m) == \sqrt{\frac{n}{(n-1)(m-n)}} \left(F(z \mid m)\left(\frac{2\, i\, \pi}{K(m)} \sum_{k=1}^{\infty} \frac{q(m)^k}{1 - q(m)^{2k}} \sin\left(\frac{k\, \pi\, a}{K(m)}\right) + \sqrt{\frac{(n-1)(m-n)}{n}}\right) - 2\, i \sum_{k=1}^{\infty} \frac{q(m)^k}{k\left(1 - q(m)^{2k}\right)} \sin\left(\frac{k\, \pi\, a}{K(m)}\right) \sin\left(\frac{k\, \pi\, F(z \mid m)}{K(m)}\right)\right) /; a == \operatorname{sn}^{-1}\left(\sqrt{\frac{n}{m}} \,\middle|\, m\right) \bigwedge -1 \le n \le 1 \bigwedge -1 \le m \le 1 \bigwedge -\frac{\pi}{2} \le z \le \frac{\pi}{2}$$

08.06.06.0105.01

$$\Pi(n; z \mid m) == (-1)^{\left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{n^k \left(k+\frac{1}{2}\right)_{i+j} \left(\frac{1}{2}\right)_i \left(\frac{1}{2}\right)_j m^j \sin^{2i+2j+2k+1}(z)}{(2k+1)\left(k+\frac{3}{2}\right)_{i+j} i!\, j!} + 2 \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \Pi(n \mid m)$$

08.06.06.0106.01

$$\Pi(n; z \mid m) = (-1)^{\left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil} \sin(z) \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{i+j+k} (1)_k \left(\frac{1}{2}\right)_i \left(\frac{1}{2}\right)_j}{\left(\frac{3}{2}\right)_{i+j+k} k!\, i!\, j!} n^k \sin^{2(i+j+k)}(z)\, m^j + 2 \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \Pi(n \mid m)$$

08.06.06.0107.01

$$\Pi(n; z \mid m) == (-1)^{\left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil} \sum_{k=0}^{\infty} \frac{n^k \sin^{2k+1}(z)}{2k+1} F_1\left(k+\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; k+\frac{3}{2}; \sin^2(z), m \sin^2(z)\right) + 2 \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \Pi(n \mid m)$$

08.06.06.0108.01

$$\Pi(n; z \mid m) = 2 \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \Pi(n \mid m) + (-1)^{\left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil} \sin(z) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \sin^{2k}(z)}{k!\,(2k+1)} F_1\left(k+\frac{1}{2}; 1, \frac{1}{2}; k+\frac{3}{2}; n \sin^2(z), m \sin^2(z)\right)$$

08.06.06.0109.01

$$\Pi(n; z \mid m) = 2 \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \Pi(n \mid m) + (-1)^{\left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil} \sin(z) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{n^k \left(\frac{1}{2}\right)_j \sin^{2j+2k+1}(z)}{(2j+2k+1)\, j!} {}_2F_1\left(\frac{1}{2}, j+k+\frac{1}{2}; j+k+\frac{3}{2}; m \sin^2(z)\right)$$

08.06.06.0110.01

$$\Pi(n; z \mid m) = 2 \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \Pi(n \mid m) + (-1)^{\left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil} \sin(z) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{m^j \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_j \sin^{2j+2k+1}(z)}{(2j+2k+1)\, k!\, j!} {}_2F_1\left(1, \frac{1}{2}+k+j; \frac{3}{2}+k+j; n \sin^2(z)\right)$$

08.06.06.0111.01

$$\Pi(n; z \mid m) == \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k m^k}{k!} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{j+k} n^j}{(j+k)!} \left(z - \frac{1}{2}\sin(2z) \sum_{i=0}^{j+k-1} \frac{i!\, \sin^{2i}(z)}{\left(\frac{3}{2}\right)_i}\right)$$

Integral representations

On the real axis

Of the direct function

08.06.07.0001.01

$$\Pi(n; z \mid m) == \int_0^z \frac{1}{\left(1 - n \sin^2(t)\right) \sqrt{1 - m \sin^2(t)}} \, dt \; /; \; -\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$$

