InverseJacobiCS

Notations

Traditional name

Inverse of the Jacobi elliptic function cs

Traditional notation

$\operatorname{cs}^{-1}(z \mid m)$

Mathematica StandardForm notation

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InverseJacobiCS[z, m]
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Primary definition

09.39.02.0001.01

$$z = \operatorname{cs}(w \mid m) \,/;\, w = \operatorname{cs}^{-1}(z \mid m)$$

09.39.02.0002.01

$$\operatorname{cs}^{-1}(z \mid m) = \int_z^\infty \frac{1}{\sqrt{t^2+1}\,\sqrt{t^2-m+1}}\,dt \,/;\, z \in \mathbb{R} \bigwedge z^2 - m > -1$$

Specific values

Specialized values

For fixed *z*

09.39.03.0001.01

$$\operatorname{cs}^{-1}(z \mid 0) = \cot^{-1}(z)$$

09.39.03.0002.01

$$\operatorname{cs}^{-1}\left(z \,\middle|\, \frac{1}{2}\right) = -i\, F\left(i \sinh^{-1}\left(\frac{1}{z}\right) \,\middle|\, \frac{1}{2}\right)$$

09.39.03.0003.01

$$\operatorname{cs}^{-1}(z \mid 1) = \operatorname{csch}^{-1}(z)$$

For fixed *m*

09.39.03.0004.01

$$\operatorname{cs}^{-1}(-1 \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) - i\, F\left(i \sinh^{-1}(1) \,\middle|\, \frac{1}{1-m}\right)\right)$$

09.39.03.0005.01

$$\mathrm{cs}^{-1}\left(-\frac{1}{2}\,\middle|\,m\right) = -i\,F\left(i\sinh^{-1}(2)\,\middle|\,1-m\right) - \frac{2\,i}{\sqrt{1-m}}\,F\left(i\sinh^{-1}\left(\frac{1}{2}\right)\,\middle|\,\frac{1}{1-m}\right)$$

09.39.03.0006.01

$$\mathrm{cs}^{-1}(0\mid m) = \frac{1}{\sqrt{1-m}}\,K\left(\frac{m}{m-1}\right)$$

09.39.03.0007.01

$$\mathrm{cs}^{-1}\left(\frac{1}{2}\,\middle|\,m\right) = -i\,F\left(i\sinh^{-1}(2)\,\middle|\,1-m\right)$$

09.39.03.0008.01

$$\mathrm{cs}^{-1}(1\mid m) = \frac{i}{\sqrt{1-m}}\left(F\left(i\sinh^{-1}(1)\,\middle|\,\frac{1}{1-m}\right) + K\left(\frac{m}{m-1}\right)\right)$$

09.39.03.0009.01

$$\mathrm{cs}^{-1}(i\mid m) = -i\,K(1-m)$$

09.39.03.0010.01

$$\mathrm{cs}^{-1}(-i\mid m) = i\,K(1-m)$$

Values at infinities

09.39.03.0011.01

$$\mathrm{cs}^{-1}(z\mid\infty) = 0$$

09.39.03.0012.01

$$\mathrm{cs}^{-1}(z\mid-\infty) = 0$$

09.39.03.0013.01

$$\mathrm{cs}^{-1}(\infty\mid m) = 0$$

09.39.03.0014.01

$$\mathrm{cs}^{-1}(-\infty\mid m) = \frac{2}{\sqrt{1-m}}\,K\left(\frac{m}{m-1}\right)$$

General characteristics

Domain and analyticity

$\mathrm{cs}^{-1}(z\mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

09.39.04.0001.01

$$(z * m) \longrightarrow \mathrm{cs}^{-1}(z\mid m) :: (\mathbb{C}\otimes\mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.39.04.0002.01

$$\mathrm{cs}^{-1}(\bar{z}\mid\bar{m}) = \overline{\mathrm{cs}^{-1}(z\mid m)}$$

Quasi-reflection symmetry

09.39.04.0003.01

$$\mathrm{cs}^{-1}(-z \mid m) = \mathrm{cs}^{-1}(z \mid m) - \frac{2\,i}{\sqrt{1-m}}\, F\!\left(i \sinh^{-1}(z) \,\middle|\, \frac{1}{1-m}\right)$$

Poles and essential singularities

With respect to *m*

The function $\mathrm{cs}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to m.

