Fibonacci2

Notations

Traditional name

Fibonacci polynomial

Traditional notation

$F_n(z)$

Mathematica StandardForm notation

Fibonacci[n, z]

Primary definition

05.12.02.0001.01

$$F_n(z) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} z^{n-2k-1} \;/;\; n \in \mathbb{N}$$

Specific values

Specialized values

For fixed n

05.12.03.0001.01

$$F_n(0) = \sin^2\left(\frac{\pi n}{2}\right)$$

05.12.03.0002.01

$$F_n(1) = F_n$$

05.12.03.0003.01

$$F_n(-1) = F_{-n}$$

05.12.03.0018.01

$$F_n(4) = \frac{1}{2} F_{3n}$$

05.12.03.0019.01

$$F_n(11) = \frac{1}{5} F_{5n}$$

05.12.03.0020.01

$$F_n(29) = \frac{1}{13} F_{7n}$$

05.12.03.0021.01

$$F_n(76) = \frac{1}{34} F_{9n}$$

05.12.03.0022.01

$$F_n(199) = \frac{1}{89} F_{11n}$$

05.12.03.0023.01

$$F_n(521) = \frac{1}{233} F_{13n}$$

05.12.03.0024.01

$$F_n(1364) = \frac{1}{610} F_{15n}$$

05.12.03.0025.01

$$F_n(3571) = \frac{1}{1597} F_{17n}$$

05.12.03.0026.01

$$F_n\left(U_{p-1}\left(\frac{\sqrt{5}}{2}\right)\right) = \frac{F_{n p}}{F_p} \;/;\; n \in \mathbb{N}^+ \land p \in \mathbb{N}^+$$

05.12.03.0004.01

$$F_n(-2 i) = -(-i)^{n+1} n$$

05.12.03.0005.01

$$F_n(2 i) = -i^{n+1} n$$

For fixed z

05.12.03.0006.01

$$F_0(z) = 0$$

05.12.03.0007.01

$$F_1(z) = 1$$

05.12.03.0008.01

$$F_2(z) = z$$

05.12.03.0009.01

$$F_3(z) = z^2 + 1$$

05.12.03.0010.01

$$F_4(z) = z^3 + 2 z$$

05.12.03.0011.01

$$F_5(z) = z^4 + 3 z^2 + 1$$

05.12.03.0012.01

$$F_6(z) = z^5 + 4 z^3 + 3 z$$

05.12.03.0013.01

$$F_7(z) == z^6 + 5 z^4 + 6 z^2 + 1$$

05.12.03.0014.01

$$F_8(z) == z^7 + 6 z^5 + 10 z^3 + 4 z$$

05.12.03.0015.01

$$F_9(z) == z^8 + 7 z^6 + 15 z^4 + 10 z^2 + 1$$

05.12.03.0016.01

$$F_{10}(z) == z^9 + 8 z^7 + 21 z^5 + 20 z^3 + 5 z$$

General characteristics

Domain and analyticity

The function $F_n(z)$is defined over $\mathbb{N} \otimes \mathbb{C}$. For fixed n, the function $F_n(z)$ is a polynomial in z of degree n.

05.12.04.0001.01

$$(n * z) \longrightarrow F_n(z) :: (\mathbb{N} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

05.12.04.0002.01

$$F_n(-z) == (-1)^{n-1} F_n(z)$$

Mirror symmetry

05.12.04.0003.01

$$F_n(\bar{z}) == \overline{F_n(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

The function $F_n(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.12.04.0004.01

$$\mathcal{S}ing_z(F_n(z)) == \{\{\tilde{\infty}, n\}\}$$

Branch points

With respect to z

The function $F_n(z)$ does not have branch points.

05.12.04.0005.01

$$\mathcal{BP}_z(F_n(z)) == \{\}$$

Branch cuts

With respect to z

The function $F_n(z)$ does not have branch cuts.