08.06.07.0004.01

$$\Pi(n; z \mid m) = (-1)^{\left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil} \int_0^z \frac{\cos(t)}{\left(1 - n \sin^2(t)\right) \sqrt{1 - m \sin^2(t)} \sqrt{\cos^2(t)}} \, dt + 2 \left\lfloor \frac{\text{Re}(z)}{\pi} \right\rceil \Pi(n \mid m)$$

08.06.07.0002.01

$$\Pi(n; z \mid m) = \int_0^{\sin(z)} \frac{1}{(1 - n t^2)\sqrt{1-t^2}\,\sqrt{1-m t^2}}\,dt \;/;\; -\frac{\pi}{2} \le z \le \frac{\pi}{2}$$

08.06.07.0005.01

$$\Pi(n; z \mid m) = (-1)^{\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil} \int_0^{\sin(z)} \frac{1}{(1 - n t^2)\sqrt{1-t^2}\,\sqrt{1-m t^2}}\,dt + 2\left\lfloor \frac{\mathrm{Re}(z)}{\pi} \right\rceil \Pi(n \mid m)$$

08.06.07.0003.01

$$\Pi(n; z \mid m) = \int_0^{F(z|m)} \frac{1}{1 - n\,\mathrm{sn}(t \mid m)^2}\,dt$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

With respect to *n*

08.06.13.0002.01

$$2(n-1)(m-n)\,n\,\frac{\partial^3 w(n)}{\partial n^3} + \left(-13 n^2 + 8 m n + 8 n - 3 m\right)\frac{\partial^2 w(n)}{\partial n^2} + 4(m - 4n + 1)\frac{\partial w(n)}{\partial n} - 2 w(n) = \frac{\sqrt{1 - m \sin^2(z)}\,\sin(2z)}{\left(n \sin^2(z) - 1\right)^3} \;/;$$

$$w(n) = \Pi(n; z \mid m)$$

With respect to *m*

08.06.13.0001.01

$$8(m-1)\,m\,(m-n)\,\frac{\partial^3 w(m)}{\partial m^3} + 4\left(11 m^2 - 6 n m - 7 m + 2 n\right)\frac{\partial^2 w(m)}{\partial m^2} + 6(7m - n - 2)\frac{\partial w(m)}{\partial m} + 3 w(m) = \frac{3 \sin(2z)}{2\sqrt{\left(1 - m \sin^2(z)\right)^5}} \;/;$$

$$w(m) = \Pi(n; z \mid m)$$

Ordinary nonlinear differential equations

08.06.13.0003.01

$$\begin{aligned}&\left(w'(z)^2 - 1\right)\left(4 w'(z)^2 (m-n)^3 + 27 m^2 n\right)^2 \left((m-1)(n-1)^2 w'(z)^2 + 1\right) w'(z)^6 + \Big(8(m-1)(m-n)^5 (n-1)\, n\, w'(z)^6 - \\ &\quad 2(m-n)^2 \left(4(n-1)(3n-2) m^4 + n(3n(5 - 11n) + 8) m^3 - 3(n-9) n^2 (2n-1) m^2 + 2(n-10) n^3 m + 4 n^4\right) w'(z)^4 - \\ &\quad 9 m^2 n \left((6n(3n-5) + 16) m^3 + 3n(n(3n-16) + 4) m^2 - 3(n-12) n^2 m - 10 n^3\right) w'(z)^2 - 243 m^4 n^2\Big) w''(z)^2 w'(z)^4 + \\ &\quad n\left(-27 n m^4 - (m-n)^2 (m(9n-8) - 10n) w'(z)^2 m^2 + (m-1)(m-n)^4 n\, w'(z)^4\right) w''(z)^4 w'(z)^2 - \\ &\quad m^4 n^2 w''(z)^6 = 0 \;/;\; w(z) = \Pi(n; z \mid m)\end{aligned}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

08.06.16.0001.01

$$\Pi(n; -z \mid m) = -\Pi(n; z \mid m)$$

08.06.16.0002.01

$$\Pi(n; z + \pi k \mid m) = 2 k \Pi(n \mid m) + \Pi(n; z \mid m) \;/;\; k \in \mathbb{Z} \wedge -1 \leq n \leq 1$$

