HypergeometricPFQ

Notations

Traditional name

Generalized hypergeometric function

Traditional notation

${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$

Mathematica StandardForm notation

`HypergeometricPFQ[{a_1, …, a_p}, {b_1, …, b_q}, z]`

Primary definition

07.31.02.0001.01

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_k \, z^k}{\prod_{j=1}^{q} (b_j)_k \, k!} \;/;$$

$$q \geq p \bigvee q = p-1 \wedge |z| < 1 \bigvee q = p-1 \bigwedge |z| = 1 \bigwedge \operatorname{Re}\left(\sum_{j=1}^{p-1} b_j - \sum_{j=1}^{p} a_j\right) > 0$$

In the cases $q < p - 1$ the series above does not converge but it (together with symbol) can be used as asymptotic series, where, when needed a Borel summation is implicitly understood.

07.31.02.0002.01

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{n} \frac{\prod_{j=1}^{p} (a_j)_k \, z^k}{\prod_{j=1}^{q} (b_j)_k \, k!} \;/;\; \exists_{a_j} \, -a_j = n \in \mathbb{N}$$

For $a_i = -n$, $b_j = -m$ /; $m \geq n$ being nonpositive integers and $\nexists_{a_k} (a_k > -n \wedge a_k \in \mathbb{N}) \bigwedge \nexists_{b_k} (b_k > -m \wedge b_k \in \mathbb{N})$ the function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ cannot be uniquely defined by a limiting procedure based on the above definition because the two variables a_i, b_j can approach nonpositive integers $-n$, $-m$; $m \geq n$ at different speeds. For the above conditions we define:

07.31.02.0003.01

$${}_pF_q(a_1, \ldots, a_i, \ldots, a_p; b_1, \ldots, b_j, \ldots, b_q; z) = \sum_{k=0}^{n} \frac{\prod_{j=1}^{p} (a_j)_k \, z^k}{\prod_{j=1}^{q} (b_j)_k \, k!} \;/;\; a_i = -n \wedge b_j = -m \wedge m \in \mathbb{N} \wedge n \in \mathbb{N} \wedge m \geq n$$

General characteristics

Some abbreviations

07.31.04.0001.01

$$\mathcal{NT}(\{a_1, \ldots, a_p\}) = \neg(-a_1 \in \mathbb{N} \lor \ldots \lor -a_p \in \mathbb{N})$$

Domain and analyticity

${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ is an analytical function of $a_1, \ldots, a_p, b_1, \ldots, b_q$ and z which is defined in $\mathbb{C}^{p+q+1}$. In the cases $p \leq q$ for fixed $a_1, \ldots, a_p, b_1, \ldots, b_q$, it is an entire function of z. If parameters $a_k$include negative integers, the function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ degenerates to a polynomial in z.

07.31.04.0002.01

$$(\{a_1 * \ldots * a_p\} * \{b_1 * \ldots * b_q\} * z) \longrightarrow {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) :: (\{\mathbb{C} \otimes \ldots \otimes \mathbb{C}\} \otimes \{\mathbb{C} \otimes \ldots \otimes \mathbb{C}\} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

07.31.04.0003.02

$${}_pF_q(\overline{a_1}, \ldots, \overline{a_p}; \overline{b_1}, \ldots, \overline{b_q}; \bar{z}) = \overline{{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)} \;/;\; \neg(z \in (1, \infty) \land p = q + 1)$$

Permutation symmetry

07.31.04.0004.01

$${}_pF_q(a_1, a_2, \ldots, a_k, \ldots, a_j, \ldots, a_p; b_1, \ldots, b_q; z) = {}_pF_q(a_1, a_2, \ldots, a_j, \ldots, a_k, \ldots, a_p; b_1, \ldots, b_q; z) \;/;\; a_k \neq a_j \land k \neq j$$

07.31.04.0005.01

$${}_pF_q(a_1, \ldots, a_p; b_1, b_2, \ldots, b_k, \ldots, b_j, \ldots, b_q; z) = {}_pF_q(a_1, \ldots, a_p; b_1, b_2, \ldots, b_j, \ldots, b_k, \ldots, b_q; z) \;/;\; b_k \neq b_j \land k \neq j$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For $p = q + 1$ and fixed a_l, b_j in nonpolynomial cases (when $\neg(-a_1 \in \mathbb{N} \lor \ldots \lor -a_p \in \mathbb{N})$), the function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ does not have poles and essential singularities.

07.31.04.0006.01

$$\mathcal{Sing}_z({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\} \;/;\; p = q + 1 \land \mathcal{NT}(\{a_1, \ldots, a_p\})$$

For $p \leq q$ and fixed a_l, b_j in nonpolynomial cases (when $\neg(-a_1 \in \mathbb{N} \lor \ldots \lor -a_p \in \mathbb{N})$), the function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ has only one singular point at $z = \tilde{\infty}$. It is an essential singular point.

07.31.04.0007.01

$$\mathcal{Sing}_z({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\{\tilde{\infty}, \infty\}\} \;/;\; p \leq q \land \mathcal{NT}(\{a_1, \ldots, a_p\})$$

If parameters a_k include r negative integers α_k, the function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ is a polynomial and has pole of order $\min(-\alpha_1, \ldots, -\alpha_r\}$ at $z = \tilde{\infty}$.

07.31.04.0008.01

$$\mathcal{Sing}_z({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\{\tilde{\infty}, -\alpha\}\} /; \neg\,(\mathcal{NT}(\{a_1, \ldots, a_p\})) \bigwedge \alpha = \min(-a_{s_1}, \ldots, -a_{s_r}\} \bigwedge -a_{s_k} \in \mathbb{N}^+$$

With respect to a_l

The function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ as a function of a_l, $1 \le l \le p$, has only one singular point at $a_l = \tilde{\infty}$. It is an essential singular point.

07.31.04.0009.01

$$\mathcal{Sing}_{a_l}({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\{\tilde{\infty}, \infty\}\} /; 1 \le l \le p$$

With respect to b_j

The function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ as a function of b_j, $1 \le j \le q$, has an infinite set of singular points:

a) $b_j = -k /; k \in \mathbb{N}$, are the simple poles with residues $\frac{(-1)^k}{k!}\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_{j-1}, -k, b_{j+1}, \ldots, b_q; z)$;

b) $b_j = \tilde{\infty}$ is the point of accumulation of poles, which is an essential singular point.

07.31.04.0010.01

$$\mathcal{Sing}_{b_j}({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\{\{-k, 1\} /; k \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\} /; 1 \le j \le q$$

07.31.04.0011.01

$$\operatorname{res}_{b_j}({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z))(-k) = \frac{(-1)^k}{k!}\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_{j-1}, -k, b_{j+1}, \ldots, b_q; z) /; k \in \mathbb{N} \land 1 \le j \le q$$

Branch points

With respect to z

The function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ does not have branch points for $p \le q$ and has two branch points: $z = 1$, $z = \tilde{\infty}$ for $p = q + 1$ in nonpolynomial case (when $\neg\,(-a_1 \in \mathbb{N} \lor \ldots \lor -a_p \in \mathbb{N})$)

07.31.04.0012.01

$$\mathcal{BP}_z({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)) = \{\} /; p \le q$$

07.31.04.0013.01

$$\mathcal{BP}_z({}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z)) = \{1, \tilde{\infty}\} /; \mathcal{NT}(\{a_1, \ldots, a_{q+1}\})$$

07.31.04.0014.01

$$\mathcal{R}_z({}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z), 1) = \log /; \psi_q \in \mathbb{Z} \bigvee \psi_q \notin \mathbb{Q} \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \mathcal{NT}(\{a_1, \ldots, a_{q+1}\})$$

07.31.04.0015.01

$$\mathcal{R}_z({}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z), 1) = s /;$$
$$\psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j = \frac{r}{s} \bigwedge r \in \mathbb{Z} \bigwedge s - 1 \in \mathbb{N}^+ \bigwedge \gcd(r, s) = 1 \bigwedge \mathcal{NT}(\{a_1, \ldots, a_{q+1}\})$$

07.31.04.0016.01

$$\mathcal{R}_z\left({}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z), \tilde{\infty}\right) == \log /; \exists_{a_i, a_j} \left(a_i - a_j \in \mathbb{Z} \land 1 \leq i \leq q+1 \land 1 \leq j \leq q+1 \land i \neq j\right) \bigwedge \left(a_1 \notin \mathbb{Q} \lor \ldots \lor a_{q+1} \notin \mathbb{Q}\right)$$

07.31.04.0017.01

$$\mathcal{R}_z\left({}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z), \tilde{\infty}\right) == \operatorname{lcm}(s_1, \ldots, s_{q+1}) /; a_l == \frac{r_l}{s_l} \bigwedge \{r_l, s_l\} \in \mathbb{Z} \bigwedge s_l > 1 \bigwedge \gcd(r_l, s_l) == 1 \bigwedge 1 \leq l \leq q+1 \bigwedge \mathcal{NT}(\{a_1, \ldots, a_{q+1}\})$$

With respect to a_l

The function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ as a function of a_l, $1 \leq l \leq p$, does not have branch points.

07.31.04.0018.01

$$\mathcal{BP}_{a_l}\left({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right) == \{\} /; 1 \leq l \leq p$$

With respect to b_j

The function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ as a function of b_j, $1 \leq j \leq q$, does not have branch points.

07.31.04.0019.01

$$\mathcal{BP}_{b_j}\left({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right) == \{\} /; 1 \leq j \leq q$$

Branch cuts

With respect to z

For all nonnegative integer a_k, the function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ in the cases $p == q+1$ is a single-valued function on the z-plane cut along the interval $(1, \infty)$, where it is continuous from below. In the cases $p \leq q$ this function does not have branch cuts.

07.31.04.0020.01

$$\mathcal{BC}_z\left({}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z)\right) == \{\{(1, \infty), i\}\} /; \mathcal{NT}(\{a_1, \ldots, a_{q+1}\})$$

07.31.04.0021.01

$$\mathcal{BC}_z\left({}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right) == \{\} /; p \leq q$$

07.31.04.0022.01

$$\lim_{\epsilon \to +0} {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; x - i\epsilon) == {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; x) /; x > 1$$

07.31.04.0026.01

$$\lim_{\epsilon \to +0} {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; x + i\epsilon) == \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} G_{q+1,q+1}^{q+1,1}\left(e^{\pi i} \frac{1}{x} \,\middle|\, \begin{matrix} 1, b_1, \ldots, b_q \\ a_1, \ldots, a_{q+1} \end{matrix}\right) /; x > 1$$

07.31.04.0023.01

$$\lim_{\epsilon\to+0} {}_{q+1}F_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; x + i\,\epsilon\right) = \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k)}{\prod_{j=1}^{q}\Gamma\left(b_j - a_k\right)} \left(\prod_{\substack{j=1\\ j\neq k}}^{q+1}\Gamma\left(a_j - a_k\right)\right) e^{\pi\,i\,a_k}\, x^{-a_k}$$

$$ {}_{q+1}F_q\left(a_k, a_k - b_1 + 1, \ldots, a_k - b_q + 1; 1 - a_1 + a_k, \ldots, 1 - a_{k-1} + a_k, 1 - a_{k+1} + a_k, \ldots, 1 - a_{q+1} + a_k; \frac{1}{x}\right) /;$$

$$\forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1}\left(a_j - a_k \notin \mathbb{Z}\right) \bigwedge x > 1$$

With respect to a_l

The function ${}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right)$ as a function of a_l, $1 \leq l \leq p$, does not have branch cuts.

07.31.04.0024.01

$$\mathcal{BC}_{a_l}\left({}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right)\right) = \{\} /; 1 \leq l \leq p$$

With respect to b_j

The function ${}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right)$ as a function of b_j, $1 \leq j \leq q$, does not have branch cuts.

07.31.04.0025.01

$$\mathcal{BC}_{b_j}\left({}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right)\right) = \{\} /; 1 \leq j \leq q$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

07.31.06.0045.01

$${}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) = \left(\prod_{j=1}^{q}\Gamma\left(b_j\right)\right) \sum_{k=0}^{\infty} \frac{z_0^{-k}}{k!}\, {}_{p+1}F_{q+1}\left(1, a_1, \ldots, a_p; 1 - k, b_1, \ldots, b_q; z_0\right)(z - z_0)^k /;$$

$$p \neq q + 1 \bigvee p == q + 1 \bigwedge z_0 \notin (1, \infty)$$

07.31.06.0046.01

$${}_{q+1}F_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z\right) ==$$

$$\frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\prod_{\substack{j=1\\ j\neq k}}^{q+1}\Gamma\left(a_j - a_k\right)}{\prod_{j=1}^{q}\Gamma\left(b_j - a_k\right)} \left(-\frac{1}{z_0}\right)^{-a_k \left\lfloor \frac{\arg(z_0 - z)}{2\pi} \right\rfloor} (-z_0)^{-a_k\left(\left\lfloor \frac{\arg(z_0 - z)}{2\pi} \right\rfloor + 1\right)} \sum_{j=0}^{\infty} \frac{\Gamma(a_k + j)\,(-z_0)^{-j}}{j!}\, {}_{q+1}F_q$$

$$\left(a_k - b_1 + 1, \ldots, a_k - b_q + 1, j + a_k; 1 - a_1 + a_k, \ldots, 1 - a_{k-1} + a_k, 1 - a_{k+1} + a_k, \ldots, 1 - a_{q+1} + a_k; \frac{1}{z_0}\right)(z - z_0)^j /;$$

$$|z_0| > 1 \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1}\left(a_j - a_k \notin \mathbb{Z}\right)$$

Expansions on branch cuts for $p == q + 1$

For the function itself

07.31.06.0047.01

$$ {}_{q+1}F_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z\right) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{k=0}^{\infty} \frac{1}{k!} G_{q+1,q+1}^{1,q+1}\left(-x\, e^{2\pi i \left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor} \,\middle|\, \begin{matrix} 1-a_1-k, \ldots, 1-a_{q+1}-k \\ 0, 1-b_1-k, \ldots, 1-b_q-k \end{matrix}\right) (z-x)^k \text{ /; } x > 1 $$