05.12.04.0006.01

$$\mathcal{BC}_z(F_n(z)) == \{\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

05.12.06.0018.01

$$F_n(z) \propto F_n(z_0) + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-j-1}{j} z_0^{n-2j-2} (z-z_0) \left(1 + \frac{z-z_0}{2 z_0} + \ldots\right) /; (z \to z_0)$$

05.12.06.0019.01

$$F_n(z) \propto F_n(z_0) + \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-j-1}{j} z_0^{n-2j-2} (z-z_0) \left(1 + \frac{z-z_0}{2 z_0} + O\left((z-z_0)^2\right)\right)$$

05.12.06.0020.01

$$F_n(z) == \sqrt{\pi} \sum_{k=0}^{n} \frac{1}{k!} \left(2^{k-2} n \cos^2\left(\frac{\pi n}{2}\right) z_0^{1-k} \, {}_3\tilde{F}_2\left(1, 1-\frac{n}{2}, \frac{n}{2}+1; 1-\frac{k}{2}, \frac{3-k}{2}; -\frac{z_0^2}{4}\right) + 2^k \sin^2\left(\frac{\pi n}{2}\right) z_0^{-k} \, {}_3\tilde{F}_2\left(1, \frac{1-n}{2}, \frac{n+1}{2}; \frac{1-k}{2}, 1-\frac{k}{2}; -\frac{z_0^2}{4}\right)\right) (z-z_0)^k$$

05.12.06.0021.01

$$F_n(z) == \sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-j-1}{j} (n-2j-k)_k \, z_0^{n-2j-k-1} (z-z_0)^k$$

05.12.06.0022.01

$$F_n(z) \propto F_n(z_0) \, (1 + O(z-z_0))$$

Expansions at $z == 0$

05.12.06.0001.01

$$F_n(z) \propto \sin^2\left(\frac{\pi n}{2}\right) + \frac{n}{2} \cos^2\left(\frac{\pi n}{2}\right) z + \frac{n^2-1}{8} \sin^2\left(\frac{\pi n}{2}\right) z^2 + \frac{\left(n^2-4\right) n}{48} \cos^2\left(\frac{\pi n}{2}\right) z^3 + \ldots /; (z \to 0)$$

05.12.06.0023.01

$$F_n(z) \propto \sin^2\left(\frac{\pi n}{2}\right) + \frac{n}{2} \cos^2\left(\frac{\pi n}{2}\right) z + \frac{n^2-1}{8} \sin^2\left(\frac{\pi n}{2}\right) z^2 + \frac{\left(n^2-4\right) n}{48} \cos^2\left(\frac{\pi n}{2}\right) z^3 + O\left(z^4\right)$$

05.12.06.0002.01

$$F_n(z) = \sin^2\left(\frac{\pi n}{2}\right) \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(\frac{n+1}{2}\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(-\frac{z^2}{4}\right)^k + \frac{n z}{2} \cos^2\left(\frac{\pi n}{2}\right) \sum_{k=0}^{\left\lfloor \frac{n}{2}-1 \right\rfloor} \frac{\left(1-\frac{n}{2}\right)_k \left(\frac{n}{2}+1\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(-\frac{z^2}{4}\right)^k$$

05.12.06.0003.01

$$F_n(z) = \sin^2\left(\frac{\pi n}{2}\right) {}_2F_1\left(\frac{1-n}{2}, \frac{n+1}{2}; \frac{1}{2}; -\frac{z^2}{4}\right) + \frac{n z}{2} \cos^2\left(\frac{\pi n}{2}\right) {}_2F_1\left(1-\frac{n}{2}, \frac{n}{2}+1; \frac{3}{2}; -\frac{z^2}{4}\right)$$

05.12.06.0004.01

$$F_n(z) \propto \sin^2\left(\frac{\pi n}{2}\right) + \frac{n}{2} \cos^2\left(\frac{\pi n}{2}\right) z (1 + O[z]) /; (z \to 0)$$

