Fibonacci

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Notations

Traditional name

Fibonacci number

Traditional notation

F_ν

Mathematica StandardForm notation

`Fibonacci[ν]`

Primary definition

04.11.02.0001.01

$$F_\nu = \frac{\phi^\nu - \cos(\nu\pi)\,\phi^{-\nu}}{\sqrt{5}}$$

Specific values

Specialized values

04.11.03.0001.01

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) /;\ n \in \mathbb{Z}$$

04.11.03.0002.01

$$F_{-n} = \frac{(-1)^{n-1}}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) /;\ n \in \mathbb{Z}$$

04.11.03.0015.01

$$F_n = \frac{1}{5}\left(\left\lfloor \phi^{n-1} \right\rceil + \left\lfloor \phi^{n+1} \right\rceil\right) /;\ n \in \mathbb{Z} \land n > 2$$

04.11.03.0016.01

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} /;\ n \in \mathbb{Z}$$

04.11.03.0017.01

$$F_{n+1} = \left\lfloor \frac{1}{2}\left(\left(1+\sqrt{5}\right) F_n + 1\right) \right\rfloor /;\ n \in \mathbb{Z} \land n > 1$$

04.11.03.0018.01

$$F_n = \frac{\phi^n - (1-\phi)^n}{\phi - (1-\phi)} \text{ /; } n \in \mathbb{Z}$$

Values at fixed points

04.11.03.0003.01

$$F_0 == 0$$

04.11.03.0004.01

$$F_1 == 1$$

04.11.03.0005.01

$$F_2 == 1$$

04.11.03.0006.01

$$F_3 == 2$$

04.11.03.0007.01

$$F_4 == 3$$

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$$F_5 == 5$$

04.11.03.0009.01

$$F_6 == 8$$

04.11.03.0010.01

$$F_7 == 13$$

04.11.03.0011.01

$$F_8 == 21$$

04.11.03.0012.01

$$F_9 == 34$$

04.11.03.0013.01

$$F_{10} == 55$$

Values at infinities

04.11.03.0014.01

$$F_\infty == \infty$$

General characteristics

Domain and analyticity

F_ν is an entire analytical function of ν which is defined over the whole complex ν-plane.

04.11.04.0001.01

$$\nu \longrightarrow F_\nu :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

04.11.04.0002.01

$$F_{-n} = (-1)^{n-1} F_n \;/;\; n \in \mathbb{Z}$$

Mirror symmetry

04.11.04.0003.01

$$F_{\overline{\nu}} = \overline{F_\nu}$$

Periodicity

No periodicity

Poles and essential singularities

The function F_ν has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

04.11.04.0004.01

$$\mathcal{S}ing_\nu(F_\nu) = \{\{\tilde{\infty}, \infty\}\}$$

Branch points

The function F_ν does not have branch points.

04.11.04.0005.01

$$\mathcal{BP}_\nu(F_\nu) = \{\}$$

Branch cuts

The function F_ν does not have branch cuts.

04.11.04.0006.01

$$\mathcal{BC}_\nu(F_\nu) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $\nu = \nu_0$

For the function itself

04.11.06.0015.01

$$F_\nu \propto F_{\nu_0} + \frac{1}{\sqrt{5}} \left(2\left(1+\sqrt{5}\right)\right)^{-\nu_0} \left(\operatorname{csch}^{-1}(2)\left(1+\sqrt{5}\right)^{2\nu_0} + 4^{\nu_0}\left(\operatorname{csch}^{-1}(2)\cos(\pi\nu_0) + \pi\sin(\pi\nu_0)\right)\right)(\nu-\nu_0) + \frac{1}{10}\left(\sqrt{5}\,\pi\left(\pi\cos(\pi\nu_0) - 2\operatorname{csch}^{-1}(2)\sin(\pi\nu_0)\right)\left(\frac{1}{2}\left(-1+\sqrt{5}\right)\right)^{\nu_0} + 5\operatorname{csch}^{-1}(2)^2 F_{\nu_0}\right)(\nu-\nu_0)^2 + \ldots \;/;\; (\nu \to \nu_0)$$

04.11.06.0016.01

$$F_\nu \propto F_{\nu_0} + \frac{1}{\sqrt{5}}\left(2\left(1+\sqrt{5}\right)\right)^{-\nu_0}\left(\operatorname{csch}^{-1}(2)\left(1+\sqrt{5}\right)^{2\nu_0} + 4^{\nu_0}\left(\operatorname{csch}^{-1}(2)\cos(\pi\,\nu_0)+\pi\sin(\pi\,\nu_0)\right)\right)(\nu-\nu_0) + \frac{1}{10}\left(\sqrt{5}\,\pi\left(\pi\cos(\pi\,\nu_0) - 2\operatorname{csch}^{-1}(2)\sin(\pi\,\nu_0)\right)\left(\frac{1}{2}\left(-1+\sqrt{5}\right)\right)^{\nu_0} + 5\operatorname{csch}^{-1}(2)^2 F_{\nu_0}\right)(\nu-\nu_0)^2 + O\left((\nu-\nu_0)^3\right)$$

04.11.06.0017.01

$$F_\nu = \sum_{k=0}^{\infty}\frac{1}{k!}\left(F_{\nu_0}\operatorname{csch}^{-1}(2)^k + \frac{1}{\sqrt{5}}\left(2^{\nu_0-1}\left(1+\sqrt{5}\right)^{-\nu_0} e^{-i\pi\nu_0}\left(\operatorname{csch}^{-1}(2)^k - (-1)^k\left(i\,\pi+\operatorname{csch}^{-1}(2)\right)^k + e^{2i\pi\nu_0}\left(\operatorname{csch}^{-1}(2)^k - \left(i\,\pi-\operatorname{csch}^{-1}(2)\right)^k\right)\right)\right)\right)(\nu-\nu_0)^k$$

04.11.06.0018.01

$$F_\nu \propto F_{\nu_0}\left(1+O(\nu-\nu_0)\right)$$

Expansions at $\nu == 0$

04.11.06.0001.01

$$F_\nu \propto \frac{2\log(\phi)\,\nu}{\sqrt{5}} + \frac{\pi^2\,\nu^2}{2\sqrt{5}} + \frac{1}{\sqrt{5}}\left(\frac{\log^3(\phi)}{3} - \frac{\pi^2}{2}\log(\phi)\right)\nu^3 + \ldots\ /;\ (\nu\to 0)$$

04.11.06.0019.01

$$F_\nu \propto \frac{2\log(\phi)\,\nu}{\sqrt{5}} + \frac{\pi^2\,\nu^2}{2\sqrt{5}} + \frac{1}{\sqrt{5}}\left(\frac{\log^3(\phi)}{3} - \frac{\pi^2}{2}\log(\phi)\right)\nu^3 + O\left(\nu^4\right)$$

04.11.06.0002.01

$$F_\nu = \frac{1}{2\sqrt{5}}\sum_{k=1}^{\infty}\frac{\left(2\operatorname{csch}^{-1}(2)^k - \left(-i\,\pi-\operatorname{csch}^{-1}(2)\right)^k - \left(i\,\pi-\operatorname{csch}^{-1}(2)\right)^k\right)\nu^k}{k!}$$