09.39.04.0004.01

$$\mathcal{S}ing_m\left(\mathrm{cs}^{-1}(z \mid m)\right) = \{\}$$

With respect to *z*

The function $\mathrm{cs}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to z.

09.39.04.0005.01

$$\mathcal{S}ing_z\left(\mathrm{cs}^{-1}(z \mid m)\right) = \{\}$$

Branch points

With respect to *m*

For fixed z, the function $\mathrm{cs}^{-1}(z \mid m)$ has two branch points: $m = 1 + z^2$, $m = \tilde{\infty}$.

09.39.04.0006.01

$$\mathcal{BP}_m\left(\mathrm{cs}^{-1}(z \mid m)\right) = \left\{1 + z^2,\ \tilde{\infty}\right\}$$

09.39.04.0007.01

$$\mathcal{R}_m\left(\mathrm{cs}^{-1}(z \mid m),\ 1 + z^2\right) = \log$$

09.39.04.0008.01

$$\mathcal{R}_m\left(\mathrm{cs}^{-1}(z \mid m),\ \tilde{\infty}\right) = \log$$

With respect to *z*

For fixed m, the function $\mathrm{cs}^{-1}(z \mid m)$ has five branch points: $z = \pm\, i$, $z = \pm \sqrt{m-1}$, $z = \tilde{\infty}$.

09.39.04.0009.01

$$\mathcal{BP}_z\left(\mathrm{cs}^{-1}(z \mid m)\right) = \left\{i,\ -i,\ \sqrt{m-1},\ -\sqrt{m-1},\ \tilde{\infty}\right\}$$

09.39.04.0010.01

$$\mathcal{R}_z\left(\mathrm{cs}^{-1}(z \mid m),\ i\right) = 2$$

09.39.04.0011.01

$$\mathcal{R}_z\left(\mathrm{cs}^{-1}(z \mid m),\ -i\right) = 2$$

09.39.04.0012.01

$$\mathcal{R}_z\left(\mathrm{cs}^{-1}(z \mid m),\ \sqrt{m-1}\right) = 2$$

09.39.04.0013.01

$$\mathcal{R}_z\left(\mathrm{cs}^{-1}(z \mid m),\ -\sqrt{m-1}\right) = 2$$

09.39.04.0014.01

$$\mathcal{R}_z(\operatorname{cs}^{-1}(z \mid m), \tilde{\infty}) == \log$$

Branch cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at $z == 0$

09.39.06.0001.02

$$\operatorname{cs}^{-1}(z \mid m) \propto i\left(\frac{1}{\sqrt{1-m}} K\left(\frac{1}{1-m}\right) - K(1-m) - \frac{z}{\sqrt{m-1}}\left(1 - \frac{m-2}{6(m-1)} z^2 + \frac{8-8m+3m^2}{40(m-1)^2} z^4 - \ldots\right)\right) /; (z \to 0)$$

09.39.06.0002.01

$$\operatorname{cs}^{-1}(z \mid m) == i\left(\frac{1}{\sqrt{1-m}} K\left(\frac{1}{1-m}\right) - K(1-m) - \sum_{k=0}^{\infty} \frac{(m-1)^{-k-\frac{1}{2}} \left(\frac{1}{2}\right)_k}{(2k+1)\,k!} \, {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; 1-m\right) z^{2k+1}\right)$$

09.39.06.0007.01

$$\operatorname{cs}^{-1}(z \mid m) \propto i\left(\frac{1}{\sqrt{1-m}} K\left(\frac{1}{1-m}\right) - K(1-m)\right)(1 + O(z))$$

Expansions at $m == 0$

09.39.06.0003.02

$$\operatorname{cs}^{-1}(z \mid m) \propto \cot^{-1}(z) + \frac{\left(z^2+1\right)\cot^{-1}(z) - z}{4\left(z^2+1\right)} m + \frac{3\left(-3z^3 - 5z + 3\left(z^2+1\right)^2 \cot^{-1}(z)\right)}{64\left(z^2+1\right)^2} m^2 + \ldots /; (m \to 0)$$

09.39.06.0004.01

$$\operatorname{cs}^{-1}(z \mid m) == \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k z^{-2k-1}}{(2k+1)\,k!} \, {}_2F_1\left(k + \frac{1}{2}, k+1; k + \frac{3}{2}; -\frac{1}{z^2}\right) m^k$$