05.12.06.0005.01

$$F_n(z) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} z^{n-2k-1}$$

Expansions at $z = 2 i$

05.12.06.0006.01

$$F_n(z) \propto -n\, i^{n+1} \left(1 - \frac{i\left(n^2-1\right)}{6} (z - 2 i) - \frac{\left(n^2-1\right)\left(n^2-4\right)}{120} (z - 2 i)^2 - \ldots\right) /; (z \to 2 i)$$

05.12.06.0024.01

$$F_n(z) \propto -n\, i^{n+1} \left(1 - \frac{i\left(n^2-1\right)}{6} (z - 2 i) - \frac{\left(n^2-1\right)\left(n^2-4\right)}{120} (z - 2 i)^2 + O\left((z - 2 i)^3\right)\right)$$

05.12.06.0007.01

$$F_n(z) = -n\, i^{n+1} \sum_{k=0}^{n-1} \frac{(1-n)_k (n+1)_k}{k! \left(\frac{3}{2}\right)_k} \left(\frac{i}{4}\right)^k (z - 2 i)^k$$

05.12.06.0008.01

$$F_n(z) = -n\, i^{n+1}\, {}_2F_1\left(1-n, n+1; \frac{3}{2}; \frac{1}{4} i (z - 2 i)\right)$$

05.12.06.0009.01

$$F_n(z) \propto -n\, i^{n+1} (1 + O(z - 2 i)) /; (z \to 2 i)$$

Expansions at $z = -2 i$

05.12.06.0010.01

$$F_n(z) \propto -(-i)^{1+n} n \left(1 + \frac{i}{6} \left(n^2-1\right) (z + 2 i) - \frac{1}{120} \left(n^2-1\right)\left(n^2-4\right) (z + 2 i)^2 + \ldots\right) /; (z \to -2 i)$$

05.12.06.0025.01

$$F_n(z) \propto -(-i)^{1+n} n \left(1 + \frac{i}{6} \left(n^2-1\right) (z + 2 i) - \frac{1}{120} \left(n^2-1\right)\left(n^2-4\right) (z + 2 i)^2 + O\left((z + 2 i)^3\right)\right)$$

05.12.06.0011.01

$$F_n(z) = -n (-i)^{n+1} \sum_{k=0}^{n-1} \frac{(1-n)_k (n+1)_k}{k! \left(\frac{3}{2}\right)_k} \left(-\frac{i}{4}\right)^k (z + 2 i)^k$$

05.12.06.0012.01

$$F_n(z) = -n\,(-i)^{n+1}\,{}_2F_1\left(1-n,\, n+1;\, \frac{3}{2};\, \frac{-i\,(z+2\,i)}{4}\right)$$

05.12.06.0013.01

$$F_n(z) \propto -n\,(-i)^{n+1}\,(1+O(z+2\,i)) \;/;\; (z \to -2\,i)$$

Expansions at $z = \infty$

05.12.06.0014.01

$$F_n(z) \propto z^{n-1}\left(1+\frac{n-2}{z^2}+\frac{(n-4)\,(n-3)}{2\,z^4}+\ldots\right) /;\; (|z| \to \infty)$$

05.12.06.0026.01

$$F_n(z) \propto z^{n-1}\left(1+\frac{n-2}{z^2}+\frac{(n-4)\,(n-3)}{2\,z^4}+O\left(\frac{1}{z^6}\right)\right)$$

05.12.06.0015.01

$$F_n(z) = \frac{2^{1-n}\sqrt{\pi}\;z^{n-1}}{\Gamma\left(\frac{n}{2}\right)}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!\,\Gamma\left(\frac{n+1}{2}-k\right)}\left(1-\frac{n}{2}\right)_k\left(-\frac{4}{z^2}\right)^k$$

05.12.06.0016.01

$$F_n(z) = z^{n-1}\,{}_2F_1\left(\frac{1-n}{2},\, 1-\frac{n}{2};\, 1-n;\, -\frac{4}{z^2}\right) /;\; n > 0$$