04.11.06.0003.01

$$F_\nu \propto \frac{2\log(\phi)}{\sqrt{5}}\,\nu + O\left(\nu^2\right)\ /;\ (\nu\to 0)$$

Asymptotic series expansions

04.11.06.0004.01

$$F_\nu \propto \frac{\phi^\nu - \cos(\nu\,\pi)\,\phi^{-\nu}}{\sqrt{5}}\ /;\ (|\nu|\to\infty)$$

04.11.06.0020.01

$$F_\nu \propto \begin{cases} \frac{\phi^\nu}{\sqrt{5}} & \operatorname{Im}(\nu)<0\land\operatorname{Re}(\nu)-\pi\,|\operatorname{Im}(\nu)|>0 \\ -\frac{e^{i\,\nu\,\pi-\nu\operatorname{csch}^{-1}(2)}}{2\sqrt{5}} & \operatorname{Im}(\nu)<0\land\pi\operatorname{Im}(\nu)+\operatorname{Re}(\nu)<0 \\ -\frac{e^{-\operatorname{csch}^{-1}(2)\,\nu-i\,\pi\,\nu}}{2\sqrt{5}} & \operatorname{Im}(\nu)>0\land\operatorname{Re}(\nu)-\pi\operatorname{Im}(\nu)<0 \\ \frac{\phi^\nu-\cos(\nu\,\pi)\,\phi^{-\nu}}{\sqrt{5}} & \text{True} \end{cases}\ /;\ (|\nu|\to\infty)$$

04.11.06.0021.01

$$F_\nu \propto \frac{\phi^\nu}{\sqrt{5}} /; (\nu \to \infty)$$

Other series representations

04.11.06.0005.01

$$F_n = \sum_{k=0}^{n-1} \binom{n-k-1}{k} /; n \in \mathbb{N}$$

04.11.06.0006.01

$$F_n = \sum_{k=0}^{n-1} \binom{k}{n-k-1} /; n \in \mathbb{N}$$

04.11.06.0007.01

$$F_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} /; n \in \mathbb{N}$$

04.11.06.0008.01

$$F_{2n+1} = \sum_{k=0}^{n} \binom{k+n}{2k} /; n \in \mathbb{N}$$

04.11.06.0009.01

$$F_{2n} = \sum_{k=0}^{n-1} \binom{k+n}{2k+1} /; n \in \mathbb{N}$$

04.11.06.0010.01

$$F_n = \frac{2^{1-n}\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)} \sum_{k=0}^{n-1} \frac{(n-k-1)!\left(1-\frac{n}{2}\right)_k (-4)^k}{k!\,\Gamma\left(\frac{n+1}{2}-k\right)} /; n \in \mathbb{N}$$

04.11.06.0011.01

$$F_n = \sum_{k=1}^{n} 5^{\frac{n-k}{2}} i^{k-1} \binom{2n-k}{k-1} \exp\left(i\,(n-k)\tan^{-1}(-2)\right) /; n \in \mathbb{N}$$

04.11.06.0012.01

$$F_n = \frac{1}{2^{n-1}} \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2k+1} 5^k /; n \in \mathbb{N}$$

04.11.06.0013.01

$$F_n = (-i)^{n-1} \sum_{k=0}^{n-1} \binom{k+n}{2k+1} (i-2)^k /; n \in \mathbb{N}$$

04.11.06.0014.01

$$F_n^2 = \frac{n}{5^{n-2\left\lfloor \frac{n+1}{2} \right\rfloor+1}} \sum_{k=0}^{n-1} \frac{1}{k+1} \binom{k+n}{2k+1} (-5)^{k\left(n-2\left\lfloor \frac{n}{2} \right\rfloor\right)} /; n \in \mathbb{N}$$

Integral representations

On the real axis

Of the direct function

04.11.07.0001.01

$$F_{2n} = \frac{n}{2}\left(\frac{3}{2}\right)^{n-1}\int_0^{\pi}\left(1+\frac{\sqrt{5}}{3}\cos(t)\right)^{n-1}\sin(t)\,dt\;/;\;n\in\mathbb{Z}$$

Limit representations

04.11.09.0001.01

$$F_n = \lim_{m\to\infty}\max\left(\left\{\frac{\log(\log(d_{n-1}(\mu)))}{\log(\log(\mu))}\right\}_{\mu,1,m}\right)/;\;n\in\mathbb{N}^+\bigwedge d_k(m) = d_{k-1}(\sigma_0(m))\bigwedge d_0(m) = \sigma_0(m)$$

Generating functions

04.11.11.0001.01

$$F_n = \left([t^n]\,\frac{t}{1-t-t^2}\right)/;\;n\in\mathbb{N}$$

Differential equations

Ordinary linear differential equations and wronskians

04.11.13.0001.01

$$w^{(3)}(\nu)+\log(\phi)\,w''(\nu)+\left(\pi^2-\log^2(\phi)\right)w'(\nu)-\log(\phi)\left(\log^2(\phi)+\pi^2\right)w(\nu) = 0\;/;\;w(\nu) = c_1\,F_\nu\;+c_2\,L_\nu+c_3\,\phi^{-\nu}\sin(\pi\,\nu)$$

Transformations

Addition formulas

04.11.16.0001.01

$$F_{m+n} = F_{n+1}\,F_m+F_{m-1}\,F_n\;/;\;m\in\mathbb{Z}\land n\in\mathbb{Z}$$

04.11.16.0002.01

$$F_{m+n} = \frac{1}{2}\,(F_n\,L_m+F_m\,L_n)\;/;\;n\in\mathbb{Z}\land m\in\mathbb{Z}$$

04.11.16.0003.01

$$F_{m+n} = \sum_{k=0}^{\infty}\binom{n}{k}F_{m-k}\;/;\;m\in\mathbb{N}\land n\in\mathbb{N}$$

04.11.16.0004.01

$$F_{m-n} = (-1)^n\,(F_m\,F_{n+1}-F_n\,F_{m+1})\;/;\;m\in\mathbb{Z}\land n\in\mathbb{Z}$$

04.11.16.0005.01

$$F_{m-n} = \frac{1}{2}\,(-1)^n\,(F_m\,L_n-F_n\,L_m)\;/;\;n\in\mathbb{Z}\land m\in\mathbb{Z}$$

04.11.16.0006.01

$$F_{m+2n} = \sum_{k=0}^{\infty} \binom{n}{k} F_{k+m} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

04.11.16.0007.01

$$F_{m+2n} = \sum_{k=0}^{\infty} 2^{n-k} \binom{n}{k} F_{m-k} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

04.11.16.0008.01

$$F_{m+3n} = \sum_{k=0}^{\infty} 2^{k} \binom{n}{k} F_{k+m} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

04.11.16.0009.01

$$F_{\nu+1} = \frac{1}{2} (F_\nu + L_\nu)$$

04.11.16.0010.01

$$F_{2n+1} = F_{n+1}^2 + F_n^2 /; n \in \mathbb{N}$$

04.11.16.0027.01

$$F_{n-1} F_{n+1} - F_n^2 = (-1)^n /; n \in \mathbb{N} \land n \geq 0$$

Cassini's formula

Multiple arguments

04.11.16.0028.01

$$F_{2\nu} = L_\nu F_\nu + \frac{\sin^2(\pi \nu) \phi^{-2\nu}}{\sqrt{5}}$$

04.11.16.0029.01

$$F_{2\nu} = F_{\nu-1} F_\nu + F_{\nu+1} F_\nu + \frac{\phi^{-2\nu} \sin^2(\pi \nu)}{\sqrt{5}}$$