Products, sums, and powers of the direct function

Sums of the direct function

08.06.16.0003.01

$$\Pi(n; z_1 \mid m) + \Pi(n; z_2 \mid m) = \Pi(n; z \mid m) - \sqrt{\frac{n}{(1-n)(n-m)}} \tan^{-1}\left(\frac{\sqrt{(1-n) n (n-m)} \sin(z) \sin(z_1) \sin(z_2)}{-n \sin^2(z) + n \cos(z) \sqrt{1 - m \sin^2(z)} \sin(z_1) \sin(z_2) + 1}\right) /;$$
$$z = \cos^{-1}\left(\frac{\cos(z_1) \cos(z_2) - \sin(z_1) \sin(z_2) \sqrt{\left(1 - m \sin^2(z_1)\right)\left(1 - m \sin^2(z_2)\right)}}{1 - m \sin^2(z_1) \sin^2(z_2)}\right) \bigwedge 0 < m < n < 1 \bigwedge 0 < z_1 < 1 \bigwedge 0 < z_2 < 1$$

Identities

Functional identities

08.06.17.0001.01

$$\Pi(n; z \mid m) = \frac{1}{\sqrt{m}} \Pi\left(\frac{n}{m}; \sin^{-1}\left(\sqrt{m} \sin(z)\right) \,\middle|\, \frac{1}{m}\right) + 2 \left\lfloor \frac{\operatorname{Re}(z)}{\pi} \right\rceil \Pi(n \mid m)$$

08.06.17.0004.01

$$\Pi(n; z \mid m) = F(z \mid m) - \Pi\left(\frac{m}{n}; z \mid m\right) + \frac{1}{2 \sqrt{\frac{(n-m)(n-1)}{n}}} \log\left(\frac{\sqrt{\frac{(n-m)(n-1)}{n}} \tan(z) + \sqrt{1 - m \sin^2(z)}}{\sqrt{1 - m \sin^2(z)} - \sqrt{\frac{(n-m)(n-1)}{n}} \tan(z)}\right)$$

08.06.17.0003.01

$$\Pi\left(n; i \sinh^{-1}(\tan(z)) \,\middle|\, 1 - m\right) = \frac{i}{1-n} (F(z \mid m) - n \Pi(1-n; z \mid m))$$

08.06.17.0002.01

$$E(\phi \mid m) F(\theta \mid m) + \cot(\theta) \left(\Pi\left(m \sin^2(\theta); \phi \mid m\right) - F(\phi \mid m)\right) \sqrt{1 - m \sin^2(\theta)} =$$
$$\cot(\phi) \sqrt{1 - m \sin^2(\phi)} \left(\Pi\left(m \sin^2(\phi); \theta \mid m\right) - F(\theta \mid m)\right) + E(\theta \mid m) F(\phi \mid m)$$

Differentiation

Low-order differentiation

With respect to *n*

08.06.20.0001.01

$$\frac{\partial \Pi(n; z \mid m)}{\partial n} = \frac{1}{2(m-n)(n-1)}\left(E(z \mid m) + \frac{m-n}{n} F(z \mid m) + \frac{n^2-m}{n} \Pi(n; z \mid m) - \frac{n\sqrt{1-m\sin^2(z)}\sin(2z)}{2\left(1-n\sin^2(z)\right)}\right)$$

08.06.20.0002.01

$$\frac{\partial^2 \Pi(n; z \mid m)}{\partial n^2} = \frac{\sin(2z)\sqrt{1-m\sin^2(z)}}{16(m-n)^2(n-1)^2\left(n\sin^2(z)-1\right)^2}\left(\left(5n-8\right)n^2 + n\left(m(2n+1)+n(2-5n)\right)\cos(2z) - m\left(2n^2+n-6\right)\right) + \frac{m(1-4n)+n(5n-2)}{4(m-n)(n-1)^2n^2} F(z \mid m) + \frac{3n^4+2m(2-5n)n+m^2(4n-1)}{4(m-n)^2(n-1)^2n^2} \Pi(n; z \mid m) - \frac{(2n+1)m+n(2-5n)}{4(m-n)^2(n-1)^2 n} E(z \mid m)$$