07.31.06.0048.01

$$ {}_{q+1}F_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z\right) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{k=0}^{\infty} \frac{1}{k!} G_{q+1,q+1}^{q+1,1}\left(e^{-\pi i \left(1+2\left\lfloor \frac{\arg(x-z)}{2\pi} \right\rfloor\right)} \frac{1}{x} \,\middle|\, \begin{matrix} 1, k+b_1, \ldots, k+b_q \\ k+a_1, \ldots, k+a_{q+1} \end{matrix}\right) (z-x)^k \text{ /; } x > 1 $$

07.31.06.0049.01

$$ {}_{q+1}F_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z\right) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{k=1}^{q+1} \sum_{u=0}^{\infty} \frac{(-x)^{-u}\, \Gamma(u+a_k)}{u!\, \prod_{j=1}^{q} \Gamma(b_j-a_k)} \left(\prod_{\substack{j=1 \\ j \neq k}}^{q+1} \Gamma(a_j-a_k) \right) e^{-\pi i a_k \left(1+2\,\mathrm{Floor}\left[\frac{\arg[x-z]}{2\pi}\right]\right)} x^{-a_k} $$
$$ {}_{q+1}F_q\left(u+a_k, a_k-b_1+1, \ldots, a_k-b_q+1; 1-a_1+a_k, \ldots, 1-a_{k-1}+a_k, 1-a_{k+1}+a_k, \ldots, 1-a_{q+1}+a_k; \frac{1}{x}\right) $$
$$ (z-x)^u \text{ /; } \forall_{\{j,k\},\{j,k\}\in\mathbb{Z} \land j\neq k \land 1\leq j\leq q+1 \land 1\leq k\leq q+1} \left(a_j-a_k \notin \mathbb{Z}\right) \land x > 1 $$

Expansions at $z = 0$

07.31.06.0001.01

$$ {}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) = 1 + \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j} z + \frac{\left(\prod_{j=1}^{p} a_j\, (a_j+1)\right)}{2 \prod_{j=1}^{q} b_j\, (b_j+1)} z^2 + \ldots \text{ /; } q = p-1 \land |z| < 1 \lor q \geq p $$

07.31.06.0002.01

$$ {}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) = \sum_{k=0}^{\infty} \frac{\left(\prod_{j=1}^{p} (a_j)_k\right) z^k}{\left(\prod_{j=1}^{q} (b_j)_k\right) k!} \text{ /; } q = p-1 \land |z| < 1 \lor q \geq p $$

07.31.06.0003.01

$$ {}_pF_q\left(a_1, \ldots, a_p; b_1, \ldots, b_q; z\right) \propto 1 + O(z) $$

Expansions at $z = 1$ for $p = q+1$

The point $z = 1$ is the end point of the branch cut for the function ${}_{q+1}F_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z\right)$, where it has a rather complicated behavior. The corresponding general formula (for noninteger $\psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j$) includes two major terms - regular and singular which are an analytical functions. Moreover, the singular term has representation of the form const $(1-z)^{\psi_q}\, (1+O(z-1))$ and regular term is bounded near point $z = 1$. A more detailed description of this behavior is presented below.

At the singular point $z = 1$ the function ${}_{q+1}F_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z\right)$ is continuous for $\mathrm{Re}\left(\psi_q\right) > 0$, bounded for $\mathrm{Re}\left(\psi_q\right) = 0$, $\psi_q \neq 0$ and has, in general, a logarithmic singularity for $\psi_q = 0$ while for $\mathrm{Re}\left(\psi_q\right) < 0$ it has a power singularity of order $-\psi_q$ to which for integer ψ_qa logarithmic singularity can also occur.

The general formulas

07.31.06.0004.01

$$_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \equiv \left(\prod_{j=1}^{q} \Gamma(b_j)\right) \mathcal{A}_{\tilde{F}}\left(\begin{matrix} a_1, \ldots, a_{q+1}; \\ b_1, \ldots, b_q; \end{matrix} \{z, 1, \infty\}\right)$$

07.31.06.0005.01

$$_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \equiv \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \left((1-z)^{\psi_q} \sum_{k=0}^{\infty} g_k(\psi_q) (1-z)^k + \sum_{k=0}^{\infty} g_k(0) (1-z)^k\right) /;$$

$$|1-z| < 1 \bigwedge q > 1 \bigwedge g_k(r) \equiv \frac{(-1)^k \Gamma(k+r+a_1) \Gamma(k+r+a_2) \Gamma(\psi_q - 2r - k)}{k!}$$

$$\sum_{j=0}^{\infty} \frac{(\psi_q - r - k)_j \mathcal{E}_j^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{\Gamma(j + a_1 + \psi_q) \Gamma(j + a_2 + \psi_q)} \equiv \frac{(-1)^k \Gamma(\psi_q - 2r - k)}{k! \prod_{j=1}^{q} \Gamma(k + r + b_j)} \left(\prod_{j=1}^{q+1} \Gamma(k + r + a_j)\right) \lim_{m \to \infty} \frac{1}{\Gamma(\psi_q - r - k)}$$

$$_{q+1}F_q(k+r+a_1, k+r+a_2, \ldots, n+r+a_{q+1}, -m; k+r+b_1, k+r+b_2, \ldots, n+r+b_q, k-m+r-\psi_q+1; 1) \bigwedge$$

$$g_0(\psi_q) \equiv \Gamma(-\psi_q) \bigwedge \mathcal{E}_{k_1}^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) \equiv \frac{(b_1 - a_3)_{k_1}}{k_1!} \left(\sum_{j=2}^{q} b_j - \sum_{j=3}^{q+1} a_j\right)_{k_1}$$

$$\sum_{k_2=0}^{k_1} \frac{(-k_1)_{k_2}}{\left(\sum_{j=2}^{q} b_j - \sum_{j=3}^{q+1} a_j\right)_{k_2} (a_3 - b_1 - k_1 + 1)_{k_2}} \frac{(b_2 - a_4)_{k_2}}{k_2!} \left(\sum_{j=3}^{q} b_j - \sum_{j=4}^{q+1} a_j\right)_{k_2}$$

$$\sum_{k_3=0}^{k_2} \frac{(-k_2)_{k_3}}{\left(\sum_{j=3}^{q} b_j - \sum_{j=4}^{q+1} a_j\right)_{k_3} (a_4 - b_2 - k_2 + 1)_{k_3}} \ldots \frac{(b_{q-2} - a_q)_{k_{q-2}}}{k_{q-2}!} \left(\sum_{j=q-1}^{q} b_j - \sum_{j=q}^{q+1} a_j\right)_{k_{q-2}}$$

$$\sum_{k_{q-1}=0}^{k_{q-2}} \frac{(-k_{q-2})_{k_{q-1}}}{\left(\sum_{j=q-1}^{q} b_j - \sum_{j=q}^{q+1} a_j\right)_{k_{q-1}} (a_q - b_{q-2} - k_{q-2} + 1)_{k_{q-1}}} \frac{(b_q - a_{q+1})_{k_{q-1}} (b_{q-1} - a_{q+1})_{k_{q-1}}}{k_{q-2}!} \bigwedge$$

$$\psi_q \equiv \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \psi_q \notin \mathbb{Z} \bigwedge \operatorname{Re}(a_3) > 0 \bigwedge \ldots \bigwedge \operatorname{Re}(a_{q+1}) > 0$$

The logarithmic cases

07.31.06.0006.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \left(\sum_{j=0}^{\psi_q - 1} k_j (1-j)^j + (1-z)^{\psi_q} \sum_{j=0}^{\infty} \left(p_j + q_j \log(1-z)\right) (1-z)^j \right) /; $$

$$ |1-z| < 1 \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge $$

$$ \left(k_j = \frac{(-1)^j \Gamma(j+a_1) \Gamma(j+a_2)}{j!} \sum_{k=0}^{\infty} \frac{(k-j+\psi_q-1)! \, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{\Gamma(k+a_1+\psi_q) \Gamma(k+a_2+\psi_q)} /; \operatorname{Re}(a_3) > -j \land \ldots \land \right. $$

$$ \left. \operatorname{Re}(a_{p+1}) > -j \right) \bigwedge \left(p_j = \frac{(-1)^{\psi_q} (a_1+\psi_q)_j (a_2+\psi_q)_j}{j! \, (j+\psi_q)!} \left((-1)^j j! \sum_{k=j+1}^{\infty} \frac{(k-j-1)! \, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1+\psi_q)_k (a_2+\psi_q)_k} + \right. \right. $$

$$ \left. \sum_{k=0}^{j} \frac{(-j)_k \, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1+\psi_q)_k (a_2+\psi_q)_k} \left(\psi(j-k+1) + \psi(j+\psi_q+1) - \psi(j+a_1+\psi_q) - \psi(j+a_2+\psi_q) \right) \right) /; $$

$$ \left. \operatorname{Re}(a_3) > -j - \psi_q \land \ldots \land \operatorname{Re}(a_{p+1}) > -j - \psi_q \right) \bigwedge q_j = \frac{(-1)^{\psi_q - 1} (a_1+\psi_q)_j (a_2+\psi_q)_j}{j! \, (j+\psi_q)!} $$

$$ \sum_{k=0}^{j} \frac{(-j)_k \, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1+\psi_q)_k (a_2+\psi_q)_k} \bigwedge \psi_q \in \mathbb{N} $$

07.31.06.0007.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = $$

$$ \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \left((z-1)^{\psi_q} \sum_{j=0}^{\infty} \frac{(a_1+\psi_q)_j (a_2+\psi_q)_j}{j! \, (j+\psi_q)!} \left(\sum_{k=j+1}^{\infty} \frac{(-1)^j j! \, (k-j-1)!}{(a_1+\psi_q)_k (a_2+\psi_q)_k} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) + \right. \right. $$

$$ \sum_{k=0}^{j} \frac{(-j)_k}{(a_1+\psi_q)_k (a_2+\psi_q)_k} \left(-\log(1-z) + \psi(j-k+1) + \psi(j+\psi_q+1) - \psi(j+a_1+\psi_q) - \psi(j+a_2+\psi_q) \right) $$

$$ \left. \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) \right) (1-z)^j + $$

$$ \left. \sum_{j=0}^{\psi_q - 1} \frac{\Gamma(j+a_1) \Gamma(j+a_2)}{j!} \sum_{k=0}^{\infty} \frac{(k-j+\psi_q-1)! \, \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{\Gamma(k+a_1+\psi_q) \Gamma(k+a_2+\psi_q)} (z-1)^j \right) /; $$

$$ |1-z| < 1 \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge $$

$$ \psi_q \in \mathbb{N} $$

07.31.06.0008.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \left((1-z)^{\psi_q} \sum_{j=0}^{-\psi_q - 1} h_j (1-j)^j + \sum_{j=0}^{\infty} \left(u_j + v_j \log(1-z)\right) (1-z)^j \right) /; |1-z| < 1 \bigwedge $$

$$ \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge h_j = \frac{(-1)^j (a_1 + \psi_q)_j (a_2 + \psi_q)_j (-j - \psi_q - 1)!}{j!} \sum_{k=0}^{j} \frac{(-j)_k \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1 + \psi_q)_k (a_2 + \psi_q)_k} \bigwedge $$

$$ \left(u_j = \frac{(a_1 + \psi_q)_{j-\psi_q} (a_2 + \psi_q)_{j-\psi_q}}{j! (j - \psi_q)!} \left((-1)^j (j - \psi_q)! \sum_{k=j-\psi_q+1}^{\infty} \frac{(k - j + \psi_q - 1)! \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1 + \psi_q)_k (a_2 + \psi_q)_k} + \right.\right. $$

$$ \left. (-1)^{\psi_q} \sum_{k=0}^{j-\psi_q} \frac{(\psi_q - j)_k \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1 + \psi_q)_k (a_2 + \psi_q)_k} \left(\psi(j+1) - \psi(j+a_1) - \psi(j+a_2) + \psi(j - k - \psi_q + 1)\right) \right) /; $$

$$ \left. \operatorname{Re}(a_3) > -j \land \ldots \land \operatorname{Re}(a_{q+1}) > -j \right) \bigwedge v_j = \frac{(-1)^{\psi_q - 1} (a_1 + \psi_q)_{j-\psi_q} (a_2 + \psi_q)_{j-\psi_q}}{j! (j - \psi_q)!} $$

$$ \sum_{k=0}^{j-\psi_q} \frac{(\psi_q - j)_k \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{(a_1 + \psi_q)_k (a_2 + \psi_q)_k} \bigwedge -\psi_q \in \mathbb{N} $$

07.31.06.0009.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = $$

$$ \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \left((1-z)^{\psi_q} \sum_{j=0}^{-\psi_q - 1} (-j - \psi_q - 1)! (a_1 + \psi_q)_j (a_2 + \psi_q)_j \sum_{k=0}^{j} \frac{(-1)^k \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) (z-1)^j}{(j-k)! (a_1 + \psi_q)_k (a_2 + \psi_q)_k} + \right. $$

$$ (-1)^{-\psi_q} \sum_{j=\psi_q}^{\infty} \frac{1}{(j - \psi_q)!} \sum_{k=0}^{\infty} \frac{\Gamma(j+k+a_1) \Gamma(j+k+a_2)}{(j+k)! \Gamma(k + a_1 + \psi_q) \Gamma(k + a_2 + \psi_q)} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) $$

$$ \left(-\log(1-z) + \psi(j+k+1) + \psi(j - \psi_q + 1) - \psi(j+k+a_1) - \psi(j+k+a_2)\right) (1-z)^j + $$

$$ \left. (-1)^{-\psi_q - 1} \sum_{k=0}^{\infty} \sum_{j=-k}^{\psi_q - 1} \frac{\Gamma(j+k+a_1) \Gamma(j+k+a_2)}{(j+k)! \Gamma(k + a_1 + \psi_q) \Gamma(k + a_2 + \psi_q)} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) (1-z)^j \right) /; $$

$$ |1-z| < 1 \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge -\psi_q \in \mathbb{N}^+ $$

07.31.06.0010.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j}{j!^2} \left(\sum_{k=0}^{j} \frac{(-j)_k (-\log(1-z) + \psi(j+1) - \psi(j+a_1) - \psi(j+a_2) + \psi(j-k+1))}{(a_1)_k (a_2)_k} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) + (-1)^j j! \sum_{k=j+1}^{\infty} \frac{(k-j-1)!}{(a_1)_k (a_2)_k} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\}) \right) (1-z)^j /; $$

$$ |1-z| < 1 \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge \psi_q = 0 $$

The major terms in the general formula for expansions of function ${}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z)$ at $z = 1$