04.11.16.0030.01

$$F_{2\nu+1} = F_{\nu-1} F_{\nu+1} + F_{\nu+2} F_\nu - \frac{\phi^{-2\nu-1} \sin^2(\pi \nu)}{\sqrt{5}}$$

04.11.16.0011.01

$$F_{2n} = F_{n-p} F_{n+p-1} + F_{n-p+1} F_{n+p} /; n \in \mathbb{N} \bigwedge p \in \mathbb{N}$$

04.11.16.0012.01

$$F_{2n} = L_n F_n /; n \in \mathbb{Z}$$

04.11.16.0013.01

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k /; n \in \mathbb{N}$$

04.11.16.0031.01

$$F_{2\nu} = 3 F_{2(\nu-1)} - F_{2(\nu-2)}$$

04.11.16.0014.01

$$F_{m\nu} = L_m F_{m(\nu-1)} - (-1)^m F_{m(\nu-2)} /; m \in \mathbb{Z}$$

04.11.16.0015.01

$$F_{m n} = \sum_{k=0}^{m} \binom{m}{k} F_n^k F_{n-1}^{m-k} F_k \;/;\; m - 1 \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge n \geq m$$

04.11.16.0016.01

$$F_{m n} = \frac{1}{2^{m-1}} \sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m}{2 k + 1} F_n^{2 k+1} L_n^{-2 k+m-1} 5^k \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

04.11.16.0017.01

$$F_{m n} = F_n \sum_{k=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} \binom{m - k - 1}{k} (-1)^{k (n-1)} L_n^{m-2 k-1} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

04.11.16.0018.01

$$F_{2 m n} = L_n \sum_{k=0}^{m-1} \binom{2 m - k - 1}{k} (-1)^{k n} F_n^{-2 k+2 m-1} 5^{-k+m-1} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

04.11.16.0019.01

$$F_{(2 m-1) n} = \sum_{k=0}^{m-1} \frac{2 m - 1}{2 m - k - 1} \binom{2 m - k - 1}{k} (-1)^{k n} F_n^{2 m-2 k-1} 5^{m-k-1} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{Z}$$

Products, sums, and powers of the direct function

Products of the direct function

04.11.16.0020.01

$$F_{\nu+1} F_{\nu-1} = F_\nu^2 + \cos(\nu \pi)$$

04.11.16.0021.01

$$F_n F_m = \frac{1}{5} (L_{m+n} - (-1)^n L_{m-n}) \;/;\; m \in \mathbb{Z} \land n \in \mathbb{Z}$$

Powers of the direct function

04.11.16.0022.01

$$F_\nu^2 = F_{\nu+1} F_{\nu-1} - \cos(\nu \pi)$$

04.11.16.0023.01

$$F_n^2 = \frac{1}{5} (L_{2 n} - 2 (-1)^n) \;/;\; n \in \mathbb{Z}$$

04.11.16.0032.01

$$F_n^3 = \frac{1}{5} \left(3 (-1)^{n+1} F_n + F_{3 n}\right) \;/;\; n \in \mathbb{Z}$$

04.11.16.0033.01

$$F_n^4 = \frac{1}{25} (4 (-1)^n F_{2 n} - F_{4 n} - 8 (-1)^n F_{2 n+1} + 2 F_{4 n+1} + 6) \;/;\; n \in \mathbb{Z}$$

04.11.16.0024.01

$$F_n^4 = F_{n-2} F_{n-1} F_{n+1} F_{n+2} + 1 \;/;\; n \in \mathbb{Z}$$

04.11.16.0025.01

$$F_n^m = \frac{1}{2}\,5^{-\left\lfloor\frac{m}{2}\right\rfloor}\sum_{k=0}^{m}\binom{m}{k}(-1)^{k(n+1)}\left((1+(-1)^m)\,F_{-2kn+mn+1}-(-1)^m\,F_{(m-2k)n}\right)/;\,n\in\mathbb{Z}\bigwedge m\in\mathbb{N}^+$$

Related transformations

04.11.16.0026.01

$$F_\nu = \frac{1}{5}\,(L_{\nu-1}+L_{\nu+1})$$

Identities

Recurrence identities

Consecutive neighbors

04.11.17.0001.01

$$F_\nu = -F_{\nu+1}+F_{\nu+2}$$

04.11.17.0002.01

$$F_\nu = F_{\nu-1}+F_{\nu-2}$$

04.11.17.0017.01

$$F_\nu = \frac{F_{\nu+1}-\phi^\nu}{1-\phi}$$

04.11.17.0018.01

$$F_\nu = (1-\phi)\,F_{\nu-1}+\phi^{\nu-1}$$

Distant neighbors

04.11.17.0003.01

$$F_\nu = i^{m+1}\,U_{\frac{m-1}{2}}\left(-\frac{3}{2}\right)F_{m+\nu}+i^m\,U_{\frac{m}{2}-1}\left(-\frac{3}{2}\right)F_{m+\nu+1}\,/;\,m\in\mathbb{N}^+$$

04.11.17.0004.01

$$F_\nu = i^{1-m}\,U_{\frac{m-1}{2}}\left(-\frac{3}{2}\right)F_{\nu-m}-(-i)^m\,U_{\frac{m}{2}-1}\left(-\frac{3}{2}\right)F_{\nu-m-1}\,/;\,m\in\mathbb{N}^+$$

Functional identities

Functional equations

04.11.17.0019.01

$$w(z) = w(z-2)+w(z-1)\,/;\,w(z) = c_1\,F_z\,+c_2\,L_z$$

Relations of special kind

04.11.17.0007.01

$$F_{\nu+1}\,F_{\nu-1}-F_\nu^2 = \cos(\nu\,\pi)$$

04.11.17.0008.01

$$F_{k+n}\,F_{l+n}-F_{k+l+n}\,F_n = (-1)^n\,F_k\,F_l\,/;\,k\in\mathbb{N}\bigwedge l\in\mathbb{N}\bigwedge n\in\mathbb{N}$$

04.11.17.0009.01

$$F_{k+n-1}\,F_{n-k}+F_{-k+n+1}\,F_{k+n} = F_{2n}\,/;\,k\in\mathbb{N}\bigwedge n\in\mathbb{N}$$

04.11.17.0010.01

$$F_n^2 - F_{m+n} F_{n-m} == (-1)^{n-m} F_m^2 \ /;\ n \in \mathbb{Z} \land m \in \mathbb{Z}$$

04.11.17.0011.01

$$4 F_{2n+1} F_{2n+2} F_{2n+3} F_{2n+4} - (2 F_{2n+2} F_{2n+3} + 1)^2 + 1 == 0 \ /;\ n \in \mathbb{N}$$

04.11.17.0012.01

$$\frac{1}{F_{a+n} F_{b+n} F_{c+n}} == \frac{(-1)^{b+n}}{F_{c-b} F_{a-b} F_{b+n}} + \frac{(-1)^{c+n}}{F_{a-c} F_{b-c} F_{c+n}} + \frac{(-1)^{a+n}}{F_{b-a} F_{c-b} F_{a+n}} \ /;$$
$$n \in \mathbb{N} \bigwedge a \in \mathbb{N}^+ \bigwedge b \in \mathbb{N}^+ \bigwedge c \in \mathbb{N}^+ \bigwedge a \neq b \bigwedge a \neq c \bigwedge b \neq c$$