05.12.06.0017.01

$$F_n(z) \propto z^{n-1}\left(1+O\left(\frac{1}{z^2}\right)\right) /;\; (|z| \to \infty)$$

Integral representations

On the real axis

05.12.07.0001.01

$$F_n(z) = \frac{n}{2^n}\int_{-1}^{1}\left(\sqrt{z^2+4}\;x+z\right)^{n-1} dx \;/;\; n \in \mathbb{Z}$$

Of negative order

05.12.07.0002.01

$$F_n(z) = \frac{n!}{(2\,n-1)!}\,\frac{1}{\sqrt{z^2+4}}\,\frac{\partial^{n-1}\left(z^2+4\right)^{n-\frac{1}{2}}}{\partial z^{n-1}} \;/;\; n \in \mathbb{N}^+$$

Generating functions

05.12.11.0001.01

$$F_n(z) = \left([t^n]\,\frac{t}{1-z\,t-t^2}\right)$$

05.12.11.0002.01

$$F_n(z) = n! \left([t^n] \left(\frac{e^{\frac{t z}{2}}}{\sqrt{z^2+4}} \left(e^{\frac{1}{2} t \sqrt{z^2+4}} - e^{\frac{1}{2} (-t) \sqrt{z^2+4}} \right) \right) \right)$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

05.12.13.0001.01

$$(z^2+4)\, w''(z) + 3\, z\, w'(z) + (1-n^2)\, w(z) = 0 \;/;\; w(z) = c_1\, F_n(z) + \frac{c_2}{\sqrt[4]{z^2+4}}\, P_{n-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i\, z}{2}\right)$$

05.12.13.0002.02

$$W_z\left(F_n(z), \frac{1}{\sqrt[4]{z^2+4}}\, P_{n-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i\, z}{2}\right)\right) = -\frac{e^{-\frac{1}{2} i n \pi} \left(3 + e^{2 i n \pi}\right) n}{\sqrt{\pi}\, (z^2+4)^{3/2}}$$

05.12.13.0003.01

$$w''(z) + \left(\frac{3\, g(z)\, g'(z)}{g(z)^2+4} - \frac{g''(z)}{g'(z)}\right) w'(z) + \frac{(1-n^2)\, g'(z)^2}{g(z)^2+4}\, w(z) = 0 \;/;\; w(z) = c_1\, F_n(g(z)) + c_2\, \frac{1}{\sqrt[4]{g(z)^2+4}}\, P_{n-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i}{2}\, g(z)\right)$$

05.12.13.0004.01

$$W_z\left(F_n(g(z)), \frac{1}{\sqrt[4]{g(z)^2+4}}\, P_{n-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}\, i\, g(z)\right)\right) = -\frac{e^{-\frac{1}{2} i n \pi} \left(3 + e^{2 i n \pi}\right) n\, g'(z)}{\sqrt{\pi}\, (g(z)^2+4)^{3/2}}$$

05.12.13.0005.01

$$w''(z) + \left(\frac{3\, g(z)\, g'(z)}{g(z)^2+4} - \frac{2\, h'(z)}{h(z)} - \frac{g''(z)}{g'(z)}\right) w'(z) + \left(\frac{(1-n^2)\, g'(z)^2}{g(z)^2+4} - \frac{3\, g(z)\, h'(z)\, g'(z)}{(g(z)^2+4)\, h(z)} + \frac{2\, h'(z)^2}{h(z)^2} + \frac{h'(z)\, g''(z)}{h(z)\, g'(z)} - \frac{h''(z)}{h(z)}\right) w(z) = 0 \;/;$$

$$w(z) = c_1\, h(z)\, F_n(g(z)) + c_2\, \frac{h(z)}{\sqrt[4]{g(z)^2+4}}\, P_{n-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i}{2}\, g(z)\right)$$

