EllipticTheta3

Notations

Traditional name

Jacobi theta function ϑ_3

Traditional notation

$\vartheta_3(z, q)$

Mathematica StandardForm notation

EllipticTheta[3, z, q]

Primary definition

09.03.02.0001.01

$$\vartheta_3(z, q) = 2\sum_{k=1}^{\infty} q^{k^2} \cos(2\,k\,z) + 1 \;/;\; |q| < 1$$

Specific values

Specialized values

For fixed *z*

09.03.03.0001.01

$$\vartheta_3(z, 0) = 1$$

For fixed *q*

09.03.03.0008.01

$$\vartheta_3(0, q) = \frac{\eta\left(-\frac{i\log(q)}{\pi}\right)^5}{\eta\left(-\frac{2\,i\log(q)}{\pi}\right)^2\,\eta\left(-\frac{i\log(q)}{2\,\pi}\right)^2}$$

09.03.03.0004.01

$$\vartheta_3\left(0, e^{\pi\,i\,\tau}\right) = \frac{\eta(\tau)^5}{\eta(2\,\tau)^2\,\eta\left(\frac{\tau}{2}\right)^2} \;/;\; \operatorname{Im}(\tau) > 0$$

09.03.03.0007.02

$$\vartheta_3(0, q) = \sqrt{\frac{2}{\pi}}\,\sqrt{K\left(q^{-1}(q)\right)}$$

09.03.03.0005.02

$$\vartheta_3(0, q) = \vartheta_4(0, -q)$$

09.03.03.0006.01

$$\vartheta_3\left(0, e^{-\frac{i\pi}{\tau}}\right) = \frac{\sqrt{\tau}}{\sqrt{i}}\, \vartheta_3\left(0, e^{i\pi\tau}\right)$$

09.03.03.0009.01

$$\vartheta_3\left(\frac{\pi}{2}, q\right) = \sqrt{\frac{2}{\pi}}\, \sqrt[4]{1 - q^{-1}(q)}\, \sqrt{K\left(q^{-1}(q)\right)}$$

09.03.03.0010.01

$$\vartheta_3(m\pi, q) = \frac{1}{\eta\left(-\frac{2 i \log(q)}{\pi}\right)^2 \eta\left(-\frac{i \log(q)}{2\pi}\right)^2}\, \eta\left(-\frac{i \log(q)}{\pi}\right)^5 \;/;\; m \in \mathbb{Z}$$

09.03.03.0011.01

$$\vartheta_3\left(\pi\left(\frac{1}{2} + m\right), q\right) = \sqrt{\frac{2}{\pi}}\, \sqrt[4]{1 - q^{-1}(q)}\, \sqrt{K\left(q^{-1}(q)\right)} \;/;\; m \in \mathbb{Z}$$

09.03.03.0002.01

$$\vartheta_3\left(\frac{\pi}{2}(\tau + 1), e^{3\pi i \tau}\right) = e^{-\frac{\pi i \tau}{12}}\, \eta(\tau) \;/;\; \operatorname{Im}(\tau) > 0$$

09.03.03.0003.01

$$\vartheta_3\left((2m + 1)\frac{\pi}{2} + (2n + 1)\frac{\pi\tau}{2}, q\right) = 0 \;/;\; \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$$

General characteristics

Domain and analyticity

$\vartheta_3(z, q)$ is an analytic function of z and q for $z, q \in \mathbb{C}$ and $|q| < 1$.

09.03.04.0001.01

$$(3 * z * q) \longrightarrow \vartheta_3(z, q) :: (\{3\} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\vartheta_3(z, q)$ is an even function with respect to z.

09.03.04.0002.01

$$\vartheta_3(-z, q) = \vartheta_3(z, q)$$

09.03.04.0003.01

$$\vartheta_3(z, -q) = \vartheta_4(z, q)$$

Mirror symmetry

09.03.04.0004.01

$$\vartheta_3(\bar{z}, \bar{q}) = \overline{\vartheta_3(z, q)}$$

Periodicity

The function $\vartheta_3(z, q)$ is a periodic function with respect to z with period π and a quasi-period $i \log(q)$.