04.11.17.0013.01

$$F_{n+1} == \left\lfloor \phi F_n + \frac{1}{2} \right\rfloor \ /;\ n - 1 \in \mathbb{N}^+$$

04.11.17.0014.01

$$F_{\gcd(m,n)} == \gcd(F_m, F_n) \ /;\ m \in \mathbb{Z} \land n \in \mathbb{Z}$$

04.11.17.0015.01

$$F_m \sum_{k=1}^{n} \frac{(-1)^k}{F_k F_{k+m}} == F_n \sum_{k=1}^{m} \frac{(-1)^k}{F_k F_{k+n}} \ /;\ m \in \mathbb{N} \land n \in \mathbb{N}$$

04.11.17.0016.01

$$\tan^{-1}\left(\frac{1}{F_{2n+1}}\right) + \tan^{-1}\left(\frac{1}{F_{2n+2}}\right) == \tan^{-1}\left(\frac{1}{F_{2n}}\right) \ /;\ n \in \mathbb{N}^+$$

Complex characteristics

Real part

04.11.19.0001.01

$$\operatorname{Re}(F_{x+iy}) == \frac{\phi^{-x}}{\sqrt{5}} \left(\phi^{2x} \cos(y \log(\phi)) - \cos(\pi x) \cosh(\pi y) \cos(y \log(\phi)) + \sin(\pi x) \sin(y \log(\phi)) \sinh(\pi y)\right)$$

04.11.19.0006.01

$$\operatorname{Re}(F_{x+iy}) =$$
$$\frac{1}{\sqrt{5}} \left(\left(\sin(\pi x) \sin\left(y \operatorname{csch}^{-1}(2)\right) \sinh(\pi y) - \cos(\pi x) \cos\left(y \operatorname{csch}^{-1}(2)\right) \cosh(\pi y)\right) \left(\frac{\sqrt{5} - 1}{2}\right)^x + \left(\frac{\sqrt{5} + 1}{2}\right)^x \cos\left(y \operatorname{csch}^{-1}(2)\right) \right)$$

Imaginary part

04.11.19.0002.01

$$\operatorname{Im}(F_{x+iy}) == \frac{\phi^{-x}}{\sqrt{5}} \left(\phi^{2x} \sin(y \log(\phi)) + \cos(\pi x) \cosh(\pi y) \sin(y \log(\phi)) + \cos(y \log(\phi)) \sin(\pi x) \sinh(\pi y)\right)$$

04.11.19.0007.01

$$\operatorname{Im}(F_{x+iy}) =$$
$$\frac{1}{\sqrt{5}} \left(\left(\cos(\pi x) \cosh(\pi y) \sin\left(y \operatorname{csch}^{-1}(2)\right) + \cos\left(y \operatorname{csch}^{-1}(2)\right) \sin(\pi x) \sinh(\pi y)\right) \left(\frac{\sqrt{5} - 1}{2}\right)^x + \left(\frac{\sqrt{5} + 1}{2}\right)^x \sin\left(y \operatorname{csch}^{-1}(2)\right) \right)$$

Absolute value

04.11.19.0003.01

$$\left|F_{x+i y}\right| == \frac{1}{\sqrt{10}} \sqrt{\left(\phi^{-2 x} \left(\cosh^2(\pi y) - 4 \phi^{2 x} \cos(\pi x) \cos(2 y \log(\phi)) \cosh(\pi y) + 2 \phi^{4 x} + \sinh^2(\pi y) + \cos(2 \pi x) + 4 \phi^{2 x} \sin(\pi x) \sin(2 y \log(\phi)) \sinh(\pi y)\right)\right)}$$

04.11.19.0008.01

$$\left|F_{x+i y}\right| = \frac{1}{\sqrt{10}} \sqrt{\left(\left(2\left(3+\sqrt{5}\right)^{2 x} + 4^x \left(\cos(2 \pi x) + \cosh(2 \pi y)\right)\right)\left(1+\sqrt{5}\right)^{-2 x} - 4 \cos(\pi x) \cos\left(2 y \operatorname{csch}^{-1}(2)\right) \cosh(\pi y) + 4 \sin(\pi x) \sin\left(2 y \operatorname{csch}^{-1}(2)\right) \sinh(\pi y)\right)}$$

Argument

04.11.19.0004.01

$$\arg\left(F_{x+i y}\right) == \tan^{-1}\left(\phi^{-x} \left(\phi^{2 x} \cos(y \log(\phi)) - \cos(\pi x) \cosh(\pi y) \cos(y \log(\phi)) + \sin(\pi x) \sin(y \log(\phi)) \sinh(\pi y)\right), \phi^{-x} \left(\phi^{2 x} \sin(y \log(\phi)) + \cos(\pi x) \cosh(\pi y) \sin(y \log(\phi)) + \cos(y \log(\phi)) \sin(\pi x) \sinh(\pi y)\right)\right)$$

04.11.19.0009.01

$$\arg\left(F_{x+i y}\right) = \tan^{-1}\left(\left(\sin(\pi x) \sin\left(y \operatorname{csch}^{-1}(2)\right) \sinh(\pi y) - \cos(\pi x) \cos\left(y \operatorname{csch}^{-1}(2)\right) \cosh(\pi y)\right)\left(\frac{\sqrt{5}-1}{2}\right)^x + \left(\frac{\sqrt{5}+1}{2}\right)^x \cos\left(y \operatorname{csch}^{-1}(2)\right), \left(\cos(\pi x) \cosh(\pi y) \sin\left(y \operatorname{csch}^{-1}(2)\right) + \cos\left(y \operatorname{csch}^{-1}(2)\right) \sin(\pi x) \sinh(\pi y)\right)\left(\frac{\sqrt{5}-1}{2}\right)^x + \left(\frac{\sqrt{5}+1}{2}\right)^x \sin\left(y \operatorname{csch}^{-1}(2)\right)\right)$$

Conjugate value

04.11.19.0005.01

$$\overline{F_{x+i y}} == \frac{\phi^{-x}}{\sqrt{5}} \left(\phi^{2 x} (\cos(y \log(\phi)) - i \sin(y \log(\phi))) + \sin(\pi x) \sinh(\pi y) (\sin(y \log(\phi)) - i \cos(y \log(\phi))) - \cos(\pi x) \cosh(\pi y) (\cos(y \log(\phi)) + i \sin(y \log(\phi)))\right)$$

04.11.19.0010.01

$$\overline{F_{x+i y}} = \frac{1}{\sqrt{5}} \left(-\cos(\pi x) \cosh(\pi y) \left(\cos\left(y \operatorname{csch}^{-1}(2)\right) + i \sin\left(y \operatorname{csch}^{-1}(2)\right)\right)\left(\frac{\sqrt{5}-1}{2}\right)^x + \sin(\pi x) \left(\sin\left(y \operatorname{csch}^{-1}(2)\right) - i \cos\left(y \operatorname{csch}^{-1}(2)\right)\right) \sinh(\pi y) \left(\frac{\sqrt{5}-1}{2}\right)^x + \left(\frac{\sqrt{5}+1}{2}\right)^x \left(\cos\left(y \operatorname{csch}^{-1}(2)\right) - i \sin\left(y \operatorname{csch}^{-1}(2)\right)\right)\right)$$