05.12.13.0006.01

$$W_z\left(h(z)\, F_n(g(z)), \frac{h(z)}{\sqrt[4]{g(z)^2+4}}\, P_{n-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}\, i\, g(z)\right)\right) = -\frac{e^{-\frac{1}{2} i n \pi} \left(3 + e^{2 i n \pi}\right) n\, h(z)^2\, g'(z)}{\sqrt{\pi}\, (g(z)^2+4)^{3/2}}$$

05.12.13.0007.01

$$z^2 (a^2 z^{2r} + 4)\, w''(z) + \left(a^2 (2r - 2s + 1)\, z^{2r} - 4 (r + 2s - 1)\right) z\, w'(z) + \left(a^2 \left((r-s)^2 - r^2 n^2\right) z^{2r} + 4 s (r+s)\right) w(z) = 0 \;/;$$

$$w(z) = c_1\, z^s\, F_n(a z^r) + c_2\, z^s\, \frac{1}{\sqrt[4]{a^2 z^{2r} + 4}}\, P_{n-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}\, i\, a\, z^r\right)$$

05.12.13.0008.01

$$W_z\left(z^s F_n(a z^r),\ \frac{z^s}{\sqrt[4]{a^2 z^{2r}+4}} P_{n-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2} i a z^r\right)\right) = -\frac{a e^{-\frac{1}{2} i \pi n} \left(3+e^{2 i \pi n}\right) r z^{r+2s-1} n}{\sqrt{\pi} \left(a^2 z^{2r}+4\right)^{3/2}}$$

05.12.13.0009.01

$$w''(z) + \frac{-2 a^2 (\log(s)-\log(r)) r^{2z} - 4 (\log(r)+2\log(s))}{a^2 r^{2z}+4} w'(z) + \frac{1}{a^2 r^{2z}+4} \left(4 \log(s) (\log(r)+\log(s)) - a^2 r^{2z} \left(\left(n^2-1\right) \log^2(r) + 2 \log(s) \log(r) - \log^2(s)\right)\right) w(z) = 0 /;$$

$$w(z) = c_1 s^z F_n(a r^z) + c_2 \frac{s^z}{\sqrt[4]{a^2 r^{2z}+4}} P_{n-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2} i a r^z\right)$$

05.12.13.0010.01

$$W_z\left(s^z F_n(a r^z),\ \frac{s^z}{\sqrt[4]{a^2 r^{2z}+4}} P_{n-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{i a r^z}{2}\right)\right) = -\frac{a e^{-\frac{1}{2} i \pi n} \left(3+e^{2 i \pi n}\right) r^z s^{2z} n \log(r)}{\sqrt{\pi} \left(a^2 r^{2z}+4\right)^{3/2}}$$

Identities

Recurrence identities

Consecutive neighbors

05.12.17.0001.01

$$F_n(z) = -z F_{n+1}(z) + F_{n+2}(z)$$

05.12.17.0002.01

$$F_n(z) = z F_{n-1}(z) + F_{n-2}(z)$$

Distant neighbors

05.12.17.0003.01

$$F_n(z) = (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} (-z)^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \left(-z^2\right)^{\frac{1-m}{2}+\left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}\left(-\frac{z^2}{2}-1\right) F_{n+m}(z) + (-1)^{\left\lfloor \frac{m-1}{2} \right\rfloor} (-z)^{1-m+2\left\lfloor \frac{m}{2} \right\rfloor} \left(-z^2\right)^{\left\lfloor \frac{m+1}{2} \right\rfloor-\frac{m}{2}} U_{\frac{m}{2}-1}\left(-\frac{z^2}{2}-1\right) F_{n+m+1}(z) /; m \in \mathbb{N}^+$$