07.31.06.0011.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; 1)(1 + O(z-1)) + \frac{\Gamma(-\psi_q) \prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} (1-z)^{\psi_q} (1 + O(z-1)) /; $$

$$ (z \to 1) \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \psi_q \notin \mathbb{Z} $$

07.31.06.0012.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto \frac{\Gamma(\psi_q) \prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=3}^{q+1} \Gamma(a_k)} \sum_{k=0}^{\infty} \frac{(\psi_q)_k \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})}{\Gamma(k + a_1 + \psi_q) \Gamma(k + a_2 + \psi_q)} (1 + O(z-1)) + \frac{\Gamma(-\psi_q) \prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=3}^{q+1} \Gamma(a_k)} (1-z)^{\psi_q} (1 + O(z-1)) /; $$

$$ (z \to 1) \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge q > 1 \bigwedge \operatorname{Re}(\psi_q) > 0 \bigwedge \operatorname{Re}(a_3) > 0 \bigwedge \ldots \bigwedge \operatorname{Re}(a_{q+1}) > 0 $$

07.31.06.0013.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; 1)(1 + O(z-1)) - \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{q+1} \Gamma(a_j)} \log(1-z)(1 + O(z-1)) /; $$

$$ (z \to 1) \bigwedge \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge \psi_q = 0 $$

Expansions at $z = \infty$ for $p = q + 1$

The general formulas

07.31.06.0014.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \left(\prod_{k=1}^{q} \Gamma(b_k) \right) \mathcal{A}_{\tilde{F}} \left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \tilde{\infty}, \infty\} \right) /; z \notin (0, 1) $$

07.31.06.0015.01

$${}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \left(\prod_{k=1}^{q} \Gamma(b_k)\right) \mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \tilde{\infty}, \infty\}\right) /; z \notin (0, 1)$$

Case of simple poles

07.31.06.0016.01

$${}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{q+1} \Gamma(a_j - a_k)}{\prod_{j=1}^{q} \Gamma(b_j - a_k)} (-z)^{-a_k} \left(1 + \frac{a_k \prod_{j=1}^{q} (a_k - b_j + 1)}{\prod_{j=1, j\neq k}^{q+1} (a_k - a_j + 1)\, z} + \frac{a_k(a_k+1) \prod_{j=1}^{q} (a_k - b_j + 1)(a_k - b_j + 2)}{2 \prod_{j=1, j\neq k}^{q+1} ((a_k - a_j + 1)(a_k - a_j + 2))\, z^2} + \ldots\right) /;$$

$$|z| > 1 \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z})$$

07.31.06.0017.01

$${}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{q+1} \Gamma(a_j - a_k)}{\prod_{j=1}^{q} \Gamma(b_j - a_k)} (-z)^{-a_k} \sum_{i=0}^{\infty} \frac{(a_k)_i \prod_{j=1}^{q} (a_k - b_j + 1)_i}{i! \prod_{j=1, j\neq k}^{q+1} (a_k - a_j + 1)_i\, z^i} /;$$

$$|z| > 1 \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z})$$

07.31.06.0018.01

$${}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{k=1}^{q+1} \sum_{i=0}^{\infty} \Gamma\text{Res}\left(\begin{matrix} 0; & 1 - a_1, \ldots, 1 - a_{q+1}; \\ ; & 1 - b_1, \ldots, 1 - b_q; \end{matrix} 1 - a_k, 1, i; -z\right) /;$$

$$|z| > 1 \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z})$$

07.31.06.0019.01

$${}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{q+1} \Gamma(a_j - a_k)}{\prod_{j=1}^{q} \Gamma(b_j - a_k)} (-z)^{-a_k}$$

$${}_{q+1}F_q\left(a_k, a_k - b_1 + 1, \ldots, a_k - b_q + 1; 1 - a_1 + a_k, \ldots, 1 - a_{k-1} + a_k, 1 - a_{k+1} + a_k, \ldots, 1 - a_{q+1} + a_k; \frac{1}{z}\right) /;$$

$$z \notin (0, 1) \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z})$$

Case of poles of order r in the points $a_r + k$ /; $r \in \{2, 3, 4\} \bigwedge k \in \mathbb{N}$

07.31.06.0020.01

$${}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \prod_{k=1}^{q} \Gamma(b_k)\, \mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \infty, \infty\}\right) /; z \notin (0, 1) \bigwedge a_k - a_{k-1} \in \mathbb{N} \bigwedge$$

$$2 \leq k \leq r \bigwedge a_k - a_1 \notin \mathbb{Z} \bigwedge r + 1 \leq k \leq q + 1 \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge r+1\leq j\leq q+1\wedge r+1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z}) \bigwedge r \in \{2, 3, 4\}$$

The major terms for expansions of function ${}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z)$ at $z = \infty$

07.31.06.0021.01

$$ {}_{q+1}F_q\left(a_1,\,\ldots,\,a_{q+1};\,b_1,\,\ldots,\,b_q;\,z\right) \propto \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k)\prod_{j=1,\,j\neq k}^{q+1}\Gamma\left(a_j-a_k\right)}{\prod_{j=1}^{q}\Gamma\left(b_j-a_k\right)} (-z)^{-a_k} \left(1+O\left(\frac{1}{z}\right)\right) \;/;$$

$$ (|z|\to\infty)\bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 1\leq j\leq q+1\wedge 1\leq k\leq q+1}\left(a_j-a_k\notin\mathbb{Z}\right) $$

07.31.06.0022.01

$$ {}_{q+1}F_q\left(a_1,\,\ldots,\,a_{q+1};\,b_1,\,\ldots,\,b_q;\,z\right) \propto $$

$$ \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{q+1}\Gamma(a_k)} \left(\sum_{k=r+1}^{q+1} \frac{\Gamma(a_k)\prod_{j=1,\,j\neq k}^{q+1}\Gamma\left(a_j-a_k\right)}{\prod_{j=1}^{q}\Gamma\left(b_j-a_k\right)} (-z)^{-a_k} \left(1+O\left(\frac{1}{z}\right)\right) - \sum_{j=2}^{r+1} \Gamma\mathrm{Res}\left(\begin{matrix} 0; & a_1,\,\ldots,\,a_{q+1}; \\ ; & b_1,\,\ldots,\,b_q; \end{matrix}\; a_{j-1},\; j-1,\,0;\,-z\right) \left(1+O\left(\frac{1}{z}\right)\right) \right) /; $$

$$ (|z|\to\infty)\bigwedge a_k-a_{k-1}\in\mathbb{N}\bigwedge 2\leq k\leq r\bigwedge a_k-a_1\notin\mathbb{Z}\bigwedge r+1\leq k\leq q+1\bigwedge $$
$$ \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge r+1\leq j\leq q+1\wedge r+1\leq k\leq q+1}\left(a_j-a_k\notin\mathbb{Z}\right)\bigwedge r\in\{2,\,3,\,4\} $$

07.31.06.0023.01

$$ {}_{q+1}F_q\left(a_1,\,\ldots,\,a_{q+1};\,b_1,\,\ldots,\,b_q;\,z\right) \propto $$

$$ \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{q+1}\Gamma(a_k)} \left(\frac{\Gamma(a_1)\prod_{k=2}^{q+1}\Gamma(a_k-a_1)}{\prod_{k=1}^{q}\Gamma(b_k-a_1)} (-z)^{-a_1} \left(1+O\left(\frac{1}{z}\right)\right) + \frac{(-1)^{a_2-a_1}\,\Gamma(a_2)\prod_{k=3}^{q+1}\Gamma(a_k-a_2)}{(a_2-a_1)!\prod_{k=1}^{q}\Gamma(b_k-a_2)} (-z)^{-a_2} \left(\log(-z)+\psi(a_2-a_1+1)-\right.\right. $$

$$ \left.\left.\psi(a_2)+\sum_{k=3}^{q+1}\psi(a_k-a_2)-\sum_{k=1}^{q}\psi(b_k-a_2)-\gamma\right)\left(1+O\left(\frac{1}{z}\right)\right)+\sum_{k=3}^{q+1}\frac{\Gamma(a_k)\prod_{j=1,\,j\neq k}^{q+1}\Gamma\left(a_j-a_k\right)}{\prod_{j=1}^{q}\Gamma\left(b_j-a_k\right)}(-z)^{-a_k}\left(1+O\left(\frac{1}{z}\right)\right)\right) /; $$

$$ (|z|\to\infty)\bigwedge a_2-a_1\in\mathbb{N}\bigwedge a_k-a_1\notin\mathbb{Z}\bigwedge 3\leq k\leq q+1\bigwedge\forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\wedge j\neq k\wedge 3\leq j\leq q+1\wedge 3\leq k\leq q+1}\left(a_j-a_k\notin\mathbb{Z}\right) $$

07.31.06.0024.01

$$
{}_{q+1}F_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z\right) \propto
\frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \left(\frac{\Gamma(a_1) \prod_{k=2}^{q+1} \Gamma(a_k - a_1)}{\prod_{k=1}^{q} \Gamma(b_k - a_1)} (-z)^{-a_1} \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{(-1)^{a_2 - a_1} \Gamma(a_2) \prod_{k=3}^{q+1} \Gamma(a_k - a_2)}{(a_2 - a_1)! \prod_{k=1}^{q} \Gamma(b_k - a_2)} (-z)^{-a_2} \left(\log(-z) + \psi(a_2 - a_1 + 1) - \psi(a_2) + \sum_{k=3}^{q+1} \psi(a_k - a_2) - \sum_{k=1}^{q} \psi(b_k - a_2) - \gamma \right) \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{(-1)^{a_2 - a_1} \Gamma(a_3) \prod_{k=4}^{q+1} \Gamma(a_k - a_3)}{2\,(a_3 - a_2)!\,(a_3 - a_1)! \prod_{k=1}^{q} \Gamma(b_k - a_3)} (-z)^{-a_3} \left(\left(\log(-z) + \psi(a_3 - a_1 + 1) + \psi(a_3 - a_2 + 1) - \psi(a_3) + \sum_{k=4}^{q+1} \psi(a_k - a_3) - \sum_{k=1}^{q} \psi(b_k - a_3) - \gamma \right)^2 + \left(\frac{5\pi^2}{6} - \psi^{(1)}(a_3 - a_1 + 1) - \psi^{(1)}(a_3 - a_2 + 1) + \psi^{(1)}(a_3) + \sum_{k=4}^{q+1} \psi^{(1)}(a_k - a_3) - \sum_{k=1}^{q} \psi^{(1)}(b_k - a_3) \right) \right) \left(O\left(\frac{1}{z}\right) + 1 \right) + \sum_{k=4}^{q+1} \frac{\Gamma(a_k) \prod_{j=1, j \neq k}^{q+1} \Gamma\left(a_j - a_k\right)}{\prod_{j=1}^{q} \Gamma\left(b_j - a_k\right)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) \right) /; (|z| \to \infty) \bigwedge a_2 - a_1 \in \mathbb{N} \bigwedge a_3 - a_2 \in \mathbb{N} \bigwedge a_k - a_1 \notin \mathbb{Z} \bigwedge 4 \le k \le q + 1 \bigwedge \forall_{\{j,k\},\{j,k\} \in \mathbb{Z} \bigwedge j \neq k \bigwedge 4 \le j \le q+1 \bigwedge 4 \le k \le q+1} \left(a_j - a_k \notin \mathbb{Z}\right)
$$

07.31.06.0025.01

$$
{}_{q+1}F_q\left(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z\right) \propto
\frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \left(\frac{\Gamma(a_1) \prod_{k=2}^{q+1} \Gamma(a_k - a_1)}{\prod_{k=1}^{q} \Gamma(b_k - a_1)} (-z)^{-a_1} \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{(-1)^{a_2 - a_1} \Gamma(a_2) \prod_{k=3}^{q+1} \Gamma(a_k - a_2)}{(a_2 - a_1)! \prod_{k=1}^{q} \Gamma(b_k - a_2)} (-z)^{-a_2} \left(\log(-z) + \psi(a_2 - a_1 + 1) - \right.\right.
$$

$$
\left.\psi(a_2) + \sum_{k=3}^{q+1} \psi(a_k - a_2) - \sum_{k=1}^{q} \psi(b_k - a_2) - \gamma \right) \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{(-1)^{a_2 - a_1} \Gamma(a_3) \prod_{k=4}^{q+1} \Gamma(a_k - a_3)}{2\,(a_3 - a_2)!\,(a_3 - a_1)! \prod_{k=1}^{q} \Gamma(b_k - a_3)}
$$

$$
(-z)^{-a_3} \left(\left(\log(-z) + \psi(a_3 - a_1 + 1) + \psi(a_3 - a_2 + 1) - \psi(a_3) + \sum_{k=4}^{q+1} \psi(a_k - a_3) - \sum_{k=1}^{q} \psi(b_k - a_3) - \gamma \right)^2 + \right.
$$

$$
\left.\left(\frac{5\pi^2}{6} - \psi^{(1)}(a_3 - a_1 + 1) - \psi^{(1)}(a_3 - a_2 + 1) + \psi^{(1)}(a_3) + \sum_{k=4}^{q+1} \psi^{(1)}(a_k - a_3) - \sum_{k=1}^{q} \psi^{(1)}(b_k - a_3) \right)\right)
$$

$$
\left(O\left(\frac{1}{z}\right) + 1 \right) + \frac{(-1)^{-a_1 + a_2 - a_3 + a_4} \Gamma(a_4) \prod_{k=5}^{q+1} \Gamma(a_k - a_4)}{6\,(a_4 - a_3)!\,(a_4 - a_2)!\,(a_4 - a_1)! \prod_{k=1}^{q} \Gamma(b_k - a_4)} (-z)^{-a_4}
$$

$$
\left(\left(\log(-z) + \psi(a_4 - a_1 + 1) + \psi(a_4 - a_2 + 1) + \psi(a_4 - a_3 + 1) - \psi(a_4) + \sum_{k=5}^{q+1} \psi(a_k - a_4) - \sum_{k=1}^{q} \psi(b_k - a_4) - \gamma \right)^3 + \right.
$$

$$
\left(3 \left(\psi^{(1)}(a_4) - \psi^{(1)}(a_4 - a_1 + 1) - \psi^{(1)}(a_4 - a_2 + 1) - \psi^{(1)}(a_4 - a_3 + 1) + \sum_{k=5}^{q+1} \psi^{(1)}(a_k - a_4) - \sum_{k=1}^{q} \psi^{(1)}(b_k - a_4) \right) + \right.
$$

$$
\left.\frac{7\pi^2}{2} \right) \left(\log(-z) + \psi(a_4 - a_1 + 1) + \psi(a_4 - a_2 + 1) + \right.
$$

$$
\left.\psi(a_4 - a_3 + 1) - \psi(a_4) + \sum_{k=5}^{q+1} \psi(a_k - a_4) - \sum_{k=1}^{q} \psi(b_k - a_4) - \gamma \right) +
$$

$$
\left.\left(\psi^{(2)}(a_4 - a_1 + 1) + \psi^{(2)}(a_4 - a_2 + 1) + \psi^{(2)}(a_4 - a_3 + 1) - \psi^{(2)}(a_4) + \sum_{k=5}^{q+1} \psi^{(2)}(a_k - a_4) - \sum_{k=1}^{q} \psi^{(2)}(b_k - a_4) - 2\,\zeta(3) \right)\right)
$$

$$
\left.\left(1 + O\left(\frac{1}{z}\right)\right) + \sum_{k=5}^{q+1} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{q+1} \Gamma\left(a_j - a_k\right)}{\prod_{j=1}^{q} \Gamma\left(b_j - a_k\right)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) \right) /;
$$

$$
(|z| \to \infty) \bigwedge a_2 - a_1 \in \mathbb{N} \bigwedge a_3 - a_2 \in \mathbb{N} \bigwedge a_4 - a_3 \in \mathbb{N} \bigwedge a_k - a_1 \notin \mathbb{Z} \bigwedge 5 \leq k \leq q+1 \bigwedge \forall_{\{j,k\},\{j,k\} \in \mathbb{Z} \bigwedge j \neq k \bigwedge 5 \leq j \leq q+1 \bigwedge 5 \leq k \leq q+1} \left(a_j - a_k \notin \mathbb{Z}\right)
$$