Signum value

04.11.19.0011.01

$$\operatorname{sgn}(F_{x+i y}) = \left(\sqrt{2}\ \phi^{-x} \left(\cos(\pi x) \cosh(\pi y) (i \sin(y \log(\phi)) - \cos(y \log(\phi))) + \phi^{2 x} (\cos(y \log(\phi)) + i \sin(y \log(\phi))) + \sin(\pi x) (i \cos(y \log(\phi)) + \sin(y \log(\phi))) \sinh(\pi y)\right)\right) \Big/ \left(\sqrt{\left(\phi^{-2 x} \left(\cosh^2(\pi y) - 4 \phi^{2 x} \cos(\pi x) \cos(2 y \log(\phi)) \cosh(\pi y) + 2 \phi^{4 x} + \sinh^2(\pi y) + \cos(2 \pi x) + 4 \phi^{2 x} \sin(\pi x) \sin(2 y \log(\phi)) \sinh(\pi y)\right)\right)}\right)$$

04.11.19.0012.01

$$\operatorname{sgn}(F_{x+i y}) = \left(\sqrt{2} \left(\sin(\pi x) \left(i \cos\left(y \operatorname{csch}^{-1}(2)\right) + \sin\left(y \operatorname{csch}^{-1}(2)\right)\right) \sinh(\pi y) \left(\frac{\sqrt{5}-1}{2}\right)^x + \left(\frac{\sqrt{5}+1}{2}\right)^x \left(\cos\left(y \operatorname{csch}^{-1}(2)\right) + i \sin\left(y \operatorname{csch}^{-1}(2)\right)\right) - \left(\frac{\sqrt{5}-1}{2}\right)^x \cos(\pi x) \cosh(\pi y) \left(\cos\left(y \operatorname{csch}^{-1}(2)\right) - i \sin\left(y \operatorname{csch}^{-1}(2)\right)\right)\right)\right) \Big/ \left(\sqrt{\left(\left(2 \left(3+\sqrt{5}\right)^{2 x} + 4^x (\cos(2 \pi x) + \cosh(2 \pi y))\right) \left(1+\sqrt{5}\right)^{-2 x} - 4 \cos(\pi x) \cos\left(2 y \operatorname{csch}^{-1}(2)\right) \cosh(\pi y) + 4 \sin(\pi x) \sin\left(2 y \operatorname{csch}^{-1}(2)\right) \sinh(\pi y)\right)}\right)$$

Differentiation

Low-order differentiation

04.11.20.0001.01

$$\frac{\partial F_\nu}{\partial \nu} = \frac{\phi^{-\nu} \left(\phi^{2 \nu} \log(\phi) + \cos(\pi \nu) \log(\phi) + \pi \sin(\pi \nu)\right)}{\sqrt{5}}$$

04.11.20.0002.01

$$\frac{\partial^2 F_\nu}{\partial \nu^2} = \frac{1}{\sqrt{5}} \left(\phi^{-\nu} \left(\cos(\pi \nu) \left(\pi^2 - \log^2(\phi)\right) + \log(\phi) \left(\phi^{2 \nu} \log(\phi) - 2 \pi \sin(\pi \nu)\right)\right)\right)$$

Symbolic differentiation

04.11.20.0003.02

$$\frac{\partial^n F_\nu}{\partial \nu^n} = \frac{1}{\sqrt{5}} \left(\phi^\nu \log^n(\phi) - \frac{1}{2} (-1)^n \phi^{-\nu} \left(e^{-i \pi \nu} (\log(\phi) + i \pi)^n + e^{i \pi \nu} (\log(\phi) - i \pi)^n\right)\right) /; n \in \mathbb{N}$$

04.11.20.0004.02

$$\frac{\partial^n F_\nu}{\partial \nu^n} = F_\nu \log^n(\phi) + \frac{\phi^{-\nu}}{\sqrt{5}} \left(\cos(\pi \nu) \log^n(\phi) - (-1)^n \sum_{k=0}^n \binom{n}{k} \pi^k \cos\left(\frac{\pi}{2} (k - 2 \nu)\right) \log^{n-k}(\phi)\right) /; n \in \mathbb{N}$$

Fractional integro-differentiation

04.11.20.0005.01

$$\frac{\partial^\alpha F_\nu}{\partial \nu^\alpha} = \frac{\nu^{-\alpha}}{2 \sqrt{5}} \left(\left(\nu \left(\pi i - \operatorname{csch}^{-1}(2)\right)\right)^\alpha \exp\left(\left(i \pi - \operatorname{csch}^{-1}(2)\right) \nu\right) \left(Q\left(-\alpha, \left(i \pi - \operatorname{csch}^{-1}(2)\right) \nu\right) - 1\right) + \exp\left(-\left(i \pi + \operatorname{csch}^{-1}(2)\right) \nu\right) \left(\nu \left(-i \pi - \operatorname{csch}^{-1}(2)\right)\right)^\alpha \left(Q\left(-\alpha, -\left(i \pi + \operatorname{csch}^{-1}(2)\right) \nu\right) - 1\right) - 2 \nu^\alpha \operatorname{csch}^{-1}(2)^\alpha \exp\left(\nu \operatorname{csch}^{-1}(2)\right) \left(Q\left(-\alpha, \nu \operatorname{csch}^{-1}(2)\right) - 1\right)\right)$$

Integration

Indefinite integration

Involving only one direct function

04.11.21.0001.01

$$\int F_{a\nu}\,d\nu = \frac{1}{\sqrt{5}\,a}\left(\frac{\phi^{-a\nu}\,(\log(\phi)\cos(\pi\,a\,\nu)-\pi\sin(\pi\,a\,\nu))}{\log^2(\phi)+\pi^2}+\frac{\phi^{a\nu}}{\log(\phi)}\right)$$

04.11.21.0002.01

$$\int F_{\nu}\,d\nu = \frac{1}{\sqrt{5}}\left(\frac{\phi^{-\nu}\,(\log(\phi)\cos(\pi\,\nu)-\pi\sin(\pi\,\nu))}{\log^2(\phi)+\pi^2}+\frac{\phi^{\nu}}{\log(\phi)}\right)$$

Involving one direct function and elementary functions

Involving power function

04.11.21.0003.01

$$\int \nu^{\alpha-1}\,F_{a\nu}\,d\nu = \frac{1}{2\sqrt{5}}\left(\nu^{\alpha}\left(-2\,(-a\,\nu)^{-\alpha}\,\Gamma\left(\alpha,\,-a\,\nu\,\operatorname{csch}^{-1}(2)\right)\operatorname{csch}^{-1}(2)^{-\alpha}+\left(a\,\nu\left(-i\,\pi+\operatorname{csch}^{-1}(2)\right)\right)^{-\alpha}\Gamma\left(\alpha,\,a\,\nu\left(-i\,\pi+\operatorname{csch}^{-1}(2)\right)\right)+\left(a\,\nu\left(i\,\pi+\operatorname{csch}^{-1}(2)\right)\right)^{-\alpha}\Gamma\left(\alpha,\,a\,\nu\left(i\,\pi+\operatorname{csch}^{-1}(2)\right)\right)\right)\right)$$