09.03.04.0005.01

$$\vartheta_3(z + \pi, q) == \vartheta_3(z, q)$$

09.03.04.0007.01

$$\vartheta_3(z + m\pi, q) == \vartheta_3(z, q) \;/;\; m \in \mathbb{Z}$$

09.03.04.0006.01

$$\vartheta_3(z + \pi\tau, q) == \frac{e^{-2iz}}{q}\vartheta_3(z, q) \;/;\; q == e^{i\pi\tau} \bigwedge \operatorname{Im}(\tau) > 0$$

09.03.04.0009.01

$$\vartheta_3(z + i\log(q), q) == \frac{e^{2iz}}{q}\vartheta_3(z, q)$$

09.03.04.0008.01

$$\vartheta_3(z + m\pi\tau, q) == q^{-m}\, e^{-i(2mz+(m-1)m\pi\tau)}\, \vartheta_3(z, q) \;/;\; m \in \mathbb{Z} \bigwedge q == e^{i\pi\tau}$$

09.03.04.0010.01

$$\vartheta_3(z + i\, m\log(q), q) == q^{-m^2}\, e^{2miz}\, \vartheta_3(z, q) \;/;\; m \in \mathbb{Z}$$

09.03.04.0011.01

$$\vartheta_3(\nu + m\pi + n\pi\tau, q) == q^{-n^2}\, e^{-2n\nu i}\, \vartheta_3(\nu, q) \;/;\; \{m, n\} \in \mathbb{Z} \bigwedge q == e^{i\pi\tau}$$

Poles and essential singularities

With respect to *q*

The function $\vartheta_3(z, q)$ does not have poles and essential singularities inside of the unit circle $|q| < 1$

09.03.04.0012.01

$$\mathcal{S}ing_q(\vartheta_3(z, q)) == \{\}$$

With respect to *z*

09.03.04.0013.01

$$\mathcal{S}ing_z(\vartheta_3(z, q)) == \{\}$$

Branch points

With respect to *q*

For fixed z, the function $\vartheta_3(z, q)$ does not have branch points.

09.03.04.0014.01

$$\mathcal{BP}_q(\vartheta_3(z, q)) == \{\}$$

With respect to *z*

For fixed q, the function $\vartheta_3(z, q)$ does not have branch points.

09.03.04.0015.01

$\mathcal{BP}_z(\vartheta_3(z, q)) == \{\}$

Branch cuts

With respect to *q*

For fixed z, the function $\vartheta_3(z, q)$ does not have branch cuts.

09.03.04.0016.01

$\mathcal{BC}_q(\vartheta_3(z, q)) == \{\}$

With respect to *z*

For fixed q, the function $\vartheta_3(z, q)$ does not have branch cuts.

09.03.04.0017.01

$\mathcal{BC}_z(\vartheta_3(z, q)) == \{\}$

Natural boundary of analyticity

The unit circle $|q| == 1$ is the natural boundary of the region of analyticity.

09.03.04.0018.01

$\mathcal{AB}_z(\vartheta_3(q, z)) == \left\{e^{i\,(-\pi,\pi)}\right\}$

Series representations

q-series

Expansions at generic point $z == z_0$

09.03.06.0020.01

$$\vartheta_3(z, q) \propto \vartheta_3(z_0, q) + \vartheta_3^{(1,0)}(z_0, q)\,(z - z_0) + \frac{\vartheta_3^{(2,0)}(z_0, q)}{2}\,(z - z_0)^2 + \frac{\vartheta_3^{(3,0)}(z_0, q)}{6}\,(z - z_0)^3 + O\big((z - z_0)^4\big)$$

09.03.06.0021.01

$$\vartheta_3(z, q) \propto \vartheta_3(z_0, q) + \vartheta_3'(z_0, q)\,(z - z_0) + \frac{\vartheta_3^{(2,0)}(z_0, q)}{2}\,(z - z_0)^2 + \frac{\vartheta_3^{(3,0)}(z_0, q)}{6}\,(z - z_0)^3 + O\big((z - z_0)^4\big)$$

09.03.06.0022.01

$$\vartheta_3(z, q) = \sum_{k=0}^{\infty} \frac{\vartheta_3^{(k,0)}(z_0, q)}{k!}\,(z - z_0)^k$$

09.03.06.0023.01

$$\vartheta_3(z, q) \propto \vartheta_3(z_0, q)\,(1 + O(z - z_0))$$

Expansions at generic point $q == q_0$

09.03.06.0024.01

$$\vartheta_3(z, q) \propto \vartheta_3(z, q_0) + \vartheta_3^{(0,1)}(z, q_0)\,(q - q_0) + \frac{\vartheta_3^{(0,2)}(z, q_0)}{2}\,(q - q_0)^2 + \frac{\vartheta_3^{(0,3)}(z, q_0)}{6}\,(q - q_0)^3 + O\big((q - q_0)^4\big)$$

09.03.06.0025.01

$$\vartheta_3(z, q) = \sum_{k=0}^{\infty} \frac{\vartheta_3^{(0,k)}(z, q_0)}{k!} (q - q_0)^k$$