09.39.06.0008.01

$$\operatorname{cs}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2} \left(\cot^{-1}(z) - \frac{1}{2} z \sum_{j=1}^{k} \frac{\left(z^2+1\right)^{-j} (j-1)!}{\left(\frac{1}{2}\right)_j}\right) m^k /; |m| < 1$$

09.39.06.0005.01

$$\operatorname{cs}^{-1}(z \mid m) == \frac{1}{z} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j (1)_{j+k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j+k}}{j!\,k!^2 \left(\frac{3}{2}\right)_{j+k}} z^{-2j-2k} m^k$$

09.39.06.0006.01

$$\operatorname{cs}^{-1}(z \mid m) = \frac{1}{z} F_{1\times 0\times 1}^{2\times 0\times 1}\left(\begin{matrix} \frac{1}{2}, 1;; \frac{1}{2}; \\ \frac{3}{2};; 1; \end{matrix} -\frac{1}{z^2}, \frac{m}{z^2}\right)$$

09.39.06.0009.01

$$\operatorname{cs}^{-1}(z \mid m) \propto \cot^{-1}(z)\,(1 + O(m))$$

Integral representations

On the real axis

Of the direct function

09.39.07.0001.01

$$\operatorname{cs}^{-1}(z \mid m) = \int_z^\infty \frac{1}{\sqrt{t^2+1}\,\sqrt{t^2-m+1}}\,dt\,/;\, z \in \mathbb{R} \bigwedge z^2 - m > -1$$

09.39.07.0002.01

$$\operatorname{cs}^{-1}(z \mid m) = \operatorname{cs}^{-1}(z_0 \mid m) - \frac{\sqrt{z^2-m+1}\,\operatorname{nd}\left(\operatorname{cs}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2+1}} \int_{z_0}^{z} \frac{1}{\sqrt{t^2+1}\,\sqrt{t^2-m+1}}\,dt\,/;$$
$$\neg \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\operatorname{Im}\left((\tau (z-z_0)+z_0)^2+1\right) = 0 \bigwedge (\tau (z-z_0)+z_0)^2+1 < 0 \bigwedge \operatorname{Im}\left((\tau (z-z_0)+z_0)^2-m+1\right) = 0 \bigwedge (\tau (z-z_0)+z_0)^2-m+1 < 0\right)$$

09.39.07.0003.01

$$\operatorname{cs}^{-1}(z \mid m) = \frac{\sqrt{z^2-m+1}\,\operatorname{nd}\left(\operatorname{cs}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2+1}} \int_{z}^{\infty} \frac{1}{\sqrt{t^2+1}\,\sqrt{t^2-m+1}}\,dt\,/;$$
$$\neg \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\operatorname{Im}\left(\left(z+\tan\left(\frac{\pi\tau}{2}\right)\right)^2+1\right) = 0 \bigwedge \left(z+\tan\left(\frac{\pi\tau}{2}\right)\right)^2+1 < 0 \bigwedge \operatorname{Im}\left(\left(z+\tan\left(\frac{\pi\tau}{2}\right)\right)^2-m+1\right) = 0 \bigwedge \left(z+\tan\left(\frac{\pi\tau}{2}\right)\right)^2-m+1 < 0\right)$$

Differential equations

Ordinary nonlinear differential equations

09.39.13.0001.01

$$\left(2 z^2 - m + 2\right) z\, w'(z)^3 + w''(z) = 0\,/;\, w(z) = \operatorname{cs}^{-1}(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.39.16.0001.01

$$\text{cs}^{-1}(-z \mid m) == \text{cs}^{-1}(z \mid m) - \frac{2\,i}{\sqrt{1-m}}\, F\!\left(i \sinh^{-1}(z) \,\middle|\, \frac{1}{1-m}\right)$$

Identities

Functional identities

09.39.17.0001.01

$$\left(z_1^2 - z_2^2\right)^2 \text{cs}(w(z_1) + w(z_2) \mid m)^4 + 2\left(-z_2^2 z_1^4 + \left(-z_2^4 + 2(m-2) z_2^2 + m - 1\right) z_1^2 + (m-1) z_2^2\right) \text{cs}(w(z_1) + w(z_2) \mid m)^2 + \left(z_1^2 z_2^2 + m - 1\right)^2 == 0 \;/;\; w(z) == \text{cs}^{-1}(z \mid m)$$