04.11.21.0004.01

$$\int \nu^{\alpha-1}\,F_{\nu}\,d\nu = \frac{\nu^{\alpha}}{2\sqrt{5}}\left(-2\,(-\nu)^{-\alpha}\,\Gamma\left(\alpha,\,-\nu\,\operatorname{csch}^{-1}(2)\right)\operatorname{csch}^{-1}(2)^{-\alpha}+\left(\nu\left(-i\,\pi+\operatorname{csch}^{-1}(2)\right)\right)^{-\alpha}\Gamma\left(\alpha,\,\nu\left(-i\,\pi+\operatorname{csch}^{-1}(2)\right)\right)+\left(\nu\left(i\,\pi+\operatorname{csch}^{-1}(2)\right)\right)^{-\alpha}\Gamma\left(\alpha,\,\nu\left(i\,\pi+\operatorname{csch}^{-1}(2)\right)\right)\right)$$

Integral transforms

Laplace transforms

04.11.22.0001.01

$$\mathcal{L}_t[F_t](z) = \frac{1}{\sqrt{5}\left(z-\operatorname{csch}^{-1}(2)\right)}-\frac{z+\operatorname{csch}^{-1}(2)}{\sqrt{5}\left(\left(z+\operatorname{csch}^{-1}(2)\right)^2+\pi^2\right)}\;/;\;\operatorname{Re}(z)>\log(\phi)$$

Summation

Finite summation

04.11.23.0010.01

$$\sum_{k=0}^{n} F_k = F_{n+2}-1$$

04.11.23.0012.01

$$\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n} /; n \in \mathbb{N}$$

04.11.23.0013.01

$$\sum_{k=0}^{n} \binom{n}{k} F_k \, 2^k = F_{3n} /; n \in \mathbb{N}$$

04.11.23.0014.01

$$\sum_{k=0}^{n} F_k \, z^k = \frac{z\,(z^n\,(z\,F_n + F_{n+1}) - 1)}{z^2 + z - 1} /; n \in \mathbb{N}$$

04.11.23.0001.01

$$\sum_{k=0}^{n} F_{k\,p+q} \, z^k = \frac{F_q - F_{(n+1)\,p+q} \, z^{n+1} + (-1)^p \, F_{n\,p+q} \, z^{n+2} - (-1)^p \, F_{q-p} \, z}{(-1)^p \, z^2 - L_p \, z + 1} /; p \in \mathbb{Z} \bigwedge q \in \mathbb{Z} \bigwedge n \in \mathbb{N}$$

04.11.23.0002.01

$$\sum_{k=0}^{n} F_k \, F_{n-k} = \frac{1}{5} \, (n\,L_n - F_n) /; n \in \mathbb{N}$$

04.11.23.0011.01

$$\sum_{k=0}^{n} F_k^2 = F_n \, F_{n+1}$$

Infinite summation

04.11.23.0003.02

$$\sum_{k=1}^{\infty} F_k \, z^k = -\frac{z}{z^2 + z - 1}$$

04.11.23.0004.01

$$\sum_{k=1}^{\infty} \frac{1}{F_{2k-1}} = \frac{1}{4} \sqrt{5} \, \vartheta_2\!\left(0, \frac{2}{3+\sqrt{5}}\right)^2$$

04.11.23.0005.01

$$\sum_{k=1}^{\infty} \frac{1}{F_k \, F_{k+2}} = 1$$

04.11.23.0006.01

$$\sum_{k=1}^{\infty} \sin\!\left(\frac{n\,\pi\,F_{k-1}}{2\,F_{k+1}\,F_k}\right) \cos\!\left(\frac{n\,\pi\,F_{k+2}}{2\,F_{k+1}\,F_k}\right) = 0 /; n \in \mathbb{Z}$$

04.11.23.0007.01

$$\sum_{k=0}^{\infty} F_k \, F_{k+1} \, F_{k+2} \, z^k = \frac{2\,z}{(-z^2 + z + 1)\,(-z^2 - 4\,z + 1)}$$

as a formal power series

04.11.23.0015.01

$$\sum_{k=1}^{\infty} |F_k \phi - F_{k+1}| = \phi$$

Multiple sums

04.11.23.0008.01

$$\sum_{m_1=0}^{n} \sum_{m_2=0}^{n} \cdots \sum_{m_k=0}^{n} \delta_{n,\sum_{j=1}^{k} m_j} \prod_{j=1}^{k} F_{m_j+1} = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{k-j+n-1}{k-1} \binom{n-j}{j} /; n \in \mathbb{N} \bigwedge k \in \mathbb{N}^+$$

04.11.23.0009.01

$$\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_p=1}^{n} \delta_{n-\sum_{j=1}^{p} k_j} \prod_{j=1}^{p} F_{k_j} = F_{n,p} /; n \in \mathbb{N} \bigwedge p \in \mathbb{N}^+ \bigwedge F_{n,p} = \frac{2}{5} \left(\frac{n-1}{p-1} + 1 \right) F_{n-1,p-1} + \frac{1}{5} \left(\frac{n}{p-1} - 1 \right) F_{n,p-1} \bigwedge F_{n,1} = F_n$$

Operations

Limit operation

04.11.25.0001.01

$$\lim_{\nu \to \infty} \frac{F_\nu}{L_\nu} = \frac{1}{\sqrt{5}}$$

04.11.25.0002.01

$$\lim_{\nu \to \infty} \frac{F_{\alpha+\nu}}{F_\nu} = \phi^\alpha$$

04.11.25.0003.01

$$\lim_{\nu \to \infty} \frac{\sum_{k=0}^{m-1} F_{\nu+k}}{F_{m+\nu} - F_\nu} = \phi /; m \in \mathbb{N}^+$$

Representations through more general functions

Through hypergeometric functions

Involving $_2F_1$

04.11.26.0001.01

$$F_\nu = \frac{\nu}{2} \cos^2\left(\frac{\pi \nu}{2}\right) {}_2F_1\left(1 - \frac{\nu}{2}, \frac{\nu}{2} + 1; \frac{3}{2}; -\frac{1}{4}\right) + \sin^2\left(\frac{\pi \nu}{2}\right) {}_2F_1\left(\frac{1-\nu}{2}, \frac{\nu+1}{2}; \frac{1}{2}; -\frac{1}{4}\right)$$

04.11.26.0002.01

$$F_\nu = (1 - \theta(-\nu)\, \delta(\sin(\nu \pi)))\, {}_2F_1\left(\frac{1-\nu}{2}, 1 - \frac{\nu}{2}; 1 - \nu; -4\right) - (1 - \theta(\nu)\, \delta(\sin(\nu \pi))) \cos(\nu \pi)\, {}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu}{2} + 1; \nu + 1; -4\right)$$

04.11.26.0003.01

$$F_\nu = {}_2F_1\left(\frac{1-\nu}{2}, 1 - \frac{\nu}{2}; 1 - \nu; -4\right) - \cos(\pi \nu)\, {}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu}{2} + 1; \nu + 1; -4\right) /; \nu \notin \mathbb{Z}$$