09.03.06.0026.01

$$\vartheta_3(z, q) \propto \vartheta_3(z, q_0)\,(1 + O(q - q_0))$$

Expansions at $q == 0$

09.03.06.0027.01

$$\vartheta_3(z, q) \propto 1 + 2\left(\cos(2z)\, q + \cos(4z)\, q^4 + \cos(6z)\, q^9 + \ldots\right) /; (q \to 0)$$

09.03.06.0001.01

$$\vartheta_3(z, q) == 2 \sum_{k=1}^{\infty} q^{k^2} \cos(2kz) + 1 /; |q| < 1$$

09.03.06.0002.01

$$\vartheta_3(z, q) == \sum_{k=-\infty}^{\infty} q^{k^2} e^{2kiz}$$

09.03.06.0003.01

$$\vartheta_3(0, q) == 1 + 2 \sum_{k=1}^{\infty} q^{k^2}$$

09.03.06.0028.01

$$\vartheta_3(z, q) \propto 1 + O(q) /; (q \to 0)$$

Expansions at $q == 1$

09.03.06.0029.01

$$\vartheta_3(z, q) \propto \frac{i\sqrt{\pi}}{\sqrt{q-1}}\, e^{-i\pi\left\lfloor \frac{3}{4} - \frac{\arg(q-1)}{2\pi} \right\rfloor} \left(1 + \frac{q-1}{4} - \frac{1}{96}\, 7\,(q-1)^2 + \ldots\right)$$
$$e^{\frac{z^2}{\log(q)}} \left(1 + 2\, e^{\frac{\pi^2}{\log(q)}} \cosh\!\left(\frac{2\pi z}{\log(q)}\right) + 2\, e^{\frac{4\pi^2}{\log(q)}} \cosh\!\left(\frac{4\pi z}{\log(q)}\right) + \ldots\right) /; (q \to 1) \land |q| < 1$$

09.03.06.0030.01

$$\vartheta_3(z, q) = \frac{i\sqrt{\pi}}{\sqrt{q-1}}\, e^{-i\pi\left\lfloor \frac{3}{4} - \frac{\arg(q-1)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \binom{k+\frac{1}{2}}{k} \sum_{j=0}^{k} \frac{(-1)^j}{2j+1} \binom{k}{j} p_{j,k}\, (q-1)^k\, e^{\frac{z^2}{\log(q)}} \left(1 + 2 \sum_{k=1}^{\infty} e^{\frac{k^2\pi^2}{\log(q)}} \cosh\!\left(\frac{2k\pi z}{\log(q)}\right)\right) /;$$
$$(|q| < 1 \land |q-1| < 1) \bigwedge c_k = \frac{(-1)^k}{k+1} \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\,m - k + m)\, c_m\, p_{j,k-m} \bigwedge k \in \mathbb{N}^+$$

09.03.06.0031.01

$$\vartheta_3(z, q) \propto \frac{i\sqrt{\pi}}{\sqrt{q-1}}\, e^{-i\pi\left\lfloor \frac{3}{4} - \frac{\arg(q-1)}{2\pi} \right\rfloor} (1 + O(q-1))\, e^{\frac{z^2}{\log(q)}} \left(1 + O\!\left(e^{\frac{\pi^2}{\log(q)}} \cosh\!\left(\frac{2\pi z}{\log(q)}\right)\right)\right) /; |q| < 1$$

Other q-series representations

09.03.06.0004.01

$$\frac{\vartheta_3'(z, q)}{\vartheta_3(z, q)} == 4 \sum_{k=1}^{\infty} (-1)^k \frac{q^k}{1 - q^{2k}} \sin(2kz)$$

09.03.06.0005.01

$$\log\left(\frac{\vartheta_3(a+b,q)}{\vartheta_3(a-b,q)}\right) = 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\,\frac{q^k}{1-q^{2k}}\sin(2\,k\,a)\sin(2\,k\,b)$$

09.03.06.0006.01

$$\log(\vartheta_3(z,q)) = \log(\vartheta_3(0,q)) + 4\sum_{k=1}^{\infty}(-1)^k\,\frac{q^k}{k\left(1-q^{2k}\right)}\sin^2(k\,z)$$

09.03.06.0007.01

$$\frac{\vartheta_1'(0,q)\,\vartheta_3(a+b,q)}{4\,\vartheta_1(a,q)\,\vartheta_3(b,q)} = \frac{\csc(a)}{4} - \sum_{k=1}^{\infty}\frac{\sin((2\,k-1)\,a+2\,b)\,q^{2k-1}+\sin((2\,k-1)\,a)\,q^{4k-2}}{1+2\cos(2\,b)\,q^{2k-1}+q^{4k-2}} \,/;\, |\mathrm{Im}(a)| < \mathrm{Im}(\tau) \bigwedge q = e^{i\pi\tau}$$