Expansions at $z == \infty$ for polynomial cases

07.31.06.0026.01

$$ {}_pF_q(-n, a_2, a_3, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{\prod_{k=2}^{p} (a_k)_n}{\prod_{k=1}^{q} (b_k)_n} (-z)^n \, {}_{q+1}F_{p-1}\left(-n, 1-n-b_1, \ldots, 1-n-b_q; 1-n-a_2, 1-n-a_3, \ldots, 1-n-a_p; \frac{(-1)^{p+q-1}}{z}\right) /; n \in \mathbb{N}^+ $$

Asymptotic series expansions

Expansions for $q == p$

07.31.06.0027.01

$$ {}_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z) \propto \left(\prod_{j=1}^{p} \Gamma(b_j)\right) \left(\mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_p; \end{matrix} \{z, \infty, \infty\}\right) + \mathcal{A}_{\tilde{F}}^{(\exp)}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_p; \end{matrix} \{z, \infty, \infty\}\right)\right) /; (|z| \to \infty) $$

07.31.06.0028.01

$$ {}_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z) \propto \left(\prod_{j=1}^{p} \frac{\Gamma(b_j)}{\Gamma(a_j)}\right) e^z z^{\chi} \left(1 + O\left(\frac{1}{z}\right)\right) + \left(\prod_{j=1}^{p} \frac{\Gamma(b_j)}{\Gamma(a_j)}\right) \sum_{k=1}^{p} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{p} \Gamma(a_j - a_k)}{\prod_{j=1}^{p} \Gamma(b_j - a_k)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) /; $$

$$ (|z| \to \infty) \bigwedge \forall_{\{j,k\}, \{j,k\} \in \mathbb{Z} \land j \neq k \land 1 \leq j \leq p \land 1 \leq k \leq p} (a_j - a_k \notin \mathbb{Z}) $$

Expansions for $q == p + 1$

07.31.06.0029.01

$$ {}_pF_{p+1}(a_1, \ldots, a_p; b_1, \ldots, b_{p+1}; z) \propto \left(\prod_{j=1}^{p+1} \Gamma(b_j)\right) \left(\mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_{p+1}; \end{matrix} \{z, \infty, \infty\}\right) + \mathcal{A}_{\tilde{F}}^{(\text{trig})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_{p+1}; \end{matrix} \{z, \infty, \infty\}\right)\right) /; (|z| \to \infty) $$

07.31.06.0030.01

$$ {}_pF_{p+1}(a_1, \ldots, a_p; b_1, \ldots, b_{p+1}; z) \propto \frac{\prod_{j=1}^{p+1} \Gamma(b_j)}{2\sqrt{\pi} \prod_{k=1}^{p} \Gamma(a_k)} (-z)^{\chi} \left(e^{i\left(\pi\chi + 2\sqrt{-z}\right)} \left(1 + O\left(\frac{1}{\sqrt{-z}}\right)\right) + e^{-i\left(\pi\chi + 2\sqrt{-z}\right)} \left(1 + O\left(\frac{1}{\sqrt{-z}}\right)\right)\right) + $$

$$ \frac{\prod_{j=1}^{p+1} \Gamma(b_j)}{\prod_{k=1}^{p} \Gamma(a_k)} \sum_{k=1}^{p} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{p} \Gamma(a_j - a_k)}{\prod_{j=1}^{p+1} \Gamma(b_j - a_k)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty) \bigwedge \forall_{\{j,k\}, \{j,k\} \in \mathbb{Z} \land j \neq k \land 1 \leq j \leq p \land 1 \leq k \leq p} (a_j - a_k \notin \mathbb{Z}) $$

07.31.06.0031.01

$$ {}_pF_{p+1}(a_1, \ldots, a_p; b_1, \ldots, b_{p+1}; z) \propto $$

$$ \frac{\prod_{j=1}^{p+1} \Gamma(b_j)}{\sqrt{\pi} \prod_{k=1}^{p} \Gamma(a_k)} (-z)^{\chi} \left(\cos\left(\pi\chi + 2\sqrt{-z}\right) \left(1 + O\left(\frac{1}{z}\right)\right) + \frac{c_1}{2\sqrt{-z}} \sin\left(\pi\chi + 2\sqrt{-z}\right) \left(1 + O\left(\frac{1}{z}\right)\right)\right) + $$

$$ \frac{\prod_{j=1}^{p+1} \Gamma(b_j)}{\prod_{k=1}^{p} \Gamma(a_k)} \sum_{k=1}^{p} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{p} \Gamma(a_j - a_k)}{\prod_{j=1}^{p+1} \Gamma(b_j - a_k)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) /; $$

$$ (|z| \to \infty) \bigwedge \chi == \frac{1}{2}\left(A_p - B_{p+1} + \frac{1}{2}\right) \bigwedge c_1 == 2\left(\mathfrak{B} - \mathfrak{A} + \frac{1}{4}(3A_p + B_{p+1} - 2)(A_p - B_{p+1}) - \frac{3}{16}\right) \bigwedge A_p == \sum_{k=1}^{p} a_k \bigwedge $$

$$ B_{p+1} == \sum_{k=1}^{p+1} b_k \bigwedge \mathfrak{A} == \sum_{s=2}^{p} \sum_{j=1}^{s-1} a_s a_j \bigwedge \mathfrak{B} == \sum_{s=2}^{p+1} \sum_{j=1}^{s-1} b_s b_j \bigwedge \forall_{\{j,k\}, \{j,k\} \in \mathbb{Z} \land j \neq k \land 1 \leq j \leq p \land 1 \leq k \leq p} (a_j - a_k \notin \mathbb{Z}) $$

Expansions for $q \geq p+2$

07.31.06.0032.01

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto \prod_{j=1}^{q} \Gamma(b_j) \left(\mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \infty, \infty\} \right) + \mathcal{A}_{\tilde{F}}^{(\exp)}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \infty, \infty\} \right) \right) /; $$

$$ q-p \geq 2 \land (|z| \to \infty) \land \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq p\land 1\leq k\leq p} (a_j - a_k \notin \mathbb{Z}) $$

07.31.06.0033.01

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto $$

$$ \frac{(2\pi)^{\frac{1-\beta}{2}} \prod_{j=1}^{q} \Gamma(b_j)}{\sqrt{\beta} \prod_{k=1}^{p} \Gamma(a_k)} z^{\chi} \exp(\beta z^{1/\beta}) \left(1 + O\left(\frac{1}{z^{1/\beta}}\right)\right) + \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{k=1}^{p} \Gamma(a_k)} \sum_{k=1}^{p} \frac{\Gamma(a_k) \prod_{j=1, j\neq k}^{p} \Gamma(a_j - a_k)}{\prod_{j=1}^{q} \Gamma(b_j - a_k)} (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) /; $$

$$ q-p \geq 2 \bigwedge (|z| \to \infty) \bigwedge \beta == q-p+1 \bigwedge \chi == \frac{1}{\beta}\left(\frac{\beta-1}{2} + \sum_{k=1}^{p} a_k - \sum_{k=1}^{q} b_k\right) \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq p\land 1\leq k\leq p} (a_j - a_k \notin \mathbb{Z}) $$

Expansions for ${}_0F_2$

07.31.06.0034.01

$$ {}_0F_2(;b_1, b_2; z) \propto \frac{\Gamma(b_1)\Gamma(b_2)}{2\sqrt{3}\,\pi} e^{3\sqrt[3]{z}} z^{\frac{1}{3}(1-b_1-b_2)} $$

$$ \left(1 + \frac{-3b_1^2 + 3(b_2+1)b_1 - 3b_2^2 + 3b_2 - 2}{9\sqrt[3]{z}} + \frac{1}{162 z^{2/3}} \left(9b_1^4 - 6(3b_2+2)b_1^3 + 3(9b_2^2 - 3b_2 + 1)b_1^2 - \right.\right. $$

$$ \left.\left. 3(6b_2^3 + 3b_2^2 - 17b_2 + 4)b_1 + 9b_2^4 - 12b_2^3 + 3b_2^2 - 12b_2 + 4\right) + \ldots\right) /; (|z| \to \infty) $$

07.31.06.0035.01

$$ {}_0F_2(;b_1, b_2; z) \propto \frac{\Gamma(b_1)\Gamma(b_2)}{2\sqrt{3}\,\pi} e^{3\sqrt[3]{z}} z^{\frac{1}{3}(1-b_1-b_2)} \left(1 + O\left(\frac{1}{\sqrt[3]{z}}\right)\right) /; (|z| \to \infty) $$

Expansions for ${}_0F_3$

07.31.06.0036.01

$$ {}_0F_3(;b_1, b_2, b_3; z) \propto $$

$$ \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{4\sqrt{2}\,\pi^{3/2}} e^{4\sqrt[4]{z}} z^{\frac{1}{4}\left(\frac{3}{2}-b_1-b_2-b_3\right)} \left(1 + \frac{-12b_1^2 + 8(b_2+b_3+1)b_1 - 12b_2^2 - 12b_3^2 + 8b_3 + 8b_2(b_3+1) - 7}{32\sqrt[4]{z}} + \right. $$

$$ \frac{1}{2048\sqrt{z}} \left(144b_1^4 - 64(3b_2+3b_3+1)b_1^3 + 8(44b_2^2 - 8(b_3+3)b_2 + 44b_3^2 - 24b_3 + 1)b_1^2 - 16\right. $$

$$ (12b_2^3 + 4(b_3+3)b_2^2 + (4b_3^2 - 40b_3 - 21)b_2 + 12b_3^3 + 12b_3^2 - 21b_3 + 11)b_1 + 144b_2^4 + 144b_3^4 - 64b_3^3 + 8b_3^2 - $$

$$ \left.\left. 176b_3 - 64b_2^3(3b_3+1) + 8b_2^2(44b_3^2 - 24b_3 + 1) - 16b_2(12b_3^3 + 12b_3^2 - 21b_3 + 11) + 121\right) + \ldots\right) /; (|z| \to \infty) $$

07.31.06.0037.01

$$ {}_0F_3(;b_1, b_2, b_3; z) \propto \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{4\sqrt{2}\,\pi^{3/2}} e^{4\sqrt[4]{z}} z^{\frac{1}{4}\left(\frac{3}{2}-b_1-b_2-b_3\right)} \left(1 + O\left(\frac{1}{\sqrt[4]{z}}\right)\right) /; (|z| \to \infty) $$

General formulas of asymptotic series expansions

07.31.06.0038.01

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto \left(\prod_{j=1}^{q} \Gamma(b_j)\right) \mathcal{A}_{\tilde{F}}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \tilde{\infty}, \infty\}\right) /; (|z| \to \infty) \land p \leq q+1 $$

07.31.06.0039.01

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto \left(\prod_{j=1}^{q} \Gamma(b_j)\right) \left((\theta(q-p) - \delta_{q,p+1}) \mathcal{A}_{\tilde{F}}^{(\exp)}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \tilde{\infty}, \infty\}\right) + \mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix} \{z, \tilde{\infty}, \infty\}\right) + \delta_{q,p+1} \mathcal{A}_{\tilde{F}}^{(\text{trig})}\left(\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_{p+1}; \end{matrix} \{z, \tilde{\infty}, \infty\}\right)\right) /; (|z| \to \infty) \land p \leq q+1 $$

Main terms of asymptotic expansions

07.31.06.0040.01

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \left(\sum_{k=1}^{p} \operatorname{res}_s\left(\frac{\Gamma(s) \prod_{j=1}^{p} \Gamma(a_j - s)}{\prod_{j=1}^{q} \Gamma(b_j - s)} (-z)^{-s} \right)(a_k) \left(1 + O\left(\frac{1}{z}\right)\right) + \delta_{q,p+1} d_1 (-z)^{\chi} \cos\left(\pi \chi + 2\sqrt{-z}\right) \left(1 + O\left(\frac{1}{\sqrt{-z}}\right)\right) + (\theta(q-p) - \delta_{q,p+1}) d_2 z^{\chi} e^{\beta z^{1/\beta}} \left(1 + O\left(\frac{1}{z^{1/\beta}}\right)\right)\right) /; $$

$$ (|z| \to \infty) \bigwedge \beta == q - p + 1 \bigwedge \chi == \frac{1}{\beta}\left(\frac{\beta - 1}{2} + \sum_{k=1}^{p} a_k - \sum_{k=1}^{q} b_k\right) \bigwedge 2 d_2 == d_1 == \frac{2 (2\pi)^{\frac{1-\beta}{2}}}{\sqrt{\beta}} $$