05.12.17.0004.01

$$F_n(z) = (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} z^{m-2\left\lfloor \frac{m}{2} \right\rfloor} \left(-z^2\right)^{\frac{1-m}{2}+\left\lfloor \frac{m}{2} \right\rfloor} U_{\frac{m-1}{2}}\left(-\frac{z^2}{2}-1\right) F_{n-m}(z) + (-1)^{\left\lfloor \frac{m-1}{2} \right\rfloor} z^{1-m+2\left\lfloor \frac{m}{2} \right\rfloor} \left(-z^2\right)^{\left\lfloor \frac{m+1}{2} \right\rfloor-\frac{m}{2}} U_{\frac{m}{2}-1}\left(-\frac{z^2}{2}-1\right) F_{n-m-1}(z) /; m \in \mathbb{N}^+$$

Functional identities

Relations of special kind

05.12.17.0005.01

$$F_{n+1}(z) F_{n-1}(z) - F_n(z)^2 = (-1)^n$$

Differentiation

Low-order differentiation

With respect to *z*

05.12.20.0001.01

$$\frac{\partial F_n(z)}{\partial z} = \frac{2\,n\,F_{n-1}(z) + z\,(n-1)\,F_n(z)}{z^2+4}$$

05.12.20.0002.01

$$\frac{\partial^2 F_n(z)}{\partial z^2} = \frac{4\,(n-1)\,n\,F_{n-2}(z) + 2\,z\,n\,(2\,n-5)\,F_{n-1}(z) + \left((n-2)\,z^2+4\right)(n-1)\,F_n(z)}{\left(z^2+4\right)^2}$$

Symbolic differentiation

With respect to *z*

05.12.20.0005.01

$$\frac{\partial^m F_n(z)}{\partial z^m} = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k} (n-2\,k-m)_m\, z^{n-2\,k-m-1} \;/;\; m \in \mathbb{N}$$

05.12.20.0003.02

$$\frac{\partial^m F_n(z)}{\partial z^m} = 2^{m-2}\sqrt{\pi}\;n\,z^{1-m}\cos^2\!\left(\frac{\pi\,n}{2}\right) {}_3\tilde{F}_2\!\left(1,\,1-\frac{n}{2},\,\frac{n}{2}+1;\,1-\frac{m}{2},\,\frac{3-m}{2};\,-\frac{z^2}{4}\right) + 2^m\sqrt{\pi}\;\sin^2\!\left(\frac{\pi\,n}{2}\right) z^{-m}\,{}_3\tilde{F}_2\!\left(1,\,\frac{1-n}{2},\,\frac{n+1}{2};\,\frac{1-m}{2},\,1-\frac{m}{2};\,-\frac{z^2}{4}\right) /;\; m \in \mathbb{N}$$

Fractional integro-differentiation

With respect to *z*

05.12.20.0004.01

$$\frac{\partial^\alpha F_n(z)}{\partial z^\alpha} = 2^{\alpha-2}\sqrt{\pi}\;z^{-\alpha}\left(z\,n\cos^2\!\left(\frac{\pi\,n}{2}\right){}_3\tilde{F}_2\!\left(1,\,1-\frac{n}{2},\,\frac{n}{2}+1;\,1-\frac{\alpha}{2},\,\frac{3-\alpha}{2};\,-\frac{z^2}{4}\right) + 4\sin^2\!\left(\frac{\pi\,n}{2}\right){}_3\tilde{F}_2\!\left(1,\,\frac{1-n}{2},\,\frac{1+n}{2};\,\frac{1-\alpha}{2},\,1-\frac{\alpha}{2};\,-\frac{z^2}{4}\right)\right)$$