09.03.06.0008.01

$$\frac{\vartheta_1'(0,q)\,\vartheta_3(a+b,q)}{4\,\vartheta_3(a,q)\,\vartheta_1(b,q)} = \frac{1}{4}\csc(b) + \sum_{k=1}^{\infty}(-1)^k\,q^k\,\frac{\sin(b+2\,a\,k)-q^{2k}\sin(2\,k\,a-b)}{1-2\cos(2\,b)\,q^{2k}+q^{4k}} \,/;\, |\mathrm{Im}(a)| < \mathrm{Im}(\tau) \bigwedge q = e^{i\pi\tau}$$

09.03.06.0009.01

$$\frac{\vartheta_1'(0,q)\,\vartheta_3(a+b,q)}{4\,\vartheta_1(a,q)\,\vartheta_3(b,q)} = \frac{1}{4}\csc(a) + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}(-1)^m\,q^{m(2k-1)}\sin((2\,k-1)\,a+2\,b\,m) \,/;$$

$$|\mathrm{Im}(a)| < \mathrm{Im}(\tau) \bigwedge |\mathrm{Im}(b)| < \mathrm{Im}(\tau) \bigwedge q = e^{i\pi\tau}$$

09.03.06.0010.01

$$\frac{\vartheta_1'(0,q)\,\vartheta_4(z,q)}{4\,\vartheta_2(0,q)\,\vartheta_3(z,q)} = \frac{1}{4} + \sum_{k=1}^{\infty}(-1)^k\,\frac{q^k}{1+q^{2k}}\cos(2\,k\,z) \,/;\, |\mathrm{Im}(z)| < \frac{1}{2}\mathrm{Im}(\tau) \bigwedge q = e^{i\pi\tau}$$

09.03.06.0011.01

$$\frac{\vartheta_1'(0,q)\,\vartheta_4(z,q)}{4\,\vartheta_2(0,q)\,\vartheta_3(z,q)} = \frac{1}{4} + \sum_{k=1}^{\infty}\frac{(-1)^k\left(q^{4k-2}+\cos(2\,z)\,q^{2k-1}\right)}{q^{4k-2}+2\cos(2\,z)\,q^{2k-1}+1} \,/;\, |\mathrm{Im}(z)| < \frac{1}{2}\mathrm{Im}(\tau) \bigwedge q = e^{i\pi\tau}$$

09.03.06.0012.01

$$\frac{\vartheta_1'(0,q)\,\vartheta_3(z,q)}{4\,\vartheta_3(0,q)\,\vartheta_1(z,q)} = \frac{1}{4}\csc(z) - \sum_{k=1}^{\infty}\frac{q^{2k-1}}{1+q^{2k-1}}\sin((2\,k-1)\,z) \,/;\, |\mathrm{Im}(z)| < \mathrm{Im}(\tau) \bigwedge q = e^{i\pi\tau}$$

09.03.06.0013.01

$$\frac{\vartheta_1'(0,q)\,\vartheta_3(z,q)}{4\,\vartheta_3(0,q)\,\vartheta_1(z,q)} = \frac{1}{4}\csc(z) + \sum_{k=1}^{\infty}\frac{(-1)^k\left(\left(1+q^{2k}\right)q^k\sin(z)\right)}{1-2\cos(2\,z)\,q^{2k}+q^{4k}} \,/;\, |\mathrm{Im}(z)| < \mathrm{Im}(\tau) \bigwedge q = e^{i\pi\tau}$$

09.03.06.0014.01

$$\frac{\vartheta_1'(0,q)\,\vartheta_1(z,q)}{4\,\vartheta_3(0,q)\,\vartheta_3(z,q)} = \sum_{k=1}^{\infty}(-1)^{k-1}\,\frac{q^{k-\frac{1}{2}}}{1+q^{2k-1}}\sin((2\,k-1)\,z) \,/;\, |\mathrm{Im}(z)| < \frac{1}{2}\mathrm{Im}(\tau) \bigwedge q = e^{i\pi\tau}$$