07.31.06.0041.01

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \propto \sum_{k=1}^{p} c_k (-z)^{-a_k} \left(1 + O\left(\frac{1}{z}\right)\right) + \delta_{q,p+1} e_1 (-z)^{\chi} \cos\left(\pi \chi + 2\sqrt{-z}\right) \left(1 + O\left(\frac{1}{\sqrt{-z}}\right)\right) + (\theta(q-p) - \delta_{q,p+1}) e_2 z^{\chi} e^{\beta z^{1/\beta}} \left(1 + O\left(\frac{1}{z^{1/\beta}}\right)\right) /; $$

$$ (|z| \to \infty) \bigwedge \beta == q - p + 1 \bigwedge \chi == \frac{1}{\beta}\left(\frac{\beta - 1}{2} + \sum_{k=1}^{p} a_k - \sum_{k=1}^{q} b_k\right) \bigwedge c_k == \frac{\Gamma(a_k) \left(\prod_{j=1}^{q} \Gamma(b_j)\right) \prod_{j=1, j \neq k}^{p} \Gamma(a_j - a_k)}{\left(\prod_{j=1}^{p} \Gamma(a_j)\right) \prod_{j=1}^{q} \Gamma(b_j - a_k)} \bigwedge $$

$$ 2 e_2 == e_1 == \frac{2 (2\pi)^{\frac{1-\beta}{2}} \prod_{k=1}^{q} \Gamma(b_k)}{\sqrt{\beta} \prod_{k=1}^{p} \Gamma(a_k)} \bigwedge \forall_{\{j,k\}, \{j,k\} \in \mathbb{Z} \land j \neq k \land 1 \leq j \leq p \land 1 \leq k \leq p} (a_j \neq a_k) $$

07.31.06.0042.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) \propto c (1 + O(z-1)) + d (1-z)^{\psi_q} (1 + O(z-1)) /; $$

$$ (z \to 1) \bigwedge \psi_q == \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge c == {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; 1) \bigwedge d == \frac{\Gamma(-\psi_q) \prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{q+1} \Gamma(a_k)} \bigwedge \psi_q \notin \mathbb{Z} $$

Residue representations

07.31.06.0043.01

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{p}\Gamma(a_k)} \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(s)\prod_{k=1}^{p}\Gamma(a_k - s)}{\prod_{k=1}^{q}\Gamma(b_k - s)}(-z)^{-s}\right)(-j) /; p < q + 1 \lor p = q + 1 \land |z| < 1$$

07.31.06.0044.01

$${}_pF_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = -\frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1}\sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(s)\prod_{k=1}^{q+1}\Gamma(a_k - s)}{\prod_{k=1}^{q}\Gamma(b_k - s)}(-z)^{-s}\right)(a_k + j) /; |z| > 1$$

Continued fraction representations

07.31.10.0001.01

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = 1 + \left(z\prod_{k=1}^{p} a_k \Big/ \left(\prod_{k=1}^{q} b_k\right)\right) \Bigg/ \left(1 + -\frac{z\prod_{j=1}^{p}(1 + a_j)}{2\prod_{j=1}^{q}(1 + b_j)} \Bigg/ \left(1 + \frac{z\prod_{j=1}^{p}(1 + a_j)}{2\prod_{j=1}^{q}(1 + b_j)} + \frac{-\frac{z\prod_{j=1}^{p}(2+a_j)}{3\prod_{j=1}^{q}(2+b_j)}}{1 + \frac{z\prod_{j=1}^{p}(2 + a_j)}{3\prod_{j=1}^{q}(2 + b_j)} + \ldots}\right)\right)$$

07.31.10.0002.01

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = 1 + \left(z\prod_{k=1}^{p} a_k\right) \Bigg/ \left(\left(\prod_{k=1}^{q} b_k\right)\left(1 + \mathrm{K}_k\left(-\frac{z\prod_{j=1}^{p}(k + a_j)}{(k+1)\prod_{j=1}^{q}(k + b_j)}, \frac{z\prod_{j=1}^{p}(k + a_j)}{(k+1)\prod_{j=1}^{q}(k + b_j)} + 1\right)_1^{\infty}\right)\right)$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

The differential equation for the function ${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ has the order $\max(p, q + 1)$. It has two $(z = 0, z = \tilde{\infty}$ for $p \le q)$ or three $(z = 0, z = 1, z = \tilde{\infty}$, for $p = q + 1)$ singular points. If $p \le q$, then the point $z = 0$ is a regular singular point, while $z = \tilde{\infty}$ is a nonregular (essential) singular point; if $p = q + 1$, then all three singular points are regular.

Representation of fundamental system solutions near point $z = 0$ for $p \le q + 1$ in the general case

07.31.13.0004.01

$$z^q w^{(q+1)}(z) + z^{q-1}\left(\frac{q(q-1)}{2} + \sum_{k=1}^{q} b_k\right) w^{(q)}(z) - z^p w^{(p)}(z) - z^{p-1}\left(\frac{p(p-1)}{2} + \sum_{l=1}^{p} a_l\right) w^{(p-1)}(z) + \left(\left(\frac{d}{dz}\prod_{k=1}^{q}\left(z\frac{d}{dz} + b_k - 1\right)\right) w(z) - \prod_{l=1}^{p}\left(z\frac{d}{dz} + a_l\right) w(z) - z^q w^{(q+1)}(z) - z^{q-1}\left(\frac{q(q-1)}{2} + \sum_{k=1}^{q} b_k\right) w^{(q)}(z) + z^p w^{(p)}(z) + z^{p-1}\left(\frac{p(p-1)}{2} + \sum_{k=1}^{p} a_k\right) w^{(p-1)}(z) + w(z)\prod_{l=1}^{p} a_l\right) - w(z)\prod_{l=1}^{p} a_l = 0 \,/;$$

$$\left(w(z) = c_1\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) + c_2 \sum_{k=1}^{q} G_{p,q+1}^{2,p}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_k, 1-b_1, \ldots, 1-b_{k-1}, 1-b_{k+1}, \ldots, 1-b_q \end{matrix}\right) + \ldots + c_q \sum_{k=1}^{q} G_{p,q+1}^{q,p}\left((-1)^q z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_{k-1}, 1-b_{k+1}, \ldots, 1-b_q, 1-b_k \end{matrix}\right) + c_{q+1}\, G_{p,q+1}^{q+1,p}\left((-1)^{q+1} z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right)\right)$$

07.31.13.0005.01

$$z^q w^{(q+1)}(z) + z^{q-1}\left(\frac{q(q-1)}{2} + \sum_{k=1}^{q} b_k\right) w^{(q)}(z) - z^p w^{(p)}(z) - z^{p-1}\left(\frac{p(p-1)}{2} + \sum_{l=1}^{p} a_l\right) w^{(p-1)}(z) + \left(\left(\frac{d}{dz}\prod_{k=1}^{q}\left(z\frac{d}{dz} + b_k - 1\right)\right) w(z) - \prod_{l=1}^{p}\left(z\frac{d}{dz} + a_l\right) w(z) - z^q w^{(q+1)}(z) - z^{q-1}\left(\frac{q(q-1)}{2} + \sum_{k=1}^{q} b_k\right) w^{(q)}(z) + z^p w^{(p)}(z) + z^{p-1}\left(\frac{p(p-1)}{2} + \sum_{k=1}^{p} a_k\right) w^{(p-1)}(z) + w(z)\prod_{l=1}^{p} a_l\right) - w(z)\prod_{l=1}^{p} a_l = 0 \,/; \left(w(z) = c_1\, {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) + \sum_{k=1}^{q} c_{k+1} z^{1-b_k}\, {}_p\tilde{F}_q(a_1 - b_k + 1, \ldots, a_p - b_k + 1; 2 - b_k, b_1 - b_k + 1, \ldots, b_{k-1} - b_k + 1, b_{k+1} - b_k + 1, \ldots, b_q - b_k + 1; z) \,/; \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq q\land 1\leq k\leq q} (b_j - b_k \notin \mathbb{Z}) \land b_k \notin \mathbb{Z}\right)$$

07.31.13.0006.01

$$W_z\Big({}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z), z^{1-b_1}\, {}_p\tilde{F}_q(a_1 - b_1 + 1, \ldots, a_p - b_1 + 1; 2 - b_1, 1 - b_1 + b_2, \ldots, 1 - b_1 + b_q; z), \ldots, z^{1-b_k}\, {}_p\tilde{F}_q(a_1 - b_k + 1, \ldots, a_p - b_k + 1; 2 - b_k, b_1 - b_k + 1, \ldots, b_{k-1} - b_k + 1, b_{k+1} - b_k + 1, \ldots, b_q - b_k + 1; z), \ldots, z^{1-b_q}\, {}_p\tilde{F}_q(a_1 - b_q + 1, \ldots, a_p - b_q + 1; 2 - b_q, b_1 - b_q + 1, \ldots, b_{q-1} - b_q + 1; z)\Big) = \pi^{-\frac{q(1+q)}{2}}\left(\prod_{k=1}^{q}\sin(\pi b_k)\right)\prod_{k=1}^{q}\prod_{j=1}^{k-1}\sin(\pi(b_j - b_k))\, z^{-\frac{1}{2}(q-1)q - \sum_{k=1}^{q} b_k}\left(\delta_{p,q+1}(1-z)^{-q-\sum_{l=1}^{q+1} a_l + \sum_{k=1}^{q} b_k} + e^z \delta_{p,q} + \theta(-p+q-1)\right)$$

07.31.13.0001.01

$$\left(\frac{d}{dz}\prod_{k=1}^{q}\left(z\frac{d}{dz}+b_k-1\right)-\prod_{l=1}^{p}\left(z\frac{d}{dz}+a_l\right)\right)w(z)=0\;/;\;\Bigg(w(z)=c_1\,{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)+\sum_{k=1}^{q}c_{k+1}\,z^{1-b_k}\,{}_pF_q(a_1-b_k+1,\ldots,a_p-b_k+1;2-b_k,b_1-b_k+1,\ldots,b_{k-1}-b_k+1,b_{k+1}-b_k+1,\ldots,b_q-b_k+1;z)\;/;\;\forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq q\land 1\leq k\leq q}\,(b_j-b_k\notin\mathbb{Z})\land b_k\notin\mathbb{Z}\Bigg)$$

07.31.13.0007.01

$$W_z\big({}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z),\,z^{1-b_1}\,{}_pF_q(a_1-b_1+1,\ldots,a_p-b_1+1;2-b_1,-b_1+b_2+1,\ldots,-b_1+b_q+1;z),\ldots,\,z^{1-b_k}\,{}_pF_q(a_1-b_k+1,\ldots,a_p-b_k+1;2-b_k,b_1-b_k+1,\ldots,b_{k-1}-b_k+1,b_{k+1}-b_k+1,\ldots,b_q-b_k+1;z),\ldots,\,z^{1-b_q}\,{}_pF_q(a_1-b_q+1,\ldots,a_p-b_q+1;2-b_q,b_1-b_q+1,\ldots,b_{q-1}-b_q+1;z)\big)=\text{const}\,z^{-\frac{1}{2}(q-1)q-\sum_{k=1}^{q}b_k}\left(\delta_{p,q+1}\,(1-z)^{-q-\sum_{l=1}^{q+1}a_l+\sum_{k=1}^{q}b_k}+e^z\,\delta_{p,q}+\theta(-p+q-1)\right)\;/;\;t=\lim_{\epsilon\to 0}\epsilon^{\frac{(q-1)q}{2}+\sum_{k=1}^{q}b_k}\,W_\epsilon\big({}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;\epsilon),\,\epsilon^{1-b_1}\,{}_pF_q(a_1-b_1+1,\ldots,a_p-b_1+1;2-b_1,-b_1+b_2+1,\ldots,-b_1+b_q+1;\epsilon),\ldots,\,\epsilon^{1-b_k}\,{}_pF_q(a_1-b_k+1,\ldots,a_p-b_k+1;2-b_k,b_1-b_k+1,\ldots,b_{k-1}-b_k+1,b_{k+1}-b_k+1,\ldots,b_q-b_k+1;\epsilon),\ldots,\,\epsilon^{1-b_q}\,{}_pF_q(a_1-b_q+1,\ldots,a_p-b_q+1;2-b_q,b_1-b_q+1,\ldots,b_{q-1}-b_q+1;\epsilon)\big)$$

07.31.13.0008.01

$$W_z\big({}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z),\,z^{1-b_1}\,{}_pF_q(a_1-b_1+1,\ldots,a_p-b_1+1;2-b_1,1-b_1+b_2,\ldots,1-b_1+b_q;z),\ldots,\,z^{1-b_k}\,{}_pF_q(a_1-b_k+1,\ldots,a_p-b_k+1;2-b_k,b_1-b_k+1,\ldots,b_{k-1}-b_k+1,b_{k+1}-b_k+1,\ldots,b_q-b_k+1;z),\ldots,\,z^{1-b_q}\,{}_pF_q(a_1-b_q+1,\ldots,a_p-b_q+1;2-b_q,b_1-b_q+1,\ldots,b_{q-1}-b_q+1;z)\big)=(-1)^q\left(\prod_{k=1}^{q}(b_k-1)\right)\left(\prod_{k=1}^{q}\prod_{j=1}^{k-1}(b_j-b_k)\right)z^{-\frac{1}{2}(q-1)q-\sum_{k=1}^{q}b_k}\left(\delta_{p,q+1}\,(1-z)^{-q-\sum_{l=1}^{q+1}a_l+\sum_{k=1}^{q}b_k}+e^z\,\delta_{p,q}+\theta(-p+q-1)\right)$$

Representation of fundamental system solutions near point $z = 1$ for $p = q+1$ in the general case

Below representation includes functions of two kinds. The function $G_{q+1,q+1}^{q+1,0}\left(z\,\middle|\,\begin{matrix}1-a_1,\ldots,1-a_{q+1}\\0,1-b_1,\ldots,1-b_q\end{matrix}\right)$ is the piecewise analytical function with a discontinuity on the unite circle $|z| = 1$. It has singularity near point $z = 1$of the form const $(1-z)^{\psi_q}\,(1+O(z-1))$, when $|z|<1$. The functions $G_{q+3,q+3}^{2,q+3}\left(z\,\middle|\,\begin{matrix}0,b_k,1-a_1,\ldots,1-a_{q+1}\\0,b_k,0,1-b_1,\ldots,1-b_q\end{matrix}\right)$ are the analytical functions and are bounded near point $z = 1$.