04.11.26.0004.01

$$F_\nu = \frac{1}{2} e^{\frac{i\pi\nu}{2}} \left(\nu \left(\frac{1}{2} i \sin(\pi\nu) - \cos(\pi\nu)\right) {}_2F_1\left(1 - \frac{\nu}{2}, \frac{\nu}{2} + 1; \frac{3}{2}; \frac{5}{4}\right) + \frac{\sin(\pi\nu)}{\sqrt{5}} {}_2F_1\left(\frac{\nu+1}{2}, \frac{1-\nu}{2}; \frac{1}{2}; \frac{5}{4}\right)\right)$$

04.11.26.0005.01

$$F_\nu = \frac{2+i}{10} e^{\frac{i\pi\nu}{2}} \left((2-i)\,\nu\,(2\,i\cos(\pi\nu) + \sin(\pi\nu))\, {}_2F_1\left(1-\nu, 1+\nu; \frac{3}{2}; \frac{2-i}{4}\right) - i\sqrt{2-i}\,\sin(\pi\nu)\, {}_2F_1\left(\frac{1}{2}-\nu, \frac{1}{2}+\nu; \frac{1}{2}; \frac{2-i}{4}\right)\right)$$

04.11.26.0006.01

$$F_\nu = \frac{1}{10} e^{-\frac{i\pi\nu}{2}} \left(\sqrt{2+i}\,(1+2\,i)\sin(\pi\nu)\, {}_2F_1\left(\frac{1}{2}-\nu, \nu+\frac{1}{2}; \frac{1}{2}; \frac{2+i}{4}\right) + 5\,\nu\,(\sin(\pi\nu) - 2\,i\cos(\pi\nu))\, {}_2F_1\left(1-\nu, \nu+1; \frac{3}{2}; \frac{2+i}{4}\right)\right)$$

04.11.26.0007.01

$$F_n = \frac{n}{2^{n-1}}\, {}_2F_1\left(\frac{1-n}{2}, 1-\frac{n}{2}; \frac{3}{2}; 5\right) /; n \in \mathbb{Z}$$

04.11.26.0008.01

$$F_n = {}_2F_1\left(\frac{1-n}{2}, 1-\frac{n}{2}; 1-n; -4\right) /; n-1 \in \mathbb{N}^+$$

04.11.26.0009.01

$$F_n = 2^{2\left\lfloor\frac{n-1}{2}\right\rfloor+1-n} (-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor} n\, {}_2F_1\left(-\left\lfloor\frac{n-1}{2}\right\rfloor, \left\lfloor\frac{n}{2}\right\rfloor+1; \frac{3}{2}; \frac{5}{4}\right) /; n \in \mathbb{Z}$$

04.11.26.0010.01

$$F_n = \left(\frac{5}{4}\right)^{\left\lfloor\frac{n-1}{2}\right\rfloor} \left(\frac{n}{2}\right)^{n-2\left\lfloor\frac{n-1}{2}\right\rfloor-1} {}_2F_1\left(-\left\lfloor\frac{n-1}{2}\right\rfloor, -\left\lfloor\frac{n-1}{2}\right\rfloor-\frac{1}{2}; 1+\frac{(-1)^n}{2}; \frac{1}{5}\right) /; n \in \mathbb{Z}$$

04.11.26.0011.01

$$F_n = \left(\frac{4}{5}\right)^{\left\lfloor\frac{n}{2}\right\rfloor+1} \left(\frac{n}{2}\right)^{n-2\left\lfloor\frac{n-1}{2}\right\rfloor-1} {}_2F_1\left(\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}, \left\lfloor\frac{n}{2}\right\rfloor+1; 1+\frac{(-1)^n}{2}; \frac{1}{5}\right) /; n \in \mathbb{Z}$$

04.11.26.0012.01

$$F_n = 5^{\left\lfloor\frac{n-1}{2}\right\rfloor} {}_2F_1\left(-\left\lfloor\frac{n-1}{2}\right\rfloor, -\left\lfloor\frac{n-1}{2}\right\rfloor-\frac{1}{2}; 1-n; \frac{4}{5}\right) /; n-1 \in \mathbb{N}^+$$

04.11.26.0013.01

$$F_n = \left(\frac{n}{2}\right)^{n-2\left\lfloor\frac{n-1}{2}\right\rfloor-1} {}_2F_1\left(-\left\lfloor\frac{n-1}{2}\right\rfloor, \left\lfloor\frac{n}{2}\right\rfloor+1; 1+\frac{(-1)^n}{2}; -\frac{1}{4}\right) /; n \in \mathbb{Z}$$

04.11.26.0014.01

$$F_n = \frac{2}{\sqrt{5}} \left(\frac{n}{2}\right)^{n-2\left\lfloor\frac{n-1}{2}\right\rfloor-1} {}_2F_1\left(-\left\lfloor\frac{n-1}{2}\right\rfloor-\frac{1}{2}, \left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}; 1+\frac{(-1)^n}{2}; -\frac{1}{4}\right) /; n \in \mathbb{Z}$$

04.11.26.0016.01

$$F_n = \left(\frac{3}{2}\right)^{\left\lfloor\frac{n-1}{2}\right\rfloor} \left(\frac{6}{5}\right)^{\frac{n}{2}-\left\lfloor\frac{n}{2}\right\rfloor} \left(\frac{n}{2}\right)^{n-2\left\lfloor\frac{n-1}{2}\right\rfloor-1} {}_2F_1\left(\frac{1+3(-1)^n}{8} - \frac{\left\lfloor\frac{n}{2}\right\rfloor}{2}, \frac{5+3(-1)^n}{8} - \frac{\left\lfloor\frac{n}{2}\right\rfloor}{2}; 1+\frac{(-1)^n}{2}; \frac{5}{9}\right) /; n \in \mathbb{Z}$$

04.11.26.0017.01

$$F_n = \left(\frac{2}{3}\right)^{-\frac{n}{2}+2\left\lfloor\frac{n}{2}\right\rfloor+1} \left(\frac{4}{5}\right)^{\frac{n}{2}-\left\lfloor\frac{n}{2}\right\rfloor} \left(\frac{n}{2}\right)^{n-2\left\lfloor\frac{n-1}{2}\right\rfloor-1} {}_2F_1\left(\frac{7+(-1)^n}{8} + \frac{\left\lfloor\frac{n}{2}\right\rfloor}{2}, \frac{3+(-1)^n}{8} + \frac{\left\lfloor\frac{n}{2}\right\rfloor}{2}; 1+\frac{(-1)^n}{2}; \frac{5}{9}\right) /; n \in \mathbb{Z}$$

04.11.26.0018.01

$$F_n = (-i)^{n-1}\, n\, {}_2F_1\left(1-n, n+1; \frac{3}{2}; \frac{2-i}{4}\right) /; n \in \mathbb{Z}$$

Involving ${}_pF_q$

04.11.26.0019.01

$$F_{2n+1}^2 = (2n+1)^2\, {}_3F_2\left(-2n, 1, 2n+2; \frac{3}{2}, 2; \frac{5}{4}\right) /; n \in \mathbb{Z}$$

04.11.26.0020.01

$$F_{2n}^2 = \frac{4}{5} n^2\, {}_3F_2\left(1-2n, 1, 2n+1; \frac{3}{2}, 2; -\frac{1}{4}\right) /; n \in \mathbb{Z}$$