09.03.06.0015.01

$$\frac{\vartheta_1'(0,q)\,\vartheta_1(z,q)}{4\,\vartheta_3(0,q)\,\vartheta_3(z,q)} = \sum_{k=1}^{\infty}(-1)^{k-1}\,\frac{\left(1-q^{2k-1}\right)q^{k-\frac{1}{2}}\sin(z)}{1+2\cos(2\,z)\,q^{2k-1}+q^{4k-2}} \,/;\, |\mathrm{Im}(z)| < \frac{1}{2}\mathrm{Im}(\tau) \bigwedge q = e^{i\pi\tau}$$

09.03.06.0016.01

$$\frac{\vartheta_1'(0,q)}{2\,\vartheta_3(z,q)} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\,q^{(k-1/2)^2}\left(1-q^{4k-2}\right)}{1+2\cos(2\,z)\,q^{2k-1}+q^{4k-2}} \,/;\, |\mathrm{Im}(z)| < \frac{\mathrm{Im}(\tau)}{2} \bigwedge q = e^{i\pi\tau}$$

09.03.06.0017.01

$$\frac{\vartheta_1'(0,q)^2}{4\,\vartheta_3(z,q)^2} = \frac{1}{\sqrt{q}}\sum_{k=1}^{\infty}(-1)^{k-1}\,(2k-1)\,q^k\,\frac{\cos(2kz)\,q^{2k-1}+\cos((2k-2)z)}{1+2\cos(2z)\,q^{2k-1}+q^{4k-2}} \;/;\; |\mathrm{Im}(z)|<\mathrm{Im}(\tau)\bigwedge q=e^{i\pi\tau}$$

Other series representations

09.03.06.0032.01

$$\vartheta_3(z,q) = \frac{\sqrt[4]{-1}\,\sqrt{\pi}}{\sqrt{-i\log(q)}}\,e^{\frac{z^2}{\log(q)}}\left(1+2\sum_{k=1}^{\infty}e^{\frac{k^2\pi^2}{\log(q)}}\cosh\left(\frac{2k\pi z}{\log(q)}\right)\right)$$

09.03.06.0018.01

$$\vartheta_3(u,q) = \exp\left(-\frac{i\,u^2}{\pi\,\tau}\right)\sum_{n=-\infty}^{\infty}\exp\left(i\,\pi\,\tau\left(n+\frac{u}{\pi\,\tau}\right)^2\right) \;/;\; q=e^{i\pi\tau}$$

09.03.06.0019.01

$$\vartheta_3(z,q) = \frac{\sqrt{i}}{\sqrt{\tau}}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{\pi\,i}{\tau}\left(\frac{z}{\pi}+n\right)^2\right) \;/;\; q=e^{i\pi\tau}$$

Product representations

09.03.08.0001.01

$$\vartheta_3(0,q) = \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+q^{2n-1}\right)^2$$

09.03.08.0002.01

$$\vartheta_3(z,q) = \prod_{k=1}^{\infty}\left(1-q^{2k}\right)\left(1+2\,q^{2k-1}\cos(2z)\,+q^{4k-2}\right)$$

Differential equations

Partial differential equations

The elliptic theta functions satisfy the (one-dimensional) heat equation:

09.03.13.0001.01

$$\frac{\partial\vartheta_3(z,q)}{\partial\tau} = -\frac{\pi\,i}{4}\,\frac{\partial^2\vartheta_3(z,q)}{\partial z^2} \;/;\; q=e^{i\pi\tau}$$

09.03.13.0002.01

$$4\,q\,\frac{\partial\vartheta_3(z,q)}{\partial q}+\frac{\partial^2\vartheta_3(z,q)}{\partial z^2} = 0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.03.16.0005.01

$$\vartheta_3(z, q) = \frac{\sqrt[4]{-1}\, e^{\frac{z^2}{\log(q)}} \sqrt{\pi}}{\sqrt{-i \log(q)}} \vartheta_3\left(\frac{i \pi z}{\log(q)}, e^{\frac{\pi^2}{\log(q)}}\right)$$

09.03.16.0001.01

$$\vartheta_3\left(\frac{z}{\tau}, e^{-\frac{i\pi}{\tau}}\right) = \frac{\sqrt{\tau}}{\sqrt{i}} \exp\left(\frac{i z^2}{\pi \tau}\right) \vartheta_3(z, q) \;/;\; q = e^{i \pi \tau}$$

n th root of q

09.03.16.0002.01

$$\vartheta_3\left(z, q^{1/n}\right) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{\frac{2r}{n}}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=-\frac{n-1}{2}}^{\frac{n-1}{2}} \vartheta_3\left(z + \frac{i r \log(q)}{n}, q\right) /;\; \frac{n+1}{2} \in \mathbb{Z}^+$$