The function $G_{q+1,q+1}^{q+1,0}\left(z\,\middle|\,\begin{matrix}1-a_1,\ldots,1-a_{q+1}\\0,1-b_1,\ldots,1-b_q\end{matrix}\right)$ inside of $|z|<1$can be reprezented through hypergeometric functions defined for all complex z.

07.31.13.0002.01

$$\left(\frac{d}{dz}\prod_{k=1}^{q}\left(z\frac{d}{dz}+b_k-1\right)-\prod_{l=1}^{q+1}\left(z\frac{d}{dz}+a_l\right)\right)w(z)=0\;/;$$

$$\left(w(z)=c_1\,G_{q+1,q+1}^{q+1,0}\left(z\left|\begin{array}{c}1-a_1,\,\ldots,\,1-a_{q+1}\\0,\,1-b_1,\,\ldots,\,1-b_q\end{array}\right.\right)+\sum_{k=1}^{q}c_{k+1}\,G_{q+3,q+3}^{2,q+3}\left(z\left|\begin{array}{c}0,\,b_k,\,1-a_1,\,\ldots,\,1-a_{q+1}\\0,\,b_k,\,0,\,1-b_1,\,\ldots,\,1-b_q\end{array}\right.\right)/;\right.$$

$$\left.|z|<1\bigwedge\psi_q=\sum_{j=1}^{q}b_j-\sum_{j=1}^{q+1}a_j\bigwedge\psi_q\notin\mathbb{Z}\bigwedge\forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq q\land 1\leq k\leq q}\left(b_j-b_k\notin\mathbb{Z}\right)\bigwedge b_k\notin\mathbb{Z}\right)$$

Representation of fundamental system solutions near point $z=\tilde{\infty}$ for $p\geq q+1$ in the general case

07.31.13.0009.01

$$\left(\frac{d}{dz}\prod_{k=1}^{q}\left(z\frac{d}{dz}+b_k-1\right)-\prod_{l=1}^{p}\left(z\frac{d}{dz}+a_l\right)\right)w(z)=0\;/;$$

$$w(z)=\sum_{k=1}^{p}c_k\,z^{-a_k}\,{}_{q+1}\tilde{F}_{p-1}\left(a_k,\,a_k-b_1+1,\,\ldots,\,a_k-b_q+1;\,1-a_1+a_k,\,\ldots,\,1-a_{k-1}+a_k,\right.$$

$$\left.1-a_{k+1}+a_k,\,\ldots,\,1-a_p+a_k;\,\frac{(-1)^{1-p+q}}{z}\right)\bigwedge\forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq p\land 1\leq k\leq p}\left(a_j-a_k\notin\mathbb{Z}\right)$$

07.31.13.0003.01

$$\left(\frac{d}{dz}\prod_{k=1}^{q}\left(z\frac{d}{dz}+b_k-1\right)-\prod_{l=1}^{p}\left(z\frac{d}{dz}+a_l\right)\right)w(z)=0\;/;$$

$$w(z)=\sum_{k=1}^{p}c_k\,z^{-a_k}\,{}_{q+1}F_{p-1}\left(a_k,\,a_k-b_1+1,\,\ldots,\,a_k-b_q+1;\,1-a_1+a_k,\,\ldots,\,1-a_{k-1}+a_k,\right.$$

$$\left.1-a_{k+1}+a_k,\,\ldots,\,1-a_p+a_k;\,\frac{(-1)^{1-p+q}}{z}\right)/;\forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\land j\neq k\land 1\leq j\leq p\land 1\leq k\leq p}\left(a_j-a_k\notin\mathbb{Z}\right)$$

Transformations

Products, sums, and powers of the direct function

Products of the direct function

07.31.16.0001.01

$${}_pF_q(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,c\,z)\,{}_rF_s(\alpha_1,\,\ldots,\,\alpha_r;\,\beta_1,\,\ldots,\,\beta_s;\,d\,z)=\sum_{k=0}^{\infty}c_k\,z^k\;/;$$

$$c_k=\frac{d^k\prod_{j=1}^{r}(\alpha_j)_k}{k!\prod_{j=1}^{s}(\beta_j)_k}\,{}_{p+s+1}F_{q+r}\left(-k,\,1-\beta_1-k,\,\ldots,\,1-\beta_s-k,\,a_1,\,\ldots,\,a_p;\,1-\alpha_1-k,\,\ldots,\,1-\alpha_r-k,\,b_1,\,\ldots,\,b_q;\,\frac{(-1)^{r+s-1}\,c}{d}\right)\bigvee$$

$$c_k=\frac{c^k\prod_{j=1}^{p}(a_j)_k}{k!\prod_{j=1}^{q}(b_j)_k}\,{}_{q+r+1}F_{p+s}\left(-k,\,1-b_1-k,\,\ldots,\,1-b_q-k,\,\alpha_1,\,\ldots,\,\alpha_r;\,1-a_1-k,\,\ldots,\,1-a_r-k,\,\beta_1,\,\ldots,\,\beta_s;\,\frac{(-1)^{p+q-1}\,d}{c}\right)$$

07.31.16.0002.01

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; c\, z)\, {}_rF_s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; d\, z) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{\left(\prod_{j=1}^{p} (a_j)_m\right)\left(\prod_{j=1}^{r} (\alpha_j)_{k-m}\right) c^m d^{k-m} z^k}{\left(\prod_{j=1}^{q} (b_j)_m\right)\left(\prod_{j=1}^{s} (\beta_j)_{k-m}\right) m!\, (k-m)!} $$

07.31.16.0003.01

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; c\, z)\, {}_rF_s(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s; d\, z) = F_{0:q;s}^{0:p;r}\left(\begin{matrix} : a_1, \ldots, a_p; \alpha_1, \ldots, \alpha_r; \\ : b_1, \ldots, b_q; \beta_1, \ldots, \beta_s; \end{matrix}\; c\, z, d\, z \right) $$

Identities

Recurrence identities

Distant neighbors with respect to *q*

07.31.17.0001.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=3}^{q+1} \Gamma(a_j)} \sum_{k=0}^{\infty} \mathcal{E}_k^{(q)}(\{a_1, \ldots, a_{q+1}\}, \{b_1, \ldots, b_q\})\, {}_2\tilde{F}_1(a_1, a_2; a_1 + a_2 + \psi_q + k; z) \;/;\; \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j $$

Functional identities

Relations between contiguous functions

07.31.17.0002.01

$$ b\, {}_pF_q(a, b+1, a_3, \ldots, a_p; b_1, \ldots, b_q; z) - a\, {}_pF_q(a+1, b, a_3, \ldots, a_p; b_1, \ldots, b_q; z) + (a-b)\, {}_pF_q(a, b, a_3, \ldots, a_p; b_1, \ldots, b_q; z) = 0 $$

07.31.17.0003.01

$$ c\, {}_pF_q(a, a_2, \ldots, a_p; c, b_2, \ldots, b_q; z) - a\, {}_pF_q(a+1, a_2, \ldots, a_p; c+1, b_2, \ldots, b_q; z) + (a-c)\, {}_pF_q(a, a_2, \ldots, a_p; c+1, b_2, \ldots, b_q; z) = 0 $$

07.31.17.0004.01

$$ d\, {}_pF_q(a_1, \ldots, a_p; c+1, d, b_3, \ldots, b_q; z) - c\, {}_pF_q(a_1, \ldots, a_p; c, d+1, b_3, \ldots, b_q; z) + (c-d)\, {}_pF_q(a_1, \ldots, a_p; c+1, d+1, b_3, \ldots, b_q; z) = 0 $$

07.31.17.0005.01

$$ c(a-b)\, {}_pF_q(a, b, a_3, \ldots, a_p; c, b_2, \ldots, b_q; z) - a(c-b)\, {}_pF_q(a+1, b, a_3, \ldots, a_p; c+1, b_2, \ldots, b_q; z) + b(c-a)\, {}_pF_q(a, b+1, a_3, \ldots, a_p; c+1, b_2, \ldots, b_q; z) = 0 $$

07.31.17.0006.01

$$ c(d-a)\, {}_pF_q(a, a_2, \ldots, a_p; c, d+1, b_3, \ldots, b_q; z) - d(c-a)\, {}_pF_q(a, a_2, \ldots, a_p; c+1, d, b_3, \ldots, b_q; z) + a(c-d)\, {}_pF_q(a+1, a_2, \ldots, a_p; c+1, d+1, b_3, \ldots, b_q; z) = 0 $$

07.31.17.0007.01

$$ \left(\prod_{k=1}^{q} b_k\right)\left({}_pF_q(a, a_2, \ldots, a_p; b_1, \ldots, b_q; z) - {}_pF_q(a+1, a_2, \ldots, a_p; b_1, \ldots, b_q; z)\right) + z\left(\prod_{j=2}^{p} a_j\right) {}_pF_q(a+1, a_2+1, \ldots, a_p+1; b_1+1, \ldots, b_q+1; z) = 0 $$

07.31.17.0008.01

$$c\,(c+1)\left(\prod_{k=2}^{q} b_k\right)\left({}_pF_q(a_1,\,\ldots,\,a_p;\,c,\,b_2,\,\ldots,\,b_q;\,z) - {}_pF_q(a_1,\,\ldots,\,a_p;\,c+1,\,b_2,\,\ldots,\,b_q;\,z)\right) - z\left(\prod_{j=1}^{p} a_j\right){}_pF_q(a_1+1,\,\ldots,\,a_p+1;\,c+2,\,b_2+1,\,\ldots,\,b_q+1;\,z) = 0$$

07.31.17.0009.01

$$\left(\prod_{k=1}^{q} b_k\right)\left({}_pF_q(a,\,b+1,\,a_3,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z) - {}_pF_q(a+1,\,b,\,a_3,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z)\right) - z\,(b-a)\left(\prod_{j=3}^{p} a_j\right){}_pF_q(a+1,\,b+1,\,a_3+1,\,\ldots,\,a_p+1;\,b_1+1,\,\ldots,\,b_q+1;\,z) = 0$$

07.31.17.0010.01

$$c\,(c+1)\left(\prod_{k=2}^{q} b_k\right)\left({}_pF_q(a,\,a_2,\,\ldots,\,a_p;\,c,\,b_2,\,\ldots,\,b_q;\,z) - {}_pF_q(a+1,\,a_2,\,\ldots,\,a_p;\,c+1,\,b_2,\,\ldots,\,b_q;\,z)\right) - z\,(c-a)\left(\prod_{j=2}^{p} a_j\right){}_pF_q(a+1,\,a_2+1,\,\ldots,\,a_p+1;\,c+2,\,b_2+1,\,\ldots,\,b_q+1;\,z) = 0$$

07.31.17.0011.01

$$\left(\prod_{k=2}^{q} b_k\right)\left(c\;{}_pF_q(a,\,b,\,a_3,\,\ldots,\,a_p;\,c,\,b_2,\,\ldots,\,b_q;\,z) - a\;{}_pF_q(a+1,\,b+1,\,a_3,\,\ldots,\,a_p;\,c+1,\,b_2,\,\ldots,\,b_q;\,z) - (c-a)\;{}_pF_q(a,\,b+1,\,a_3,\,\ldots,\,a_p;\,c+1,\,b_2,\,\ldots,\,b_q;\,z)\right) + z\,a\left(\prod_{j=3}^{p} a_j\right){}_pF_q(a+1,\,b+1,\,a_3+1,\,\ldots,\,a_p+1;\,c+1,\,b_2+1,\,\ldots,\,b_q+1;\,z) = 0$$

07.31.17.0012.01

$${}_pF_q(a+1,\,b+1,\,a_3,\,\ldots,\,a_p;\,c+1,\,d+1,\,e+1,\,b_4,\,\ldots,\,b_q;\,z) - \frac{c\,d\,(a-e)\,(b-e)}{a\,b\,(c-e)\,(d-e)}\;{}_pF_q(a,\,b,\,a_3,\,\ldots,\,a_p;\,c,\,d,\,e+1,\,b_4,\,\ldots,\,b_q;\,z) - \frac{c\,e\,(a-d)\,(b-d)}{a\,b\,(c-d)\,(e-d)}\;{}_pF_q(a,\,b,\,a_3,\,\ldots,\,a_p;\,c,\,d+1,\,e,\,b_4,\,\ldots,\,b_q;\,z) - \frac{d\,e\,(a-c)\,(b-c)}{a\,b\,(d-c)\,(e-c)}\;{}_pF_q(a,\,b,\,a_3,\,\ldots,\,a_p;\,c+1,\,d,\,e,\,b_4,\,\ldots,\,b_q;\,z) = 0$$

07.31.17.0013.01

$${}_pF_q(a,\,b,\,c,\,a_4,\,\ldots,\,a_p;\,d,\,e,\,b_3,\,\ldots,\,b_q;\,z) - \frac{a\,b\,(d-c)\,(e-c)}{d\,e\,(a-c)\,(b-c)}\;{}_pF_q(a+1,\,b+1,\,c,\,a_4,\,\ldots,\,a_p;\,d+1,\,e+1,\,b_3,\,\ldots,\,b_q;\,z) - \frac{a\,c\,(d-b)\,(e-b)}{d\,e\,(a-b)\,(c-b)}\;{}_pF_q(a+1,\,b,\,c+1,\,a_4,\,\ldots,\,a_p;\,d+1,\,e+1,\,b_3,\,\ldots,\,b_q;\,z) - \frac{b\,c\,(d-a)\,(e-a)}{d\,e\,(b-a)\,(c-a)}\;{}_pF_q(a,\,b+1,\,c+1,\,a_4,\,\ldots,\,a_p;\,d+1,\,e+1,\,b_3,\,\ldots,\,b_q;\,z) = 0$$

07.31.17.0014.01

$$\left(a+z\sum_{j=1}^{q}\left(a_{j+1}-b_j\right)\right){}_{q+1}F_q\left(a,a_2,\ldots,a_{q+1};b_1,\ldots,b_q;z\right)+ z\sum_{j=1}^{q}\frac{\left(b_j-a\right)\prod_{k=1}^{q}\left(b_j-a_{k+1}\right)}{b_j\prod_{\substack{k=1\\k\neq j}}^{q}\left(b_j-b_k\right)}\,{}_{q+1}F_q\left(a,a_2,\ldots,a_{q+1};b_1,\ldots,b_{j-1},b_j+1,b_{j+1},\ldots,b_q;z\right)= a(1-z)\,{}_{q+1}F_q\left(a+1,a_2,\ldots,a_{q+1};b_1,\ldots,b_q;z\right)$$