Integration

Indefinite integration

Involving only one direct function

05.12.21.0001.01

$$\int F_n(a\,z)\,dz = \frac{2\,F_{n+1}(a\,z) - z\,F_n(a\,z)}{a\,n}$$

05.12.21.0002.01

$$\int F_n(z)\,dz = \frac{2\,F_{n+1}(z) - z\,F_n(z)}{n}$$

Involving one direct function and elementary functions

Involving power function

05.12.21.0003.01

$$\int z^{\alpha-1}\,F_n(z)\,dz = -\Bigg(2^{\alpha-n-1}\,z^{\alpha}\left(z+\sqrt{z^2+4}\right)^{1-n}\left(-z\left(z+\sqrt{z^2+4}\right)\right)^{-\alpha}$$
$$\Bigg((\alpha+n-1)\,{}_2F_1\!\left(\frac{1}{2}(n-\alpha+1),\,1-\alpha;\,\frac{1}{2}(n-\alpha+3);\,\frac{1}{4}\left(z+\sqrt{z^2+4}\right)^2\right)\left(z+\sqrt{z^2+4}\right)^{2n}+$$
$$4^n\,(n-\alpha+1)\cos(\pi\,n)\,{}_2F_1\!\left(\frac{1}{2}(1-\alpha-n),\,1-\alpha;\,\frac{1}{2}(3-\alpha-n);\,\frac{1}{4}\left(z+\sqrt{z^2+4}\right)^2\right)\Bigg)\Bigg)\Big/((-\alpha+n+1)\,(\alpha+n-1))$$

Summation

Multiple sums

05.12.23.0001.01

$$\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\ldots\sum_{k_p=1}^{n}\delta_{n-\sum_{j=1}^{p}k_j}\prod_{j=1}^{p}F_{k_j+1}(z) = \sum_{j=0}^{\lfloor n/2\rfloor}\binom{p-j+n-1}{j}\binom{p-2\,j+n-1}{p-1}z^{n-2\,j}\;/;\;n\in\mathbb{N}\bigwedge p\in\mathbb{N}^+$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2F_1$

05.12.26.0001.01

$$F_n(z) = \sin^2\!\left(\frac{\pi\,n}{2}\right){}_2F_1\!\left(\frac{1-n}{2},\,\frac{n+1}{2};\,\frac{1}{2};\,-\frac{z^2}{4}\right) + \frac{n\,z}{2}\cos^2\!\left(\frac{\pi\,n}{2}\right){}_2F_1\!\left(1-\frac{n}{2},\,1+\frac{n}{2};\,\frac{3}{2};\,-\frac{z^2}{4}\right)$$

05.12.26.0002.01

$$F_n(z) = \frac{n}{2}\left(\sqrt{-z^2}\,\sin^3\!\left(\frac{\pi\,n}{2}\right) - z\cos^3\!\left(\frac{\pi\,n}{2}\right)\right){}_2F_1\!\left(1-\frac{n}{2},\,\frac{n}{2}+1;\,\frac{3}{2};\,\frac{z^2}{4}+1\right)$$

05.12.26.0003.01

$$F_n(z) = -n\,i^{n+1}\,{}_2F_1\!\left(1-n,\,1+n;\,\frac{3}{2};\,\frac{2+i\,z}{4}\right)$$

05.12.26.0004.01

$$F_n(z) = -n\,(-i)^{n+1}\,{}_2F_1\!\left(\frac{1}{2}-n,\,n+\frac{1}{2};\,\frac{1}{2};\,\frac{2-i\,z}{4}\right)$$

05.12.26.0005.01

$$F_n(z) = z^{n-1}\,{}_2F_1\!\left(\frac{1-n}{2},\,1-\frac{n}{2};\,1-n;\,-\frac{4}{z^2}\right)\;/;\;n>0$$

Involving ${}_pF_q$

05.12.26.0006.01

$$F_n(z) = \frac{1}{2\sqrt{z^2+4}} \left(2\,{}_0F_0\left(;;n\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right) - {}_0F_0\left(;;n\left(-i\pi-\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right)\right) - {}_0F_0\left(;;n\left(i\pi-\log\left(\frac{1}{2}\left(z+\sqrt{z^2+4}\right)\right)\right)\right)\right)$$

