InverseJacobiNC

Notations

Traditional name

Inverse of the Jacobi elliptic function nc

Traditional notation

$\operatorname{nc}^{-1}(z \mid m)$

Mathematica StandardForm notation

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InverseJacobiNC[z, m]
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Primary definition

09.43.02.0001.01

$$z = \operatorname{nc}(w \mid m) \;/;\; w = \operatorname{nc}^{-1}(z \mid m)$$

09.43.02.0002.01

$$\operatorname{nc}^{-1}(z \mid m) = \int_1^z \frac{1}{\sqrt{t^2 - 1}\,\sqrt{(1-m)\,t^2 + m}}\,dt \;/;\; z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge (1-m)\,z^2 + m > 0$$

Specific values

Specialized values

For fixed z

09.43.03.0001.01

$$\operatorname{nc}^{-1}(z \mid 0) = \sec^{-1}(z)$$

09.43.03.0002.01

$$\operatorname{nc}^{-1}\left(z \,\middle|\, \frac{1}{2}\right) = i\sqrt{2}\left(F\left(\sin^{-1}(z) \mid -1\right) - \frac{1}{4\sqrt{2\pi}}\,\Gamma\left(\frac{1}{4}\right)^2\right) \;/;\; z > 1$$

09.43.03.0003.01

$$\operatorname{nc}^{-1}(z \mid 1) = \cosh^{-1}(z)$$

For fixed m

09.43.03.0004.01

$$\operatorname{nc}^{-1}(-1 \mid m) = \frac{2\,i}{\sqrt{m}}\,K\left(\frac{m-1}{m}\right)$$

09.43.03.0005.01

$$\mathrm{nc}^{-1}\left(-\frac{1}{2}\,\middle|\,m\right) = \frac{i}{\sqrt{m}}\left(K\left(\frac{m-1}{m}\right) + F\left(\frac{\pi}{6}\,\middle|\,\frac{m-1}{m}\right)\right)$$

09.43.03.0006.01

$$\mathrm{nc}^{-1}(0 \mid m) = \frac{i}{\sqrt{m}} K\left(\frac{m-1}{m}\right)$$

09.43.03.0007.01

$$\mathrm{nc}^{-1}\left(\frac{1}{2}\,\middle|\,m\right) = \frac{i}{\sqrt{m}}\left(K\left(\frac{m-1}{m}\right) - F\left(\frac{\pi}{6}\,\middle|\,\frac{m-1}{m}\right)\right)$$

09.43.03.0008.01

$$\mathrm{nc}^{-1}(1 \mid m) = 0$$

09.43.03.0009.01

$$\mathrm{nc}^{-1}(i \mid m) = \frac{i}{\sqrt{m}}\left(K\left(\frac{m-1}{m}\right) - F\left(\sin^{-1}(i)\,\middle|\,\frac{m-1}{m}\right)\right)$$

09.43.03.0010.01

$$\mathrm{nc}^{-1}(-i \mid m) = \frac{i}{\sqrt{m}}\left(F\left(\sin^{-1}(i)\,\middle|\,\frac{m-1}{m}\right) + K\left(\frac{m-1}{m}\right)\right)$$

Values at infinities

09.43.03.0011.01

$$\mathrm{nc}^{-1}(z \mid \infty) = 0$$

09.43.03.0012.01

$$\mathrm{nc}^{-1}(z \mid -\infty) = 0$$

09.43.03.0013.01

$$\mathrm{nc}^{-1}(\infty \mid m) = -\frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right)$$

09.43.03.0014.01

$$\mathrm{nc}^{-1}(-\infty \mid m) = \frac{1}{\sqrt{1-m}} K\left(\frac{m}{m-1}\right) + \frac{2\,i}{\sqrt{m}} K\left(\frac{m-1}{m}\right)$$

General characteristics

Domain and analyticity

$\mathrm{nc}^{-1}(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

09.43.04.0001.01

$$(z * m) \longrightarrow \mathrm{nc}^{-1}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.43.04.0002.01

$$\mathrm{nc}^{-1}(\bar{z} \mid \bar{m}) = \overline{\mathrm{nc}^{-1}(z \mid m)}$$

Quasi-reflection symmetry

09.43.04.0003.01

$$\mathrm{nc}^{-1}(-z \mid m) = \frac{2\,i}{\sqrt{m}}\, F\!\left(\sin^{-1}(z) \,\middle|\, \frac{m-1}{m}\right) + \mathrm{nc}^{-1}(z \mid m) \;/;\; m < 1$$

Poles and essential singularities

With respect to *m*

The function $\mathrm{nc}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to m.