Relations of special kind

07.31.17.0015.01

$${}_pF_q\left(a_1,\ldots,a_p;c,1-c,b_3,\ldots,b_q;z\right)+{}_pF_q\left(a_1,\ldots,a_p;-c,1+c,b_3,\ldots,b_q;z\right)=2\,{}_pF_q\left(a_1,\ldots,a_p;1+c,1-c,b_3,\ldots,b_q;z\right)$$

07.31.17.0016.01

$${}_pF_q\left(a,a_2,\ldots,a_p;-a,1+a,b_3,\ldots,b_q;z\right)-2\,{}_pF_q\left(a,a_2,\ldots,a_p;1-a,1+a,b_3,\ldots,b_q;z\right)= -{}_{p-1}F_{q-1}\left(a_2,\ldots,a_p;1-a,b_3,\ldots,b_q;z\right)$$

07.31.17.0017.01

$${}_pF_q\left(a,a_2,\ldots,a_p;1+a,b_2,\ldots,b_q;z\right)+{}_pF_q\left(-a,a_2,\ldots,a_p;1-a,b_2,\ldots,b_q;z\right)= 2\,{}_{p+1}F_{q+1}\left(a,-a,a_2,\ldots,a_p;1+a,1-a,b_2,\ldots,b_q;z\right)$$

07.31.17.0018.01

$${}_pF_q\left(a,1-a,a_3,\ldots,a_p;b_1,\ldots,b_q;z\right)+{}_pF_q\left(-a,1+a,a_3,\ldots,a_p;b_1,\ldots,b_q;z\right)=2\,{}_{p+1}F_{q+1}\left(a,-a,a_3,\ldots,a_p;b_1,\ldots,b_q;z\right)$$

Division on even and odd parts and generalization

07.31.17.0019.01

$${}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right)=A^+(z)+A^-(z)\;/;\;A^+(z)=\frac{1}{2}\left({}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right)+{}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;-z\right)\right)\bigwedge A^-(z)=\frac{1}{2}\left({}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right)-{}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;-z\right)\right)$$

07.31.17.0020.01

$${}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right)=A^+(z)+A^-(z)\;/;$$
$$A^+(z)={}_{2p}F_{2q+1}\left(\frac{a_1}{2},\ldots,\frac{a_p}{2},\frac{a_1+1}{2},\ldots,\frac{a_p+1}{2};\frac{1}{2},\frac{b_1}{2},\ldots,\frac{b_q}{2},\frac{b_1+1}{2},\ldots,\frac{b_q+1}{2};4^{p-q-1}z^2\right)\bigwedge$$
$$A^-(z)=\frac{z\prod_{j=1}^{p}a_j}{\prod_{j=1}^{q}b_j}\,{}_{2p}F_{2q+1}\left(\frac{a_1+1}{2},\ldots,\frac{a_p+1}{2},\frac{a_1+2}{2},\ldots,\frac{a_p+2}{2};\frac{3}{2},\frac{b_1+1}{2},\ldots,\frac{b_q+1}{2},\frac{b_1+2}{2},\ldots,\frac{b_q+2}{2};4^{p-q-1}z^2\right)$$

07.31.17.0021.01

$${}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right)= \sum_{k=0}^{n-1}\frac{z^k\prod_{j=1}^{p}\left(a_j\right)_k}{k!\prod_{j=1}^{q}\left(b_j\right)_k}\,{}_{n\,p+1}F_{n\,q+n}\left(1,\frac{a_1+k}{n},\ldots,\frac{a_1+k+n-1}{n},\ldots,\frac{a_p+k}{n},\ldots,\frac{a_p+k+n-1}{n};\frac{k+1}{n},\ldots,\frac{k+n}{n},\frac{b_1+k}{n},\ldots,\frac{b_1+k+n-1}{n},\ldots,\frac{b_q+k}{n},\ldots,\frac{b_q+k+n-1}{n};n^{n(p-q-1)}z^n\right)$$

Case ${}_{q+1}F_q$

07.31.17.0022.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{k=1}^{q+1} \frac{\Gamma(a_k)\prod_{j=1, j\neq k}^{q+1}\Gamma(a_j - a_k)}{\prod_{j=1}^{q}\Gamma(b_j - a_k)} (-z)^{-a_k} $$

$$ {}_{q+1}F_q\left(a_k, a_k - b_1 + 1, \ldots, a_k - b_q + 1; 1 - a_1 + a_k, \ldots, 1 - a_{k-1} + a_k, 1 - a_{k+1} + a_k, \ldots, 1 - a_{q+1} + a_k; \frac{1}{z}\right) /; $$

$$ z \notin (0, 1) \bigwedge \forall_{\{j,k\},\{j,k\}\in\mathbb{Z}\bigwedge j\neq k\bigwedge 1\leq j\leq q+1\bigwedge 1\leq k\leq q+1} (a_j - a_k \notin \mathbb{Z}) $$

07.31.17.0023.01

$$ \sum_{k=1}^{m} \frac{\prod_{j=1, j\neq k}^{m}\Gamma(b_j - b_k)\prod_{j=1}^{n}\Gamma(1 - a_j + b_k)}{\prod_{j=n+1}^{q+1}\Gamma(a_j - b_k)\prod_{j=m+1}^{q+1}\Gamma(1 - b_j + b_k)} z^{b_k} $$

$$ {}_{q+1}F_q(1 - a_1 + b_k, \ldots, 1 - a_{q+1} + b_k; 1 - b_1 + b_k, \ldots, 1 - b_{k-1} + b_k, 1 - b_{k+1} + b_k, \ldots, 1 - b_{q+1} + b_k; (-1)^{q-m-n+1} z) = $$

$$ \sum_{k=1}^{n} \frac{\prod_{j=1, j\neq k}^{n}\Gamma(a_k - a_j)\prod_{j=1}^{m}\Gamma(1 - a_k + b_j)}{\prod_{j=m+1}^{q+1}\Gamma(a_k - b_j)\prod_{j=n+1}^{q+1}\Gamma(a_j - a_k + 1)} z^{a_k - 1} $$

$$ {}_{q+1}F_q\left(1 - a_k + b_1, \ldots, 1 - a_k + b_{q+1}; 1 + a_1 - a_k, \ldots, 1 + a_{k-1} - a_k, 1 + a_{k+1} - a_k, \ldots, 1 + a_{q+1} - a_k; \frac{(-1)^{q-m-n+1}}{z}\right) /; $$

$$ m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge q \in \mathbb{N} \bigwedge m \leq q + 1 \bigwedge n \leq q + 1 \bigwedge (m + n - q > 2 \bigvee (m + n - q = 2 \bigwedge z \notin (-1, 0))) $$

07.31.17.0024.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; w z) = (1 - z)^{-a_1} \sum_{k=0}^{\infty} \frac{(a_1)_k}{k!} {}_{q+1}F_q(-k, a_2, \ldots, a_{q+1}; b_1, \ldots, b_q; w) \left(\frac{z}{z - 1}\right)^k $$

Differentiation

Low-order differentiation

With respect to a_1

07.31.20.0001.01

$$ {}_pF_q^{(\{1,0,\ldots,0\},\{0,\ldots,0\},0)}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{\psi(k + a_1)\left(\prod_{j=1}^{p}(a_j)_k\right) z^k}{k!\prod_{j=1}^{q}(b_j)_k} - \psi(a_1)\, {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) /; $$

$$ q = p - 1 \bigwedge |z| < 1 \bigvee q \geq p $$

07.31.20.0002.01

$$ {}_pF_q^{(\{1,0,\ldots,0\},\{0,\ldots,0\},0)}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{z\prod_{j=2}^{p} a_j}{\prod_{j=1}^{q} b_j} F_{q+1\times 0\times 1}^{p\,1\times 2}\left(\begin{matrix} a_1 + 1, \ldots, a_p + 1; 1; 1, a_1; \\ 2, b_1 + 1, \ldots, b_q + 1;; a_1 + 1; \end{matrix}\; z, z\right) $$

With respect to b_1

07.31.20.0003.01

$$ {}_pF_q^{(\{0,\ldots,0\},\{1,0,\ldots,0\},0)}(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z) = \psi(b_1)\;{}_pF_q(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z) - \sum_{k=0}^{\infty} \frac{\psi(k+b_1)\left(\prod_{j=1}^{p}(a_j)_k\right) z^k}{k!\,\prod_{j=1}^{q}(b_j)_k} \;/;\; q = p-1 \land |z|<1 \lor q \geq p $$

07.31.20.0004.01

$$ {}_pF_q^{(\{0,\ldots,0\},\{1,0,\ldots,0\},0)}(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z) = -\frac{z\prod_{j=1}^{p} a_j}{b_1\prod_{j=1}^{q} b_j} F_{q+1\times 0\times 1}^{p\,1\times 2}\left(\begin{matrix} a_1+1,\,\ldots,\,a_p+1;\,1;\,1,\,b_1; \\ 2,\,b_1+1,\,\ldots,\,b_q+1;;\,b_1+1; \end{matrix}\; z,\,z\right) $$

With respect to element of parameters ||| With respect to element of parameters

07.31.20.0005.01

$$ \frac{\partial\,{}_pF_q(a,\,a_2,\,\ldots,\,a_p;\,a+1,\,b_2,\,\ldots,\,b_q;\,z)}{\partial a} = \frac{z\prod_{j=2}^{p} a_j}{(a+1)^2\prod_{j=2}^{q} b_j}\;{}_{p+1}F_{q+1}(a+1,\,a+1,\,a_2+1,\,\ldots,\,a_p+1;\,a+2,\,a+2,\,b_2+1,\,\ldots,\,b_q+1;\,z) $$

07.31.20.0006.01

$$ \frac{\partial\,{}_pF_q(a+1,\,a_2,\,\ldots,\,a_p;\,a,\,b_2,\,\ldots,\,b_q;\,z)}{\partial a} = -\frac{z\prod_{j=2}^{p} a_j}{a^2\prod_{j=2}^{q} b_j}\;{}_{p-1}F_{q-1}(a_2+1,\,\ldots,\,a_p+1;\,b_2+1,\,\ldots,\,b_q+1;\,z) $$

With respect to z

07.31.20.0007.01

$$ \frac{\partial\,{}_pF_q(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z)}{\partial z} = \frac{\prod_{j=1}^{p} a_j}{\prod_{j=1}^{q} b_j}\;{}_pF_q(a_1+1,\,\ldots,\,a_p+1;\,b_1+1,\,\ldots,\,b_q+1;\,z) $$

07.31.20.0008.01

$$ \frac{\partial^2\,{}_pF_q(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z)}{\partial z^2} = \frac{\prod_{j=1}^{p} a_j\,(a_j+1)}{\prod_{j=1}^{q} b_j\,(b_j+1)}\;{}_pF_q(a_1+2,\,\ldots,\,a_p+2;\,b_1+2,\,\ldots,\,b_q+2;\,z) $$

Symbolic differentiation

With respect to a_1

07.31.20.0009.01

$$ {}_pF_q^{(\{n,0,\ldots,0\},\{0,\ldots,0\},0)}(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z) = \sum_{k=0}^{\infty} \frac{\prod_{j=2}^{p}(a_j)_k}{k!\,\prod_{j=1}^{q}(b_j)_k}\,\frac{\partial^n (a_1)_k}{\partial a_1^n}\,z^k \;/;\; n\in\mathbb{N}^+ \land q = p-1 \land |z|<1 \lor q \geq p $$

With respect to b_1

07.31.20.0010.01

$$ {}_pF_q^{(\{0,\ldots,0\},\{n,0,\ldots,0\},0)}(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_k}{k!\,\prod_{j=2}^{q}(b_j)_k}\,\frac{\partial^n \frac{1}{(b_1)_k}}{\partial b_1^n}\,z^k \;/;\; |z|<1 \land n\in\mathbb{N}^+ \land q = p-1 \land |z|<1 \lor q \geq p $$

With respect to element of parameters ||| With respect to element of parameters

07.31.20.0021.01

$$\frac{\partial^n {}_pF_q(a, a_2, \ldots, a_p; a+1, b_2, \ldots, b_q; z)}{\partial a^n} = \frac{(-1)^{n-1}\, n!\, z\left(\prod_{j=2}^p a_j\right)}{(a+1)^{n+1} \prod_{j=2}^q b_j} \, {}_{p+n}F_{q+n}(a+1, \ldots, a+1, a_2+1, \ldots, a_p+1; a+2, \ldots, a+2, b_2+1, \ldots, b_q+1; z) \,/;\, n \in \mathbb{N}^+$$

07.31.20.0022.01

$$\frac{\partial^n {}_pF_q(a+1, a_2, \ldots, a_p; a, b_2, \ldots, b_q; z)}{\partial a^n} = \frac{(-1)^n\, n!}{a^{n+1}} \left(a\, {}_{p-1}F_{q-1}(a_2, \ldots, a_p; b_2, \ldots, b_q; z) + \frac{z \prod_{j=2}^p a_j}{\prod_{j=2}^q b_j} \, {}_{p-1}F_{q-1}(a_2+1, \ldots, a_p+1; b_2+1, \ldots, b_q+1; z) \right) + \frac{(-1)^{n-1}\, n!}{a^n} \, {}_{p-1}F_{q-1}(a_2, \ldots, a_p; b_2, \ldots, b_q; z) \,/;\, n \in \mathbb{N}^+$$

With respect to z

07.31.20.0011.01

$$\frac{\partial^n {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{\partial z^n} = \frac{\prod_{j=1}^p (a_j)_n}{\prod_{j=1}^q (b_j)_n} \, {}_pF_q(a_1+n, \ldots, a_p+n; b_1+n, \ldots, b_q+n; z) \,/;\, n \in \mathbb{N}^+$$

07.31.20.0012.01

$$\frac{\partial^n {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)}{\partial z^n} = z^{-n} \prod_{j=1}^q \Gamma(b_j) \, {}_{p+1}\tilde{F}_{q+1}(1, a_1, \ldots, a_p; 1-n, b_1, \ldots, b_q; z) \,/;\, n \in \mathbb{N}^+$$