With respect to z

08.06.20.0003.01

$$\frac{\partial \Pi(n; z \mid m)}{\partial z} = \frac{1}{\sqrt{1-m\sin^2(z)}\left(1-n\sin^2(z)\right)}$$

08.06.20.0004.01

$$\frac{\partial^2 \Pi(n; z \mid m)}{\partial z^2} = \frac{\cos(z)\sin(z)\left(-3mn\sin^2(z)+m+2n\right)}{\left(1-m\sin^2(z)\right)^{3/2}\left(1-n\sin^2(z)\right)^2}$$

With respect to m

08.06.20.0005.01

$$\frac{\partial \Pi(n; z \mid m)}{\partial m} = \frac{1}{2(n-m)}\left(\frac{1}{m-1} E(z \mid m) + \Pi(n; z \mid m) - \frac{m\sin(2z)}{2(m-1)\sqrt{1-m\sin^2(z)}}\right)$$

08.06.20.0006.01

$$\frac{\partial^2 \Pi(n; z \mid m)}{\partial m^2} = \frac{4m^2-(n+2)m-n}{4(m-1)^2 m(m-n)^2} E(z \mid m) + \frac{1}{4(m-1)m(m-n)} F(z \mid m) + \frac{3}{4(m-n)^2} \Pi(n; z \mid m) + \frac{\left(m\left(4m^2-(n+2)m-n\right)\sin^2(z)+m+2n-3m^2\right)\sin(2z)}{8(m-1)^2(m-n)^2\left(1-m\sin^2(z)\right)^{3/2}}$$

Symbolic differentiation

With respect to n

08.06.20.0007.02

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial n^p} = \sin^{2p+1}(z)\sum_{k=0}^{\infty}\frac{(k+p)!}{(2k+2p+1)\,k!} F_1\left(k+p+\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; k+p+\frac{3}{2}; \sin^2(z), m\sin^2(z)\right) n^k \sin^{2k}(z) \;/;\; p \in \mathbb{N}$$

08.06.20.0013.01

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial n^p} = \frac{\sin^{2p+1}(z)\,p!}{2p+1}\sum_{k_1=p}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{\left(p+\frac{1}{2}\right)_{k_1+k_2+k_3}(p+1)_{k_1}\left(\frac{1}{2}\right)_{k_2}\left(\frac{1}{2}\right)_{k_3} n^{k_1}\sin^{2(k_1+k_2+k_3)}(z)\,m^{k_3}}{\left(p+\frac{3}{2}\right)_{k_1+k_2+k_3} k_1!\,k_2!\,k_3!} \;/;\; p \in \mathbb{N}$$

08.06.20.0014.01

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial n^p} = \frac{\sin^{2p+1}(z)\, p!}{2p+1} F_D^{(3)}\left(p+\frac{1}{2}, p+1, \frac{1}{2}, \frac{1}{2}; p+\frac{3}{2}; n\sin^2(z), \sin^2(z), m\sin^2(z)\right) /; p \in \mathbb{N}$$

With respect to z

08.06.20.0015.01

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial z^p} = \Pi(n; z \mid m)\,\delta_p + \sum_{j=1}^{p} \frac{1}{j!} \sum_{k_1=0}^{j-1} \binom{j}{k_1} \sum_{k_2=0}^{j-k_1} (-1)^{k_1} 2^{k_1-j} \sin^{k_1}(z)\,(k_1+2k_2-j)^p\, e^{-\frac{1}{2} i (\pi (j+p-k_1-2k_2)+2(-j+k_1+2k_2) z)} \binom{j-k_1}{k_2} \sum_{i=0}^{j-1} \frac{(1-j)_{2(j-i)-2}}{(j-i-1)!\,(2\sin(z))^{j-2i-1}} \sum_{i_1=0}^{i}\sum_{i_2=0}^{i}\sum_{i_3=0}^{i} (-1)^{i_1}\, \delta_{-i+i_1+i_2+i_3}\,(i_1+i_2+i_3; i_1, i_2, i_3)\, n^{i_1} m^{i_3} (-i_1)_{i_1} \left(\frac{1}{2}\right)_{i_2} \left(\frac{1}{2}\right)_{i_3} \left(1-n\sin^2(z)\right)^{-i_1-1} \cos^{-2i_2-1}(z) \left(1-m\sin^2(z)\right)^{-i_3-\frac{1}{2}} /; p \in \mathbb{N}$$