Through other functions

Involving some hypergeometric-type functions

05.12.26.0007.01

$$F_n(z) = i^{n-1}\,U_{n-1}\left(-\frac{i\,z}{2}\right)$$

Representations through equivalent functions

With elementary functions

05.12.27.0001.01

$$F_n(z) = \frac{2^{-n}\left(z+\sqrt{z^2+4}\right)^n - (-1)^n\,2^n\left(z+\sqrt{z^2+4}\right)^{-n}}{\sqrt{z^2+4}}$$

05.12.27.0002.01

$$F_n(z) = \frac{e^{n\log(w)} - (-1)^n\,e^{-n\log(w)}}{\sqrt{z^2+4}} \;/;\; w = \frac{1}{2}\left(z+\sqrt{z^2+4}\right)$$

05.12.27.0003.01

$$F_n(z) = -\frac{2\,i^{n+1}}{\sqrt{z^2+4}}\sin\left(n\cos^{-1}\left(-\frac{i\,z}{2}\right)\right)$$

05.12.27.0004.01

$$F_n(z) = \frac{\exp\left(2\,n\sinh^{-1}\left(\frac{z}{2}\right)\right) - (-1)^n}{\exp\left(n\sinh^{-1}\left(\frac{z}{2}\right)\right)\sqrt{z^2+4}}$$

05.12.27.0005.01

$$F_n(z) = \frac{1}{\sqrt{z^2+4}}\left((1-(-1)^n)\cosh\left(n\sinh^{-1}\left(\frac{z}{2}\right)\right) + (1+(-1)^n)\sinh\left(n\sinh^{-1}\left(\frac{z}{2}\right)\right)\right)$$

05.12.27.0006.01

$$F_n(z) = \frac{1}{\sqrt{z^2+4}}\left(2\sin\left(\frac{\pi n}{2}\right)\sin\left(n\sin^{-1}\left(\frac{\sqrt{z^2+4}}{2}\right)\right) + (1+(-1)^n)\sinh\left(n\sinh^{-1}\left(\frac{z}{2}\right)\right)\right)$$

05.12.27.0007.01

$$F_n(z) = \frac{1}{\sqrt{z^2+4}} \left(\frac{\sin(\pi n)}{\sqrt{z}} \left(\sqrt{z} \sin\left(\frac{\pi n}{2}\right) - \sqrt{-z} \cos\left(\frac{\pi n}{2}\right)\right) \cos\left(n \sin^{-1}\left(\frac{\sqrt{z^2+4}}{2}\right)\right) + \frac{2}{\sqrt{-z}} \left(\sqrt{-z} \sin^3\left(\frac{\pi n}{2}\right) - \sqrt{z} \cos^3\left(\frac{\pi n}{2}\right)\right) \sin\left(n \sin^{-1}\left(\frac{\sqrt{z^2+4}}{2}\right)\right)\right)$$

05.12.27.0008.01

$$F_n(z) = \frac{2(-1)^{n+1/4}}{\sqrt{2-i z}\,\sqrt{z-2 i}} e^{-\frac{i \pi n}{2}} \sin\left(2 n \csc^{-1}\left(\frac{2(-1)^{3/4}}{\sqrt{z-2 i}}\right)\right)$$

05.12.27.0009.01

$$F_n(z) = -\frac{2(-i)^{n+1}}{\sqrt{z^2+4}} \sin\left(2 n \sin^{-1}\left(\frac{1}{2}\sqrt{2-i z}\right)\right)$$

05.12.27.0010.01

$$F_n(z) = -\frac{2\, i^{n+1}}{\sqrt{z^2+4}} \sin\left(2 n \sin^{-1}\left(\frac{1}{2}\sqrt{2+i z}\right)\right)$$

Zeros

05.12.30.0001.01

$$F_n(z) = 0 \;/;\; \left(2 i \cos\left(\frac{j \pi}{n}\right) /;\; n \in \mathbb{Z} \land j \in \mathbb{Z}\right)$$

History

– M. Bicknell (1970)

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