Multiple angle formulas

09.03.16.0003.01

$$\vartheta_3(n z, q^n) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{2nr}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=0}^{n-1} \vartheta_3\left(z + \frac{r \pi}{n}, q\right) /;\; \frac{n+1}{2} \in \mathbb{Z}^+$$

09.03.16.0004.01

$$\vartheta_3(n z, q^n) = \left(\prod_{r=1}^{\infty} \frac{1 - q^{2nr}}{\left(1 - q^{2r}\right)^n}\right) \prod_{r=-\frac{n-1}{2}}^{\frac{n-1}{2}} \vartheta_3\left(z + \frac{r \pi}{n}, q\right) /;\; n \in \mathbb{Z}^+$$

Identities

Functional identities

09.03.17.0001.01

$$\frac{\vartheta_3\left(0, q^3\right)^4}{\vartheta_3\left(0, q^9\right)^4} = \left(\frac{\vartheta_3(0, q)}{\vartheta_3\left(0, q^9\right)} - 1\right)^3 + 1$$

Derivative identities

09.03.17.0002.01

$$(-1 + (-1)^n)\, \theta^{(n)}(1) + (2n - 3)!!\, \theta(1) \left(-\frac{1}{2}\right)^n + \sum_{j=1}^{n-1} \binom{n}{j} \left((2(n-j) - 3)!! \left(-\frac{1}{2}\right)^{n-j} + (-1)^n \frac{(n-1)!}{(j-1)!}\right) \theta^{(j)}(1) = 0 \;/;$$

$$\theta(x) = \vartheta_3(0, e^{-x}) \bigwedge n \in \mathbb{N}^+$$

Differentiation

Low-order differentiation

With respect to z

09.03.20.0001.01

$$\frac{\partial \vartheta_3(z, q)}{\partial z} = \vartheta_3'(z, q)$$

09.03.20.0002.01

$$\frac{\partial^2 \vartheta_3(z, q)}{\partial z^2} = -8 \sum_{k=0}^{\infty} q^{k^2} k^2 \cos(2 k z) /; |q| < 1$$

With respect to *q*

09.03.20.0009.01

$$\frac{\partial \vartheta_3(z, q)}{\partial q} = -\frac{1}{4 q} \vartheta_2(0, q)^2 \vartheta_3(0, q)^2 \frac{\vartheta_3(z, q)^2 \vartheta_4(z, q)}{\vartheta_1(z, q)^2} - \frac{\vartheta_1'(z, q)^2 \vartheta_3(z, q)}{4 q \vartheta_1(z, q)^2} + \frac{1}{2 q} \vartheta_3(0, q)^2 \frac{\vartheta_1'(z, q)}{\vartheta_1(z, q)^2} \vartheta_2(z, q) \vartheta_4(z, q) + \frac{1}{q \pi^2} \vartheta_3(z, q) \left(\frac{\pi^2}{12} \left(\vartheta_3(0, q)^4 + \vartheta_4(0, q)^4\right) + \zeta\left(1; g_2\left(1, \frac{\log(q)}{\pi i}\right), g_3\left(1, \frac{\log(q)}{\pi i}\right)\right)\right)$$

09.03.20.0003.01

$$\frac{\partial \vartheta_3(z, q)}{\partial q} = 2 \sum_{k=1}^{\infty} q^{k^2-1} k^2 \cos(2 k z) /; |q| < 1$$

09.03.20.0004.01

$$\frac{\partial^2 \vartheta_3(z, q)}{\partial q^2} = \frac{2}{q^2} \sum_{k=2}^{\infty} q^{k^2} k^2 \left(k^2 - 1\right) \cos(2 k z) /; |q| < 1$$

Symbolic differentiation

With respect to *z*

09.03.20.0005.01

$$\frac{\partial^n \vartheta_3(z, q)}{\partial z^n} = 2^{n+1} \sum_{k=0}^{\infty} q^{k^2} k^n \cos\left(\frac{\pi n}{2} + 2 k z\right) /; |q| < 1 \bigwedge n \in \mathbb{N}^+$$

With respect to *q*

09.03.20.0006.01

$$\frac{\partial^n \vartheta_3(z, q)}{\partial q^n} = 2 \sum_{k=1}^{\infty} q^{k^2-n} \left(k^2 - n + 1\right)_n \cos(2 k z) /; |q| < 1 \bigwedge n \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to *z*

09.03.20.0007.01

$$\frac{\partial^\alpha \vartheta_3(z, q)}{\partial z^\alpha} = \frac{z^{-\alpha}}{\Gamma(1-\alpha)} + 2^{\alpha+1} \sqrt{\pi}\, z^{-\alpha} \sum_{k=1}^{\infty} q^{k^2} {}_1\tilde{F}_2\left(1; \frac{1-\alpha}{2}, 1 - \frac{\alpha}{2}; -k^2 z^2\right) /; |q| < 1$$