Differentiation

Low-order differentiation

With respect to *z*

09.39.20.0001.02

$$\frac{\partial \text{cs}^{-1}(z \mid m)}{\partial z} == -\frac{\text{nd}\left(\text{cs}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2 + 1}$$

09.39.20.0002.01

$$\frac{\partial \text{cs}^{-1}(z \mid m)}{\partial z} == -\frac{1}{\sqrt{z^2+1}\,\sqrt{z^2-m+1}} \;/;\; z \in \mathbb{R} \bigwedge z^2 - m > -1$$

09.39.20.0003.02

$$\frac{\partial^2 \text{cs}^{-1}(z \mid m)}{\partial z^2} == \frac{z\left(2 z^2 - m + 2\right) \text{nd}\left(\text{cs}^{-1}(z \mid m) \,\middle|\, m\right)}{\left(z^2+1\right)^2 \left(z^2 - m + 1\right)}$$

09.39.20.0011.01

$$\frac{\partial^2 \text{cs}^{-1}(z \mid m)}{\partial z^2} = -\frac{\sqrt{z^2-m+1}\;\text{nd}\left(\text{cs}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2+1}} \frac{\partial \frac{1}{\sqrt{z^2+1}\,\sqrt{z^2-m+1}}}{\partial z}$$

With respect to *m*

09.39.20.0004.02

$$\frac{\partial \text{cs}^{-1}(z \mid m)}{\partial m} == -\frac{\left(z^2+1\right) E\left(\text{am}\left(\text{cs}^{-1}(z \mid m) \,\middle|\, m\right) \,\middle|\, m\right) + (m-1)\left(z^2+1\right) \text{cs}^{-1}(z \mid m) - m\, z\, \text{nd}\left(\text{cs}^{-1}(z \mid m) \,\middle|\, m\right)}{2(m-1)\, m \left(z^2+1\right)}$$

09.39.20.0005.01

$$\frac{\partial \text{cs}^{-1}(z \mid m)}{\partial m} == \frac{1}{2\, m\,(m-1)} \left(\frac{\sqrt{z^2+1}\; z}{\sqrt{z^2-m+1}} - E(m) + i\sqrt{1-m}\; E\!\left(i \sinh^{-1}(z) \,\middle|\, \frac{1}{1-m}\right) + (1-m)\, K(m)\right) /;\; z \in \mathbb{R} \land m < 1$$

09.39.20.0006.02

$$\frac{\partial^2 \mathrm{cs}^{-1}(z \mid m)}{\partial m^2} = \frac{1}{4 (m-1)^2 m^2 \left(z^2 - m + 1\right)} \left(3 \left(z^2 - m + 1\right) \mathrm{cs}^{-1}(z \mid m) (m-1)^2 + \left(z^2 - m + 1\right) \left((4 m - 2) E\left(\mathrm{am}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1) F\left(\mathrm{am}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right) \mid m\right)\right) + \frac{m z \left(z^2 + m \left(-3 z^2 + 4 m - 5\right) + 1\right) \mathrm{nd}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right)}{z^2 + 1}\right)$$

09.39.20.0012.01

$$\frac{\partial^3 \mathrm{cs}^{-1}(z \mid m)}{\partial m^3} = -\frac{1}{4 (m-1)^3 m^3 \left(z^2 - m + 1\right)^2} \left(\frac{1}{2} \left(-z^2 + m - 1\right) \left((23 (m-1) m + 8) \left(-z^2 + m - 1\right) E\left(\mathrm{am}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1) \left(m \sqrt{1 - \frac{m}{z^2+1}}\, z + (11 m - 7) \left(-z^2 + m - 1\right) F\left(\mathrm{am}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right) \mid m\right)\right)\right) + \frac{1}{2 \left(z^2 + 1\right)} \left(15 (m-1)^3 \left(z^2 + 1\right) \left(z^2 - m + 1\right)^2 \mathrm{cs}^{-1}(z \mid m) - m z \left((3 m (5 m - 4) + 5) z^4 - 5 (m-1) (m (7 m - 5) + 2) z^2 + (m-1)^2 (m (23 m - 13) + 5)\right) \mathrm{nd}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right)\right)\right)$$