09.43.04.0004.01

$$\mathcal{S}ing_m\!\left(\mathrm{nc}^{-1}(z \mid m)\right) = \{\}$$

With respect to *z*

The function $\mathrm{nc}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to z.

09.43.04.0005.01

$$\mathcal{S}ing_z\!\left(\mathrm{nc}^{-1}(z \mid m)\right) = \{\}$$

Branch points

With respect to *m*

For fixed z, the function $\mathrm{nc}^{-1}(z \mid m)$ has two branch points: $m = \frac{z^2}{z^2-1}$, $m = \tilde{\infty}$.

09.43.04.0006.01

$$\mathcal{BP}_m\!\left(\mathrm{nc}^{-1}(z \mid m)\right) = \left\{\frac{z^2}{z^2-1}, \tilde{\infty}\right\}$$

09.43.04.0007.01

$$\mathcal{R}_m\!\left(\mathrm{nc}^{-1}(z \mid m), \frac{z^2}{z^2-1}\right) = \log$$

09.43.04.0008.01

$$\mathcal{R}_m\!\left(\mathrm{nc}^{-1}(z \mid m), \tilde{\infty}\right) = 2$$

With respect to *z*

For fixed m, the function $\mathrm{nc}^{-1}(z \mid m)$ has five branch points: $z = \pm 1$, $z = \pm \sqrt{\frac{m}{m-1}}$, $z = \tilde{\infty}$.

09.43.04.0009.01

$$\mathcal{BP}_z\!\left(\mathrm{nc}^{-1}(z \mid m)\right) = \left\{1, -1, \sqrt{\frac{m}{m-1}}, -\sqrt{\frac{m}{m-1}}, \tilde{\infty}\right\}$$

09.43.04.0010.01

$$\mathcal{R}_z\!\left(\mathrm{nc}^{-1}(z \mid m), 1\right) = 2$$

09.43.04.0011.01

$$\mathcal{R}_z\!\left(\mathrm{nc}^{-1}(z \mid m), -1\right) = 2$$

09.43.04.0012.01

$$\mathcal{R}_z\left(\text{nc}^{-1}(z \mid m), \sqrt{\frac{m}{m-1}}\right) == 2$$

09.43.04.0013.01

$$\mathcal{R}_z\left(\text{nc}^{-1}(z \mid m), -\sqrt{\frac{m}{m-1}}\right) == 2$$

09.43.04.0014.01

$$\mathcal{R}_z\left(\text{nc}^{-1}(z \mid m), \tilde{\infty}\right) == \log$$

Branch cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at $z == 0$

09.43.06.0001.02

$$\text{nc}^{-1}(z \mid m) \propto i\, K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{m}\,\sqrt{z^2-1}}\left(z + \frac{2m-1}{6m} z^3 + \frac{8m^2-8m+3}{40m^2} z^5 + \ldots\right) /; (z \to 0)$$

09.43.06.0002.01

$$\text{nc}^{-1}(z \mid m) == i\, K(1-m) + \frac{\sqrt{1-z^2}}{\sqrt{z^2-1}\,\sqrt{m}} \sum_{k=0}^{\infty} \frac{\left(\frac{m-1}{m}\right)^k \left(\frac{1}{2}\right)_k}{(2k+1)\,k!}\, {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2}-k; \frac{m}{m-1}\right) z^{2k+1}$$

09.43.06.0007.01

$$\text{nc}^{-1}(z \mid m) \propto i\, K(1-m) + \frac{z\sqrt{1-z^2}}{\sqrt{m}\,\sqrt{z^2-1}}\left(1 + O\left(z^2\right)\right)$$

Expansions at $m == 0$

09.43.06.0003.02

$$\text{nc}^{-1}(z \mid m) \propto \sec^{-1}(z) + \frac{1}{4z}\left(z \sec^{-1}(z) - \sqrt{1-\frac{1}{z^2}}\right) m + \frac{3}{64z^3}\left(3\sec^{-1}(z)\, z^3 + \left(2-5z^2\right)\sqrt{1-\frac{1}{z^2}}\right) m^2 + \ldots /; (m \to 0)$$