08.06.20.0008.02

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial z^p} = \Pi(n; z \mid m)\,\delta_p + \frac{i^{p+1}\, 2^{p-2}\, e^{2iz} \sqrt{\pi}}{\sqrt{1-n}\,\sqrt{1-m\sin^2(z)}} \sqrt{\frac{2\left(\sqrt{1-m}+1\right)+\left(e^{2iz}-1\right)m}{\sqrt{1-m}}} \left(\frac{e^{2iz}\left(m-2\left(\sqrt{1-m}+1\right)\right)}{m}\right)^{-3/2} \sum_{k=0}^{p-1} e^{2ikz}\, S_{p-1}^{(k)} \sum_{j=0}^{k} \binom{k}{j} \frac{(j-k)_{k-j}}{\Gamma\left(\frac{1}{2}-j\right)} \left(\frac{m}{\left(e^{2iz}-1\right)m-2\sqrt{1-m}+2}\right)^{j} \left(\left(\frac{2\left(1+\sqrt{1-n}\right)}{n}+e^{2iz}-1\right)^{j-k-1} - \left(\frac{2\left(1-\sqrt{1-n}\right)}{n}+e^{2iz}-1\right)^{j-k-1}\right) F_1\left(\frac{1}{2}; \frac{1}{2}, -\frac{3}{2}; \frac{1}{2}-j; \frac{m\left(1-e^{2iz}\right)+2\sqrt{1-m}-2}{4\sqrt{1-m}}, \frac{m\left(1-e^{2iz}\right)+2\sqrt{1-m}-2}{m+2\sqrt{1-m}-2}\right) /; p \in \mathbb{N}$$

With respect to m

08.06.20.0009.02

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial m^p} = \sin^{2p+1}(z) \left(\frac{1}{2}\right)_p \sum_{k=0}^{\infty} \frac{\left(p+\frac{1}{2}\right)_k}{(2k+2p+1)\,k!} F_1\left(k+p+\frac{1}{2}; \frac{1}{2}, 1; k+p+\frac{3}{2}; \sin^2(z), n\sin^2(z)\right) m^k \sin^{2k}(z) /; p \in \mathbb{N}$$

08.06.20.0016.01

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial m^p} = \frac{\sin^{2p+1}(z) \left(\frac{1}{2}\right)_p}{2p+1} \sum_{k_1=p}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty} \frac{\left(p+\frac{1}{2}\right)_{k_1+k_2+k_3} (1)_{k_1} \left(\frac{1}{2}\right)_{k_2} \left(p+\frac{1}{2}\right)_{k_3} n^{k_1} \sin^{2(k_1+k_2+k_3)}(z)\, m^{k_3}}{\left(p+\frac{3}{2}\right)_{k_1+k_2+k_3} k_1!\, k_2!\, k_3!} /; p \in \mathbb{N}$$

08.06.20.0017.01

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial m^p} = \frac{\sin^{2p+1}(z) \left(\frac{1}{2}\right)_p}{2p+1} F_D^{(3)}\left(p+\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}+p; p+\frac{3}{2}; n\sin^2(z), \sin^2(z), m\sin^2(z)\right) /; p \in \mathbb{N}$$

08.06.20.0018.01

$$\frac{\partial^p \Pi(n; z \mid m)}{\partial m^p} = m^{-p}\left(\frac{\tanh^{-1}\left(\sqrt{n-1}\,\tan(z)\right)\sqrt{\cos^2(z)}\,\sec(z)}{\sqrt{n-1}}\,{}_2\tilde{F}_1\left(\frac{1}{2}, 1; 1-p; \frac{m}{n}\right) - \frac{m\sin(z)}{4n}\sum_{k=0}^{\infty}\frac{(k+1)!\left(\frac{3}{2}\right)_k 2^{-k} m^k}{(k-p+1)!}\sum_{j=0}^{k}\frac{\left(\frac{n-2}{n}\right)^j}{(k-j)!}\right.$$