07.31.20.0013.01

$$\frac{\partial^n \left(z^\alpha {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)\right)}{\partial z^n} = (-1)^n (-\alpha)_n z^{\alpha-n} \, {}_{p+1}F_{q+1}(\alpha+1, a_1, \ldots, a_p; \alpha-n+1, b_1, \ldots, b_q; z) \,/;\, n \in \mathbb{N}^+$$

07.31.20.0014.01

$$\frac{\partial^n \left(z^{a+n-1} {}_pF_q(a, a_2, \ldots, a_p; b_1, \ldots, b_q; z)\right)}{\partial z^n} = (a)_n z^{a-1} \, {}_pF_q(a+n, a_2, \ldots, a_p; b_1, \ldots, b_q; z) \,/;\, n \in \mathbb{N}^+$$

07.31.20.0015.01

$$\frac{\partial^n \left(z^{c-1} {}_pF_q(a_1, \ldots, a_p; c, b_2, \ldots, b_q; z)\right)}{\partial z^n} = (c-n)_n z^{c-n-1} \, {}_pF_q(a_1, \ldots, a_p; c-n, b_2, \ldots, b_q; z) \,/;\, n \in \mathbb{N}^+$$

07.31.20.0016.01

$$\frac{\partial^n \left(z^n {}_pF_q\left(-n, a_2, \ldots, a_p; \frac{1}{2}, b_2, \ldots, b_q; z\right)\right)}{\partial z^n} = n! \, {}_{p+1}F_{q+1}\left(-n, n+1, a_2, \ldots, a_p; \frac{1}{2}, 1, b_2, \ldots, b_q; z\right) \,/;\, n \in \mathbb{N}^+$$

07.31.20.0017.01

$$\frac{\partial^n \left(z^\alpha {}_pF_q(-n, a_2, \ldots, a_p; b_1, \ldots, b_q; z)\right)}{\partial z^n} = (-1)^n (-\alpha)_n z^{\alpha-n} \, {}_{p+1}F_{q+1}(-n, \alpha+1, a_2, \ldots, a_p; \alpha-n+1, b_1, \ldots, b_q; z) \,/;\, n \in \mathbb{N}^+$$

07.31.20.0018.01

$$\frac{\partial^n\left(z^\alpha\,{}_pF_q\left(-\frac{n}{r},\frac{-n+1}{r},\ldots,\frac{-n+r-1}{r},a_{r+1},\ldots,a_p;b_1,\ldots,b_q;z^m\right)\right)}{\partial z^n}=(-1)^n(-\alpha)_n z^{\alpha-n}\,{}_{p+m}F_{q+m}\left(-\frac{n}{r},\frac{-n+1}{r},\ldots,\frac{-n+r-1}{r},\frac{\alpha+1}{m},\frac{\alpha+2}{m},\ldots,\frac{\alpha+m}{m},a_{r+1},\ldots,a_p;\frac{\alpha-n+1}{m},\frac{\alpha-n+2}{m},\ldots,\frac{\alpha-n+m}{m},b_1,\ldots,b_q;z^m\right)\,/;\,r\in\mathbb{N}^+\land m\in\mathbb{N}^+\land n\in\mathbb{N}^+$$

07.31.20.0019.01

$$\frac{\partial^n\left(e^{-z}\,{}_pF_q(-n,a_2,\ldots,a_p;b_1,\ldots,b_q;z)\right)}{\partial z^n}=(-1)^n e^{-z}\sum_{k=0}^{n}\frac{(-n)_k z^k}{k!\prod_{j=1}^{q}(b_j)_k}\,{}_{p+1}F_q(-n,k-n,a_2+k,\ldots,a_p+k;b_1+k,\ldots,b_q+k;z)\,/;\,n\in\mathbb{N}^+$$

Fractional integro-differentiation

With respect to z

07.31.20.0020.01

$$\frac{\partial^\alpha\,{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)}{\partial z^\alpha}=z^{-\alpha}\prod_{j=1}^{q}\Gamma(b_j)\,{}_{p+1}\tilde{F}_{q+1}(1,a_1,\ldots,a_p;1-\alpha,b_1,\ldots,b_q;z)$$

Integration

Indefinite integration

Involving only one direct function

07.31.21.0001.01

$$\int {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)\,dz=\frac{\prod_{j=1}^{q}(b_j-1)}{\prod_{j=1}^{p}(a_j-1)}\,{}_pF_q(a_1-1,\ldots,a_p-1;b_1-1,\ldots,b_q-1;z)$$

Involving one direct function and elementary functions

Involving power function

07.31.21.0002.01

$$\int z^{\alpha-1}\,{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)\,dz=\frac{z^\alpha}{\alpha}\,{}_{p+1}F_{q+1}(\alpha,a_1,\ldots,a_p;\alpha+1,b_1,\ldots,b_q;z)$$

Definite integration

For the direct function itself

07.31.21.0003.01

$$\int_0^\infty t^{\alpha-1}\,{}_pF_q(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,-t)\,dt = \frac{\left(\prod_{k=1}^q\Gamma(b_k)\right)\left(\Gamma(\alpha)\prod_{k=1}^p\Gamma(a_k-\alpha)\right)}{\left(\prod_{k=1}^p\Gamma(a_k)\right)\prod_{k=1}^q\Gamma(b_k-\alpha)}\ /;$$

$$0<\operatorname{Re}(\alpha)<\min(\operatorname{Re}(a_1),\,\ldots,\,\operatorname{Re}(a_p))\land p-1\le q\le p\ \bigvee$$

$$0<\operatorname{Re}(\alpha)<\min\left(\operatorname{Re}(a_1),\,\ldots,\,\operatorname{Re}(a_p),\,\frac{1}{4}-\frac{1}{2}\operatorname{Re}\left(\sum_{j=1}^p a_j-\sum_{k=1}^q b_k\right)\right)\bigwedge q=p+1$$

Summation

Finite summation

07.31.23.0001.01

$$\sum_{k=0}^{n-1}\frac{\prod_{j=1}^p(a_j)_k}{\prod_{j=1}^q(b_j)_k}\,{}_{n\,p+1}F_{n\,q+n}\left(1,\,\frac{a_1+k}{n},\,\ldots,\,\frac{a_1+k+n-1}{n},\,\ldots,\,\frac{a_p+k}{n},\,\ldots,\,\frac{a_p+k+n-1}{n};\,\frac{k+1}{n},\,\ldots,\,\frac{k+n}{n},\right.$$

$$\left.\frac{b_1+k}{n},\,\ldots,\,\frac{b_1+k+n-1}{n},\,\frac{b_q+k}{n},\,\ldots,\,\frac{b_q+k+n-1}{n};\,n^{n(p-q-1)}z^n\right)\frac{z^k}{k!} = {}_pF_q(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z)$$

Infinite summation

07.31.23.0002.01

$$\sum_{k=0}^\infty\frac{(a_1)_k}{k!}\,{}_{q+1}F_q(-k,\,a_2,\,\ldots,\,a_{q+1};\,b_1,\,\ldots,\,b_q;\,w)\,z^k = \left(\frac{1}{1-z}\right)^{a_1}{}_{q+1}F_q\left(a_1,\,\ldots,\,a_{q+1};\,b_1,\,\ldots,\,b_q;\,\frac{z\,w}{z-1}\right)$$

Operations

Limit operation

07.31.25.0001.01

$$\lim_{z\to1}(1-z)^{-\psi_q}\,{}_{q+1}F_q(a_1,\,\ldots,\,a_{q+1};\,b_1,\,\ldots,\,b_q;\,z) = \frac{\Gamma(-\psi_q)\prod_{j=1}^q\Gamma(b_j)}{\prod_{j=1}^{q+1}\Gamma(a_j)}\ /;\ \psi_q=\sum_{j=1}^q b_j-\sum_{j=1}^{q+1}a_j\bigwedge\operatorname{Re}(\psi_q)<0$$

07.31.25.0002.01

$$\lim_{z\to1}\frac{1}{\log(1-z)}\,{}_{q+1}F_q(a_1,\,\ldots,\,a_{q+1};\,b_1,\,\ldots,\,b_q;\,z) = -\frac{\prod_{j=1}^q\Gamma(b_j)}{\prod_{j=1}^{q+1}\Gamma(a_j)}\ /;\ \psi_q=\sum_{j=1}^q b_j-\sum_{j=1}^{q+1}a_j\bigwedge\psi_q=0$$

07.31.25.0003.01

$$\lim_{b_1\to-n}\frac{{}_pF_q(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z)}{\Gamma(b_1)} =$$

$$z^{n+1}\prod_{j=1}^p(a_j)_{n+1}\,{}_p\tilde{F}_q(n+a_1+1,\,\ldots,\,n+a_p+1;\,n+2,\,n+b_2+1,\,\ldots,\,n+b_q+1;\,z)\ /;\ n\in\mathbb{N}^+$$

07.31.25.0004.01

$$\lim_{a\to\infty}{}_pF_q\left(a,\,a_2,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,\frac{z}{a}\right) = {}_{p-1}F_q(a_2,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z)$$

07.31.25.0005.01

$$\lim_{a\to\infty} {}_pF_q(a_1, \ldots, a_p; b, b_2, \ldots, b_q; b\, z) = {}_pF_{q-1}(a_1, \ldots, a_p; b_2, \ldots, b_q; z)$$

07.31.25.0006.01

$$\lim_{a\to n} \frac{a^{-q}}{\Gamma(1-a)}\, {}_{q+1}F_q(a, a_2, a_3, \ldots, a_{q+1}; a_2+1, a_3+1, \ldots, a_{q+1}+1; 1) = (-1)^{n-1}\, S_n^{(q)} \;/;\; a_2 = a_3 = \ldots = a_{q+1} = a \bigwedge q \in \mathbb{N} \bigwedge n \in \mathbb{N}$$

07.31.25.0007.01

$$\lim_{z\to 1} {}_{q+1}F_q(1-m, a_2+1, a_3+1, \ldots, a_{q+1}+1; a_2, a_3, \ldots, a_{q+1}; z) = (-1)^{m-1}\, \Gamma(m)\, \mathcal{S}_{q+1}^{(m)} \;/;\; a_2 = a_3 = \ldots = a_{q+1} = 1 \bigwedge m \in \mathbb{N}^+$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_p\tilde{F}_q$

07.31.26.0001.01

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \left(\prod_{k=1}^{q} \Gamma(b_k)\right) {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$$

Through hypergeometric functions of two variables

07.31.26.0002.01

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = F_{0\,q\,0}^{0\,p\,0}\left(\begin{matrix} ;\, a_1, \ldots, a_p;; \\ ;\, b_1, \ldots, b_q;; \end{matrix}\; z, 0\right)$$

07.31.26.0003.01

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \left(\prod_{k=1}^{q} \Gamma(b_k)\right) \tilde{F}_{0\,q\,0}^{0\,p\,0}\left(\begin{matrix} ;\, a_1, \ldots, a_p;; \\ ;\, b_1, \ldots, b_q;; \end{matrix}\; z, 0\right)$$

Through Meijer *G*

Classical cases for the direct function itself

07.31.26.0004.01

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{p} \Gamma(a_k)}\, G_{p,q+1}^{1,p}\left(-z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_p \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right)$$

07.31.26.0005.01

$$ {}_{q+1}F_q(a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\pi \sin(\psi_q \pi) \prod_{k=1}^{q+1}\Gamma(a_k)} \sum_{j=1}^{q} \frac{\prod_{k=1}^{q+1}\sin(\pi(b_j-a_k))}{\prod_{k=1, k\neq j}^{q}\sin(\pi(b_j-b_k))} G_{q+1,q+1}^{2,q+1}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_{q+1} \\ 0, 1-b_j, 1-b_1, \ldots, 1-b_{j-1}, 1-b_{j+1}, \ldots, 1-b_q \end{matrix}\right) - \frac{\pi \prod_{k=1}^{q}\Gamma(b_k)}{\sin(\psi_q \pi)\prod_{k=1}^{q+1}\Gamma(a_k)} \left((1-z)^{\psi_q}(z-1)^{-\psi_q} G_{q+1,q+1}^{0,q+1}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_{q+1} \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right) + G_{q+1,q+1}^{q+1,0}\left(z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_{q+1} \\ 0, 1-b_1, \ldots, 1-b_q \end{matrix}\right)\right) /; $$

$$ \psi_q = \sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j \bigwedge z \notin (-1, 0) \bigwedge \psi_q \notin \mathbb{Z} $$

07.31.26.0006.01

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \frac{\prod_{k=1}^{q}\Gamma(b_k)}{\prod_{k=1}^{p}\Gamma(a_k)} G_{4,3}^{3,1}\left(-\frac{1}{z} \,\middle|\, \begin{matrix} 1, b_1, \ldots, b_q \\ a_1, \ldots, a_p \end{matrix}\right) /; z \notin (0, \infty) $$

Theorems

Connections between series and continued fraction representations

Euler established that the convergent series $\sum_{k=1}^{\infty} a_k$ can be expressed in a continued fraction form

$$ \sum_{k=1}^{\infty} a_k = a_1 \left(1 + \text{ContinueFraction}\left[\left\{-\frac{a_{k+1}}{a_k}, 1 + \frac{a_{k+1}}{a_k}\right\}, \{k, 1, \infty\}\right]\right)^{-1}. $$

In particular the following representation takes place:

$$ {}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = 1 + \frac{z \prod_{k=1}^{p} a_k}{\prod_{k=1}^{q} b_k} \left(1 + \text{ContinueFraction}\left[\left\{-\frac{z \prod_{j=1}^{p}(k+a_j)}{(k+1)\prod_{j=1}^{q}(k+b_j)}, 1 + \frac{z \prod_{j=1}^{p}(k+a_j)}{(k+1)\prod_{j=1}^{q}(k+b_j)}\right\}, \{k, 1, \infty\}\right]\right)^{-1}. $$

History

– J. F. Pfaff (1797)
– T. Clausen (1828); J. Thomae (1870, 1879) studied differential equation
– S. Pincherle (1886, 1888)
– L. Pochhammer (1888)
– E. W. Barnes (1906–1908)
– T. W.Chaundy (1943)
– N. E. Nörlund (1955)
– A. P. Prudnikov, Y.A. Brychkov and O.I. Marichev (1986)

Copyright