With respect to *q*

09.03.20.0008.01

$$\frac{\partial^\alpha \vartheta_3(z, q)}{\partial q^\alpha} == 2\, q^{-\alpha} \sum_{k=1}^{\infty} \frac{q^{k^2}\, \Gamma\left(k^2+1\right) \cos(2\, k\, z)}{\Gamma\left(k^2-\alpha+1\right)} + \frac{q^{-\alpha}}{\Gamma(1-\alpha)} \;/;\; |q| < 1$$

Integration

Indefinite integration

Involving only one direct function

09.03.21.0001.01

$$\int \vartheta_3(z, q)\, dz == z + \sum_{k=1}^{\infty} \frac{q^{k^2} \sin(2\, k\, z)}{k} \;/;\; |q| < 1$$

Involving only one direct function with respect to *q*

09.03.21.0002.01

$$\int \vartheta_3(z, q)\, dq == q + 2 \sum_{k=1}^{\infty} \frac{q^{k^2+1} \cos(2\, k\, z)}{k^2+1} \;/;\; |q| < 1$$

Definite integration

For the direct function itself

09.03.21.0003.01

$$\int_0^\infty t^{\alpha-1} \left(\vartheta_3\left(0, e^{-\pi t}\right) - 1\right) dt == \frac{2\, \Gamma(\alpha)}{\pi^\alpha} \zeta(2\, \alpha) \;/;\; \mathrm{Re}(\alpha) > \frac{1}{2}$$

Representations through equivalent functions

With related functions

Involving theta functions

Involving $\vartheta_1(z, q)$

09.03.27.0007.02

$$\vartheta_3(z, q) == \sqrt[4]{q}\; e^{-i z}\, \vartheta_1\left(z - \frac{1}{2} \pi\, (\tau+1), q\right) /;\; q == e^{i \pi \tau}$$

09.03.27.0012.01

$$\vartheta_3(z, q) == e^{-i\,(2 m+1)\, z}\, q^{(m+1/2)^2}\; \vartheta_1\left(z - \frac{1}{2} \pi\, ((2\, m+1)\, \tau + 1), q\right) /;\; m \in \mathbb{Z} \bigwedge q == e^{i \pi \tau}$$

09.03.27.0013.01

$$\vartheta_3(z, q) == e^{-i z} \sqrt[4]{q}\; \vartheta_1\left(z + \frac{1}{2} (i \log(q) + \pi), q\right)$$

09.03.27.0014.01

$$\vartheta_3(z, q) == -e^{-i\,(2 m+1)\, z}\, q^{\left(m+\frac{1}{2}\right)^2}\; \vartheta_1\left(z - \frac{1}{2} (\pi - i\,(2\, m+1) \log(q)), q\right) /;\; m \in \mathbb{Z}$$

Involving $\vartheta_2(z, q)$

09.03.27.0003.02

$$\vartheta_3(z, q) = -\sqrt[4]{q}\; e^{-i z}\, \vartheta_2\left(z - \frac{\pi \tau}{2}, q\right) /; q = e^{i \pi \tau}$$

09.03.27.0004.02

$$\vartheta_3(z, q) = -q^{\left(m+\frac{1}{2}\right)^2} e^{-i ((2 m+1) z)}\, \vartheta_2\left(z - \frac{1}{2} (2 m + 1) \pi \tau, q\right) /; m \in \mathbb{Z} \bigwedge q = e^{i \pi \tau}$$

09.03.27.0005.02

$$\vartheta_3(z, q) = \sqrt[4]{q}\; e^{i z}\, \vartheta_2\left(z - \frac{1}{2} i \log(q), q\right)$$

09.03.27.0006.02

$$\vartheta_3(z, q) = q^{m^2+m+\frac{1}{4}} e^{(2 m+1) i z}\, \vartheta_2\left(z - \frac{1}{2} (2 m + 1) (i \log(q)), q\right) /; m \in \mathbb{Z}$$

Involving $\vartheta_4(z, q)$

09.03.27.0001.02

$$\vartheta_3(z, q) = \vartheta_4\left(z - \frac{\pi}{2}, q\right)$$

09.03.27.0015.01

$$\vartheta_3(z, q) = \vartheta_4\left(z + \frac{\pi}{2}, q\right)$$

09.03.27.0002.02

$$\vartheta_3(z, q) = \vartheta_4\left(\frac{1}{2} \pi (2 m + 1) + z, q\right) /; m \in \mathbb{Z}$$