Symbolic differentiation

With respect to z

09.39.20.0013.01

$$\frac{\partial^n \mathrm{cs}^{-1}(z \mid m)}{\partial z^n} = \delta_n \mathrm{cs}^{-1}(z \mid m) - \frac{\mathrm{nd}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right)}{z^2 + 1} \sum_{j=0}^{n-1} \frac{(-1)^j (1-n)_{2(n-j)-2}}{(n-j-1)!\, (2 z)^{-2 j + n - 1}} \sum_{k=0}^{j} \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} \left(z^2 + 1\right)^{-k} \left(z^2 - m + 1\right)^{k-j} /; n \in \mathbb{N}$$

09.39.20.0014.01

$$\frac{\partial^n \mathrm{cs}^{-1}(z \mid m)}{\partial z^n} = \delta_n \mathrm{cs}^{-1}(z \mid m) - \frac{\mathrm{nd}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right)}{z^2 + 1} \sum_{j=0}^{n-1} \frac{(-1)^j 2^{2 j - n + 1} z^{2 j - n + 1} \left(z^2 - m + 1\right)^{-j} \left(\frac{1}{2}\right)_j (1-n)_{2(n-j)-2}}{(n-j-1)!} \, {}_2F_1\left(\frac{1}{2}, -j; \frac{1}{2} - j; \frac{z^2 - m + 1}{z^2 + 1}\right) /; n \in \mathbb{N}$$

09.39.20.0015.01

$$\frac{\partial^n \mathrm{cs}^{-1}(z \mid m)}{\partial z^n} = \delta_n \mathrm{cs}^{-1}(z \mid m) - \frac{\sqrt{z^2 - m + 1}\, \mathrm{nd}\left(\mathrm{cs}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2 + 1}} \frac{\partial^{n-1} \frac{1}{\sqrt{z^2+1} \sqrt{z^2-m+1}}}{\partial z^{n-1}} /; n \in \mathbb{N}^+$$

09.39.20.0007.01

$$\frac{\partial^n \operatorname{cs}^{-1}(z\mid m)}{\partial z^n} = -\frac{2^{n-1}\,\pi\, z^{n-1}\,(n-1)!\,\operatorname{nd}\left(\operatorname{cs}^{-1}(z\mid m)\,\middle|\, m\right)}{z^2+1}\sum_{j=0}^{n-1}\frac{\left(z^2+1\right)^{-j}\left(z^2-m+1\right)^{j-n+1}}{j!\,(n-j-1)!\,\Gamma\left(\frac{1}{2}-j\right)\Gamma\left(j-n+\frac{3}{2}\right)}\, {}_2F_1\left(\frac{1-j}{2}, -\frac{j}{2}; \frac{1}{2}-j; 1+\frac{1}{z^2}\right) {}_2F_1\left(\frac{j-n+2}{2}, \frac{j-n+1}{2}; j-n+\frac{3}{2}; \frac{1-m}{z^2}+1\right) /; n\in\mathbb{N}^+$$

With respect to m

09.39.20.0008.02

$$\frac{\partial^n \operatorname{cs}^{-1}(z\mid m)}{\partial m^n} = \frac{(-1)^n\sqrt{\pi}\, z^{-2n-1}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; -\frac{1}{z^2}, -\frac{1-m}{z^2}\right) /; n\in\mathbb{N}$$

Fractional integro-differentiation

With respect to z

09.39.20.0009.01

$$\frac{\partial^\alpha \operatorname{cs}^{-1}(z\mid m)}{\partial z^\alpha} = \frac{z^{-\alpha}}{\Gamma(1-\alpha)}\left(\frac{i}{\sqrt{m-1}} K\left(\frac{1}{1-m}\right) - K(1-m)\right) - \frac{i\, z^{1-\alpha}\sqrt{\pi}}{\sqrt{m-1}}\tilde{F}^{2\times 1\times 1}_{2\times 0\times 0}\left(\begin{matrix}\frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2}; \\ \frac{3-\alpha}{2}, 1-\frac{\alpha}{2};;;\end{matrix}\; -z^2, \frac{z^2}{m-1}\right) /;$$
$$-1 < z < 1 \land m < 0$$