$$\sum_{i=0}^{j+1}\frac{(-1)^{i+j-1}}{i!\,(j-i+1)!}\left(\sum_{q=0}^{\left\lfloor\frac{i-1}{2}\right\rfloor}\frac{\binom{i-1}{2q}\left(\frac{1}{2}\right)_q}{q!}\left(\frac{n}{2-n}\right)^{2q}\left(\frac{2\sin^{-1}(\sin(z))}{\sin(z)} + \frac{\sqrt{\cos^2(z)}}{\cos(2z)}\sum_{p=1}^{q}\frac{(p-1)!\cos^{2p}(2z)}{\left(\frac{1}{2}\right)_p}\right) + \right.$$

$$\left.\left.\frac{2n\sqrt{\cos^2(z)}}{2-n}\sum_{q=0}^{\left\lfloor\frac{i}{2}\right\rfloor-1} q!\binom{i-1}{2q+1}\cos^{2q}(2z)\left(\frac{n}{2-n}\right)^{2q}\sum_{p=0}^{q}\frac{\tan^{2p}(2z)}{(q-p)!\left(\frac{3}{2}\right)_p}\right)\right) /; p \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *n*

08.06.20.0010.01

$$\frac{\partial^\alpha \Pi(n; z \mid m)}{\partial n^\alpha} = n^{-\alpha}\sin(z)\sum_{k=0}^{\infty}\frac{k!\left(n\sin^2(z)\right)^k}{(2k+1)\,\Gamma(k-\alpha+1)}\,F_1\left(k+\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; k+\frac{3}{2}; \sin^2(z), m\sin^2(z)\right)$$

With respect to *z*

08.06.20.0011.01

$$\frac{\partial^\alpha \Pi(n; z \mid m)}{\partial z^\alpha} = \sqrt{\pi}\, z^{1-\alpha}\, 2^{\alpha-1}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k+l}\, 2^{-2j-2k-2l}\, m^l\, n^k}{(2j+2k+2l+1)\, j!\, l!}\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_l$$

$$\sum_{p=0}^{j+k+l}(-1)^p\binom{2j+2k+2l+1}{p}(2j+2k+2l-2p+1)\,{}_1\tilde{F}_2\left(1; 1-\frac{\alpha}{2}, \frac{3-\alpha}{2}; -\frac{1}{4}\left((2j+2k+2l-2p+1)^2 z^2\right)\right)$$

With respect to *m*

08.06.20.0012.01

$$\frac{\partial^\alpha \Pi(n; z \mid m)}{\partial m^\alpha} = m^{-\alpha}\sin(z)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\left(m\sin^2(z)\right)^k}{(2k+1)\,\Gamma(k-\alpha+1)}\,F_1\left(k+\frac{1}{2}; \frac{1}{2}, 1; k+\frac{3}{2}; \sin^2(z), n\sin^2(z)\right)$$

Integration

Indefinite integration

Involving only one direct function

08.06.21.0001.01

$$\int \Pi(n; z \mid m)\, dz = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{j+k+l-1} m^l n^k 2^{-2j-2k-2l}}{(2j+2k+2l+1)\, j!\, l!} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_l \sum_{p=0}^{j+k+l} \frac{(-1)^p \cos((2j+2k+2l-2p+1)z)}{2j+2k+2l-2p+1} \binom{2j+2k+2l+1}{p}$$

Involving one direct function and elementary functions

Involving trigonometric functions

Involving sin

08.06.21.0002.01

$$\int \sin(z)\, \Pi(n; z \mid m)\, dz = \frac{\sqrt{2}\, \tan^{-1}\left(\frac{\sqrt{m(2-n)-(2-m)n}\, \sin(z)}{\sqrt{\cos(2z)\, m-m+2}}\right)}{\sqrt{m(2-n)-(2-m)n}} - \cos(z)\, \Pi(n; z \mid m)$$

Involving cos

08.06.21.0003.01

$$\int \cos(z)\, \Pi(n; z \mid m)\, dz = \frac{\tan^{-1}\left(\frac{2\sqrt{n-m}\, \cos(z)}{\sqrt{2-2n}\, \sqrt{\cos(2z)\, m-m+2}}\right)}{\sqrt{1-n}\, \sqrt{n-m}} + \Pi(n; z \mid m) \sin(z)$$