09.43.06.0004.01

$$\text{nc}^{-1}(z \mid m) == \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{k!}\left(\frac{\sqrt{\pi}}{2\,k!}\,\Gamma\left(k+\frac{1}{2}\right) - \frac{1}{z}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}-k; \frac{3}{2}; \frac{1}{z^2}\right)\right) m^k /; |m| < 1$$

09.43.06.0008.01

$$\mathrm{nc}^{-1}(z\mid m)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}\left(\sec^{-1}(z)-\frac{1}{2z\sqrt{1-\frac{1}{z^2}}}\sum_{j=1}^{k}\frac{\left(1-\frac{1}{z^2}\right)^j(j-1)!}{\left(\frac{1}{2}\right)_j}\right)m^k\;/;\;|m|<1$$

09.43.06.0005.01

$$\mathrm{nc}^{-1}(z\mid m)=K(m)-\frac{1}{z}\left(\frac{1}{2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^k z^{-2k}\left(\frac{3}{2}\right)_{j+k}^2 m^{j+k+1}}{(2k+1)\left(\frac{3}{2}\right)_j k!\,(j+k+1)!}+\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^k z^{-2j-2k}\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_k^2 m^k}{(2j+2k+1)\,k!\,(j+k)!}\right)$$

09.43.06.0006.01

$$\mathrm{nc}^{-1}(z\mid m)=K(m)-\frac{1}{z}\left(F_{2\times0\times0}^{1\times2\times2}\left(\begin{matrix}\frac{1}{2};\frac{1}{2},\frac{1}{2};\frac{1}{2},1;\\ \frac{3}{2},1;;;\end{matrix}\;-\frac{m}{z^2},\frac{1}{z^2}\right)+\frac{m}{2}F_{1\times1\times1}^{2\times1\times1}\left(\begin{matrix}\frac{3}{2},\frac{3}{2};\frac{1}{2};1;\\ 2;\frac{3}{2};\frac{3}{2};\end{matrix}\;-\frac{m}{z^2},m\right)\right)$$

09.43.06.0009.01

$$\mathrm{nc}^{-1}(z\mid m)\propto\sec^{-1}(z)\,(1+O(m))$$

Integral representations

On the real axis

Of the direct function

09.43.07.0001.01

$$\mathrm{nc}^{-1}(z\mid m)=\int_1^z\frac{1}{\sqrt{t^2-1}\sqrt{(1-m)t^2+m}}\,dt\;/;\;z\in\mathbb{R}\bigwedge z^2>1\bigwedge(1-m)z^2+m>0$$

09.43.07.0002.01

$$\mathrm{nc}^{-1}(z\mid m)=\frac{\sqrt{z^2-1}\,\mathrm{ds}\left(\mathrm{nc}^{-1}(z\mid m)\mid m\right)}{\sqrt{-mz^2+z^2+m}}\int_1^z\frac{1}{\sqrt{t^2-1}\sqrt{(1-m)t^2+m}}\,dt\;/;$$

$$\neg\exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}}\left(\mathrm{Im}\left(((z-1)\tau+1)^2-1\right)=0\bigwedge((z-1)\tau+1)^2-1<0\bigwedge\right.$$
$$\left.\mathrm{Im}\left((1-m)((z-1)\tau+1)^2+m\right)=0\bigwedge(1-m)((z-1)\tau+1)^2+m<0\right)$$

09.43.07.0003.01

$$\mathrm{nc}^{-1}(z\mid m)=\mathrm{nc}^{-1}(z_0\mid m)+\frac{\sqrt{z^2-1}\,\mathrm{ds}\left(\mathrm{nc}^{-1}(z\mid m)\mid m\right)}{\sqrt{-mz^2+z^2+m}}\int_{z_0}^z\frac{1}{\sqrt{t^2-1}\sqrt{(1-m)t^2+m}}\,dt\;/;$$

$$\neg\exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}}\left(\mathrm{Im}\left((\tau(z-z_0)+z_0)^2-1\right)=0\bigwedge(\tau(z-z_0)+z_0)^2-1<0\bigwedge\right.$$
$$\left.\mathrm{Im}\left((1-m)(\tau(z-z_0)+z_0)^2+m\right)=0\bigwedge(1-m)(\tau(z-z_0)+z_0)^2+m<0\right)$$