With respect to m

09.39.20.0010.01

$$\frac{\partial^\alpha \operatorname{cs}^{-1}(z\mid m)}{\partial m^\alpha} = \frac{m^{-\alpha}\sqrt{\pi}}{2\sqrt{z^2+1}}\tilde{F}^{1\times 1\times 2}_{1\times 0\times 1}\left(\begin{matrix}\frac{1}{2}; \frac{1}{2}; \frac{1}{2}, 1; \\ \frac{3}{2};; 1-\alpha;\end{matrix}\; \frac{1}{z^2+1}, \frac{m}{z^2+1}\right) /; z > 0 \land m < 1$$

Integration

Indefinite integration

Involving only one direct function

09.39.21.0001.01

$$\int \operatorname{cs}^{-1}(z\mid m)\, dz = z\operatorname{cs}^{-1}(z\mid m) + \log\left(\operatorname{ds}\left(\operatorname{cs}^{-1}(z\mid m)\,\middle|\, m\right) + \operatorname{ns}\left(\operatorname{cs}^{-1}(z\mid m)\,\middle|\, m\right)\right)$$

Representations through more general functions

Through hypergeometric functions of two variables

09.39.26.0001.01

$$\operatorname{cs}^{-1}(z\mid m) = \frac{1}{z} F^{2\times 0\times 1}_{1\times 0\times 1}\left(\begin{matrix}\frac{1}{2}, 1;; \frac{1}{2}; \\ \frac{3}{2};; 1;\end{matrix}\; -\frac{1}{z^2}, \frac{m}{z^2}\right)$$

Through other functions

Involving some hypergeometric-type functions

09.39.26.0002.01

$$\operatorname{cs}^{-1}(z \mid m) = \frac{1}{z} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -\frac{1}{z^2}, \frac{m-1}{z^2}\right) /; z \in \mathbb{R} \bigwedge m - z^2 < 1$$

Representations through equivalent functions

With inverse function

09.39.27.0001.01

$$\operatorname{cs}\left(\operatorname{cs}^{-1}(z \mid m) \,\middle|\, m\right) = z$$

With related functions

Involving cd^{-1}

09.39.27.0002.01

$$\operatorname{cs}^{-1}(z \mid m) = i\, K(1-m) - \frac{i}{\sqrt{1-m}} \operatorname{cd}^{-1}\left(i\, z \,\middle|\, \frac{1}{1-m}\right)$$

Involving cn^{-1}

09.39.27.0003.01

$$\operatorname{cs}^{-1}(z \mid m) = i \operatorname{cn}^{-1}\left(\frac{\sqrt{z^2+1}}{z} \,\middle|\, 1-m\right) /; z > 0 \land m > 0$$

Involving dc^{-1}

09.39.27.0004.01

$$\operatorname{cs}^{-1}(z \mid m) = i\left(\frac{1}{\sqrt{1-m}} \operatorname{dc}^{-1}\left(\frac{i}{z} \,\middle|\, \frac{1}{1-m}\right) - K(1-m)\right) /; 0 < z < 1 \land 0 < m < 1$$

Involving dn^{-1}

09.39.27.0005.01

$$\operatorname{cs}^{-1}(z \mid m) = i\left(\frac{1}{\sqrt{m-1}} \operatorname{dn}^{-1}\left(\frac{i}{z} \,\middle|\, \frac{m}{m-1}\right) - K(1-m)\right) /; z > 0 \land m < 0$$

Involving ds^{-1}

09.39.27.0006.01

$$\operatorname{cs}^{-1}(z \mid m) = \frac{i}{\sqrt{m}} \operatorname{ds}^{-1}\left(\frac{i\, z}{\sqrt{m}} \,\middle|\, \frac{m-1}{m}\right) /; z \in \mathbb{R} \land m < 0$$

Involving nc^{-1}

09.39.27.0007.01

$$\operatorname{cs}^{-1}(z \mid m) = i\, K(1-m) - \frac{i}{\sqrt{m}} \operatorname{nc}^{-1}\left(i\, z \,\middle|\, 1 - \frac{1}{m}\right) /; z \in \mathbb{R} \land m < 1$$

Involving nd^{-1}

09.39.27.0008.01

$$\mathrm{cs}^{-1}(z \mid m) \equiv \mathrm{nd}^{-1}\left(\frac{i}{z} \,\middle|\, m\right) - i\,K(1-m) \;/;\; z > 0 \land m \in \mathbb{R}$$