Involving only one direct function with respect to *n*

08.06.21.0004.01

$$\int \Pi(n; z \mid m)\, dn = \sin(z)\, n \sum_{k=0}^{\infty} \frac{\left(n \sin^2(z)\right)^k}{(2k+1)(k+1)} F_1\left(k+\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; k+\frac{3}{2}; \sin^2(z), m \sin^2(z)\right)$$

Involving only one direct function with respect to *m*

08.06.21.0005.01

$$\int \Pi(n; z \mid m)\, dm = \sqrt{2\cos(2z)\, m - 2m + 4}\, \cot(z) + 2E(z \mid m) - 2F(z \mid m) + 2(m-n)\, \Pi(n; z \mid m)$$

Involving one direct function and elementary functions with respect to *m*

Involving power function

08.06.21.0006.01

$$\int m\, \Pi\left(n; z \mid m^2\right) dm = \Pi\left(n; z \mid m^2\right) m^2 + E\left(z \mid m^2\right) - F\left(z \mid m^2\right) - n\, \Pi\left(n; z \mid m^2\right) + \cot(z) \sqrt{1 - m^2 \sin^2(z)}$$

Representations through more general functions

Through hypergeometric functions of several variables

08.06.26.0003.01

$$\Pi(n;z\mid m)=(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\sin(z)\,F_D^{(3)}\left(\frac{1}{2},1,\frac{1}{2},\frac{1}{2};\frac{3}{2};n\sin^2(z),\sin^2(z),m\sin^2(z)\right)+2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil\Pi(n\mid m)$$

Through hypergeometric functions of two variables

08.06.26.0001.01

$$\Pi(n;z\mid m)=(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\sum_{k=0}^{\infty}\frac{\sin^{2k+1}(z)}{2k+1}F_1\left(k+\frac{1}{2};\frac{1}{2},\frac{1}{2};k+\frac{3}{2};\sin^2(z),m\sin^2(z)\right)n^k+2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil\Pi(n\mid m)$$

08.06.26.0004.01

$$\Pi(n;z\mid m)=-\frac{2^{-\frac{5}{2}}\sqrt{\pi}\,n}{\left(\frac{m-2\left(\sqrt{1-m}+1\right)}{m}\right)^{3/2}\sqrt{1-n}}\sqrt{1+\frac{1}{\sqrt{1-m}}}$$

$$\sum_{q=0}^{\infty}\frac{(-1)^q\,2^{2q+1}}{(2q+1)!}\sum_{k=0}^{2q}\mathcal{S}_{2q}^{(k)}\sum_{j=0}^{k}\frac{2^{-k}(-1)^j\,j!\,n^j}{\Gamma\left(j-k+\frac{1}{2}\right)}\binom{k}{k-j}\left(\frac{m}{1-\sqrt{1-m}}\right)^{k-j}\left(\left(\sqrt{1-n}+1\right)^{-j-1}-\left(1-\sqrt{1-n}\right)^{-j-1}\right)$$

$$F_1\left(\frac{1}{2};\frac{1}{2},-\frac{3}{2};j-k+\frac{1}{2};\frac{1}{2}-\frac{1}{2\sqrt{1-m}},\frac{2\left(\sqrt{1-m}+1\right)}{m}\right)z^{2q+1}\;/;\;|z|<1$$

08.06.26.0002.01

$$\Pi(n;z\mid m)=(-1)^{\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil}\sin(z)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k\sin^{2k}(z)}{(2k+1)\,k!}F_1\left(k+\frac{1}{2};\frac{1}{2},1;k+\frac{3}{2};\sin^2(z),n\sin^2(z)\right)m^k++2\left\lfloor\frac{\operatorname{Re}(z)}{\pi}\right\rceil\Pi(n\mid m)$$

Theorems

The length of the geodesics of an ellipsoid

The length's of the geodesics of an ellipsoid can be expressed in terms of $\Pi(n;z\mid m)$.

History

– A. M. Legendre (1811)

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