Through Meijer *G*

Classical cases for the direct function itself

04.11.26.0021.01

$$F_\nu = \frac{\sin(\pi\nu)}{2\sqrt{\pi}}\, G_{3,3}^{2,2}\left(\frac{1}{4} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1-\nu}{2}, \frac{\nu}{2} \\ 0, \frac{1}{2}, \frac{\nu}{2} \end{matrix}\right) /; \nu \notin \mathbb{Z}$$

04.11.26.0022.01

$$F_\nu = \frac{1}{2^\nu \sqrt{\pi}}\, G_{2,2}^{1,2}\left(4 \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{\nu}{2} \\ 0, \nu \end{matrix}\right) - \frac{\cos(\nu\pi)}{2^{-\nu}\sqrt{\pi}}\, G_{2,2}^{1,2}\left(4 \,\middle|\, \begin{matrix} \frac{1-\nu}{2}, -\frac{\nu}{2} \\ 0, -\nu \end{matrix}\right) /; \nu \notin \mathbb{Z}$$

Generalized cases for the direct function itself

04.11.26.0023.01

$$F_\nu = \frac{\sin(\pi\nu)}{2\sqrt{\pi}}\, G_{3,3}^{2,2}\left(\frac{1}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{1-\nu}{2}, \frac{\nu}{2} \\ 0, \frac{1}{2}, \frac{\nu}{2} \end{matrix}\right) /; \nu \notin \mathbb{Z}$$

Through other functions

Involving some hypergeometric-type functions

04.11.26.0024.01

$$F_\nu = F_\nu(1)$$

04.11.26.0025.01

$$F_n = i^{n-1}\, U_{n-1}\left(-\frac{i}{2}\right) /; n \in \mathbb{N}$$

Representations through equivalent functions

With elementary functions

04.11.27.0001.01

$$F_\nu = \frac{2\, e^{\nu \log(w)} - e^{(\pi i - \log(w))\nu} - e^{-(\pi i + \log(w))\nu}}{2\sqrt{5}} /; w = \frac{1+\sqrt{5}}{2}$$

04.11.27.0002.01

$$F_\nu = \frac{2\phi^\nu - e^{\nu(-i\pi-\log(\phi))} - e^{\nu(i\pi-\log(\phi))}}{2\sqrt{5}}$$

04.11.27.0003.01

$$F_\nu = \frac{2\phi^\nu - e^{\nu(-i\pi-\log(\phi))} - e^{\nu(i\pi-\log(\phi))}}{2\sqrt{5}}$$

04.11.27.0004.01

$$F_\nu = \frac{1}{\sqrt{5}} \exp\left(-\nu\,\mathrm{csch}^{-1}(2)\right)\left(\exp\left(2\nu\,\mathrm{csch}^{-1}(2)\right) - \cos(\pi\nu)\right)$$

04.11.27.0005.01

$$F_\nu = -\frac{i}{\sqrt{5}} e^{\frac{i\pi\nu}{2}} \left(\sin(\pi\nu)\cos\left(\nu\sin^{-1}\left(\frac{\sqrt{5}}{2}\right)\right) - (2\cos(\pi\nu) - i\sin(\pi\nu))\sin\left(\nu\sin^{-1}\left(\frac{\sqrt{5}}{2}\right)\right)\right)$$

04.11.27.0006.01

$$F_\nu = \frac{1}{\sqrt{5}}((1-\cos(\pi\nu))\cosh(\nu\log(\phi)) + (\cos(\pi\nu)+1)\sinh(\nu\log(\phi)))$$

04.11.27.0007.01

$$F_\nu = \frac{1}{\sqrt{5}}\left(2\sin\left(\frac{\pi\nu}{2}\right)\sin\left(\nu\csc^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) + \left(1+e^{i\pi\nu}\right)\sinh\left(\nu\,\mathrm{csch}^{-1}(2)\right)\right)$$

04.11.27.0008.01

$$F_\nu = \frac{e^{\frac{i\pi\nu}{2}}}{\sqrt{5}}\left((2i\cos(\pi\nu)+\sin(\pi\nu))\sin\left(2\nu\sin^{-1}\left(\frac{\sqrt{2-i}}{2}\right)\right) - i\sin(\pi\nu)\cos\left(2\nu\sin^{-1}\left(\frac{\sqrt{2-i}}{2}\right)\right)\right)$$

04.11.27.0009.01

$$F_\nu = \frac{1}{\sqrt{5}} e^{-\frac{i\pi\nu}{2}}\left(i\sin(\pi\nu)\cos\left(2\nu\sin^{-1}\left(\frac{\sqrt{2+i}}{2}\right)\right) + (\sin(\pi\nu) - 2i\cos(\pi\nu))\sin\left(2\nu\sin^{-1}\left(\frac{\sqrt{2+i}}{2}\right)\right)\right)$$

04.11.27.0010.01

$$F_n = \frac{i^{n-1}\sin(nz)}{\sin(z)} /;\ z = i\log\left(\frac{\sqrt{5}+1}{2}\right) + \frac{\pi}{2} \bigwedge n \in \mathbb{Z}$$

With Lucas numbers

04.11.27.0011.01

$$F_\nu = \frac{1}{5}(L_{\nu-1} + L_{\nu+1})$$

04.11.27.0012.01

$$F_n = \frac{(-1)^m(2L_{n-m+1} - L_{n-m}) + 2L_{m+n+1} - L_{m+n}}{5L_m} /;\ m \in \mathbb{Z} \land n \in \mathbb{Z}$$

04.11.27.0013.01

$$F_\nu = \frac{2L_{2\nu+1} - L_{2\nu}}{5L_\nu} - \frac{\phi^{-2\nu}\sin^2(\pi\nu)}{\sqrt{5}\,L_\nu}$$

Other identities

Identities involving determinants

04.11.32.0001.01

$$F_n == \left| \left(\begin{cases} 1 & \text{if } k == l \\ i & \text{if } |k-l| == 1 \\ 0 & \text{else} \end{cases} \right)_{\substack{1 \leq k \leq n \\ 1 \leq l \leq n}} \right|$$

Theorems

Zeckendorf theorem

Every positive integer can be decomposed in a unique way as a sum of Fibonacci numbers, such that no two of these numbers are consecutive in the Fibonacci sequence.

Fibonacci substitution

After acting on A n times with the Fibonacci substitution $\{A \to A\,B,\, B \to A\}$ the resulting sequence contains F_{n+1} As and F_n Bs .

A transcendental number

$\sum_{n=1}^{\infty} F_n^{-2}$ is a transcendental number.

The numbers of primary and secondary spirals in the positions of leaves

The numbers of primary and secondary spirals in the positions of leaves or scales along a plant stem are nearly always two consecutive Fibonacci numbers.

Hirmer's conjecture

The number of the largest set of nonintersecting circles arranged along the circumference of a given circle and angle $2\,\pi\,(1 - \text{GoldenRatio})$ between consecutive midpoints is given by the Fibonacci numbers F_n.

History

– J. Kepler (1608)
– A. Girard (1634); R. Simpson (1753)
– É. Léger (1837)
– É. Lucas (1870, 1876–1880)
– G.H. Hardy and E.M. Wright (1938)

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