Involving ns^{-1}

09.39.27.0009.01

$$\mathrm{cs}^{-1}(z \mid m) \equiv -i\,\mathrm{ns}^{-1}(-i\,z \mid 1-m)$$

Involving sc^{-1}

09.39.27.0010.01

$$\mathrm{cs}^{-1}(z \mid m) \equiv \mathrm{sc}^{-1}\left(\frac{1}{z} \,\middle|\, m\right)$$

Involving sd^{-1}

09.39.27.0011.01

$$\mathrm{cs}^{-1}(z \mid m) \equiv \frac{1}{\sqrt{m}}\,\mathrm{sd}^{-1}\left(\frac{\sqrt{m}}{z} \,\middle|\, \frac{1}{m}\right) \;/;\; z > 0 \land m \in \mathbb{R}$$

Involving sn^{-1}

09.39.27.0012.01

$$\mathrm{cs}^{-1}(z \mid m) \equiv -i\,\mathrm{sn}^{-1}\left(\frac{i}{z} \,\middle|\, 1-m\right) \;/;\; z > 0 \land m \in \mathbb{R}$$

Involving elliptic integrals

09.39.27.0013.01

$$\mathrm{cs}^{-1}(z \mid m) \equiv -i\,F\left(i\,\sinh^{-1}\left(\frac{1}{z}\right) \,\middle|\, 1-m\right) \;/;\; z > 0 \land m \in \mathbb{R}$$

09.39.27.0014.01

$$\mathrm{cs}^{-1}(z \mid m) \equiv i\left(\frac{1}{\sqrt{1-m}}\,K\left(\frac{1}{1-m}\right) - K(1-m)\right) - \frac{1}{\sqrt{m-1}}\,F\left(i\,\sinh^{-1}(z) \,\middle|\, \frac{1}{1-m}\right) \;/;\; z \in \mathbb{R} \land 0 < m < 1$$

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$$\mathrm{cs}^{-1}(z \mid m) = \mathrm{cs}^{-1}(z_0 \mid m) + \frac{i\,\sqrt{z^2-m+1}\;\mathrm{nd}\left(\mathrm{cs}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2+1}} \left(\frac{\sqrt{\frac{-z^2+m-1}{m-1}}\;F\left(i\,\sinh^{-1}(z) \,\middle|\, \frac{1}{1-m}\right)}{\sqrt{z^2-m+1}} - \frac{\sqrt{\frac{-z_0^2+m-1}{m-1}}\;F\left(i\,\sinh^{-1}(z_0) \,\middle|\, \frac{1}{1-m}\right)}{\sqrt{z_0^2-m+1}}\right) /;$$

$$\neg\,\exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}}\left(\mathrm{Im}\left((\tau\,(z-z_0)+z_0)^2+1\right)=0 \bigwedge (\tau\,(z-z_0)+z_0)^2+1<0 \bigwedge \mathrm{Im}\left((\tau\,(z-z_0)+z_0)^2-m+1\right)=0 \bigwedge (\tau\,(z-z_0)+z_0)^2-m+1<0\right)$$

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$$\operatorname{cs}^{-1}(z \mid m) = -\frac{i\, z^2 \operatorname{nd}\left(\operatorname{cs}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2+1} \sqrt{\frac{z^2+1}{z^2}} \sqrt{\frac{z^2-m+1}{z^2}}\, F\left(i \operatorname{csch}^{-1}(z) \,\middle|\, 1-m\right) /;$$

$$\neg \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\operatorname{Im}\left(\left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2+1\right)=0 \bigwedge \left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2+1<0 \bigwedge\right.$$

$$\left.\operatorname{Im}\left(\left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2-m+1\right)=0 \bigwedge \left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2-m+1<0\right)$$

Involving other related functions

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$$\operatorname{cs}^{-1}(z \mid m) == -\frac{\sqrt{z_2^2}}{z_2} \operatorname{elog}(z_1,\, z_2;\, a,\, b) /; \{a,\, b,\, z_1\} == \left\{2-m,\, 1-m,\, z^2\right\} \bigwedge z_1^3 + a\, z_1^2 + b\, z_1 - z_2^2 == 0 \bigwedge z > 0 \bigwedge m < 1$$

History

– N. H. Abel (1826)
– A. G. Greenhill (1892)
– L. M. Milne–Thompson (1948)

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