Involving Jacobi functions

09.03.27.0016.01

$$\frac{\vartheta_3(z, q(m))}{\vartheta_1(z, q(m))} = \frac{1}{(m (1 - m))^{1/4}} \operatorname{ds}\left(\frac{2 K(m) z}{\pi} \,\middle|\, m\right)$$

09.03.27.0017.01

$$\frac{\vartheta_3(z, q(m))}{\vartheta_2(z, q(m))} = \frac{1}{\sqrt[4]{m}} \operatorname{dc}\left(\frac{2 K(m) z}{\pi} \,\middle|\, m\right)$$

09.03.27.0008.01

$$\frac{\vartheta_3 (z, q (m))}{\vartheta_4 (z, q (m))} = \frac{1}{\sqrt[4]{1 - m}} \operatorname{dn}\left(\frac{2 K (m) z}{\pi} \,\middle|\, m\right)$$

Involving Weierstrass functions

09.03.27.0009.01

$$\vartheta_3(z, q) = \left(\prod_{n=1}^{\infty} \left(1 - q^{2 n}\right)\right) \left(\prod_{n=1}^{\infty} \left(1 + q^{2 n-1}\right)\right)^2 \exp\left(-\frac{2 \eta_1 \omega_1 z^2}{\pi^2}\right) \sigma_2\left(\frac{2 \omega_1 z}{\pi}; g_2, g_3\right) /;$$

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \eta_1 = \zeta(\omega_1; g_2, g_3) \bigwedge q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

09.03.27.0010.01

$$\frac{\vartheta_3(z,q)}{\vartheta_3(0,q)} == \exp\left(-\frac{2\eta_1\omega_1 z^2}{\pi^2}\right)\sigma_2\left(\frac{2\omega_1 z}{\pi}; g_2, g_3\right) /;$$

$$\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \eta_1 == \zeta(\omega_1; g_2, g_3) \bigwedge q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

09.03.27.0011.01

$$\zeta(z+\omega_2; g_2, g_3) - \eta_2 - \frac{\eta_1 z}{\omega_1} == \frac{\pi}{2\omega_1}\frac{\vartheta_3'\left(\frac{\pi z}{2\omega_1}, q\right)}{\vartheta_3\left(\frac{\pi z}{2\omega_1}, q\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} == \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge q == \exp\left(\frac{\pi i \omega_3}{\omega_1}\right) \bigwedge \eta_n == \zeta(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

Zeros

09.03.30.0002.01

$$\vartheta_3\left(\frac{\pi}{2}(1+\tau), q\right) == 0 /; q == e^{i\pi\tau}$$

09.03.30.0001.01

$$\vartheta_3\left((2m+1)\frac{\pi}{2} + (2n+1)\frac{\pi\tau}{2}, q\right) == 0 /; \{m, n\} \in \mathbb{Z} \bigwedge q == e^{i\pi\tau}$$

Theorems

Green's function for one-dimensional heat equation

The Green's function for the one-dimensional heat equation $\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)G(x,t) == \delta(x)\,\delta(t)$, with conditions $G(a,t) == G(b,t) == 0$ on the interval $a < x < b$, has the representation

$$G(x,t) == \frac{1}{2(b-a)}\left(\Theta_3\left(\exp\left(-\frac{\pi^2}{(b-a)^2}t\right), \frac{\pi(x-y)}{2(b-a)}\right) - \Theta_3\left(\exp\left(-\frac{\pi^2}{(b-a)^2}t\right), \frac{\pi(x+y)}{2(b-a)}\right)\right).$$

The number of representations of *n* as a sum of four squares

The number ν_n of representations of n as a sum of four squares (including sign and permutation) is $\nu_n == [q^n]\,\theta_3^4(0,q)$.

The Bloch functions composed of Gaussian orbitals of standard deviation of a one-dimensional crystal

The Bloch functions $\psi_k^\beta(x)$ composed of Gaussian orbitals of standard deviation $\frac{1}{2\sqrt{\beta}}$ of a one-dimensional crystal can be expressed in the form

$$\psi_k^\beta(x) == \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\vartheta_3\left(\frac{k}{2}, \frac{i\beta}{2\pi}\right)}} e^{-\beta x^2}\vartheta_3\left(\frac{k}{2} - i\beta x, \frac{i\beta}{\pi}\right),$$

where k is the wave vector.

History

J. Bernoulli (1713)
–L. Euler; J. Fourier
–C. G. J. Jacobi (1827)
–C. W. Borchardt (1838)
–K. Weierstrass (1862–1863)

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