Differential equations

Ordinary nonlinear differential equations

09.43.13.0001.01

$$w''(z) + \left(2 (1-m) z^2 + 2 m - 1\right) z\, w'(z)^3 = 0 \;/;\; w(z) = \operatorname{nc}^{-1}(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.43.16.0001.01

$$\operatorname{nc}^{-1}(-z \mid m) = \frac{2 i}{\sqrt{m}} F\!\left(\sin^{-1}(z) \,\middle|\, \frac{m-1}{m}\right) + \operatorname{nc}^{-1}(z \mid m) \;/;\; m < 1$$

Identities

Functional identities

09.43.17.0001.01

$$\left((m-1)\left(z_2^2-1\right) z_1^2 - (m-1) z_2^2 + m\right) \operatorname{nc}(w(z_1)+w(z_2) \mid m)^2 - 2 z_1 z_2 \operatorname{nc}(w(z_1)+w(z_2) \mid m) + \left(z_2^2-1\right) m + z_1^2 \left(m - (m-1) z_2^2\right) = 0 \;/;\; w(z) = \operatorname{nc}^{-1}(z \mid m)$$

Differentiation

Low-order differentiation

With respect to *z*

09.43.20.0001.02

$$\frac{\partial \operatorname{nc}^{-1}(z \mid m)}{\partial z} = \frac{\operatorname{ds}\left(\operatorname{nc}^{-1}(z \mid m) \,\middle|\, m\right)}{-m z^2 + z^2 + m}$$

09.43.20.0002.01

$$\frac{\partial \operatorname{nc}^{-1}(z \mid m)}{\partial z} = \frac{1}{\sqrt{z^2-1}\,\sqrt{(1-m) z^2 + m}} \;/;\; z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge (1-m) z^2 + m > 0$$

09.43.20.0003.02

$$\frac{\partial^2 \operatorname{nc}^{-1}(z \mid m)}{\partial z^2} = \frac{z\left(-2 z^2 + 2 m \left(z^2-1\right) + 1\right) \operatorname{ds}\left(\operatorname{nc}^{-1}(z \mid m) \,\middle|\, m\right)}{\left(z^2-1\right)\left(-m z^2 + z^2 + m\right)^2}$$

09.43.20.0011.01

$$\frac{\partial^2 \operatorname{nc}^{-1}(z \mid m)}{\partial z^2} = \frac{\sqrt{z^2-1}\, \operatorname{ds}\left(\operatorname{nc}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{-m z^2 + z^2 + m}} \frac{\partial \frac{1}{\sqrt{z^2-1}\,\sqrt{(1-m) z^2 + m}}}{\partial z}$$

With respect to *m*

09.43.20.0004.02

$$\frac{\partial \operatorname{nc}^{-1}(z \mid m)}{\partial m} = \frac{m \operatorname{sd}\left(\operatorname{nc}^{-1}(z \mid m) \,\middle|\, m\right) - z\left(E\left(\operatorname{am}\left(\operatorname{nc}^{-1}(z \mid m) \,\middle|\, m\right) \,\middle|\, m\right) + (m-1) \operatorname{nc}^{-1}(z \mid m)\right)}{2 (m-1) m z}$$

09.43.20.0005.01

$$\frac{\partial \operatorname{nc}^{-1}(z \mid m)}{\partial m} = \frac{i}{2(m-1)m}\left(\frac{(1-m)\sqrt{1-z^2}\,z}{\sqrt{-m z^2+z^2+m}} + \sqrt{m}\left(E\left(\sin^{-1}(z) \,\middle|\, \frac{m-1}{m}\right) - E\left(\frac{m-1}{m}\right) - F\left(\sin^{-1}(z) \,\middle|\, \frac{m-1}{m}\right) + K\left(\frac{m-1}{m}\right)\right)\right) /;$$

$z > 1 \land m > 0$

09.43.20.0006.02

$$\frac{\partial^2 \operatorname{nc}^{-1}(z \mid m)}{\partial m^2} = \frac{1}{4(m-1)^2 m^2}\Big(3 \operatorname{nc}^{-1}(z \mid m)(m-1)^2 + F\left(\operatorname{am}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right) \mid m\right)(m-1) +$$

$$(4m-2) E\left(\operatorname{am}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right) \mid m\right) + \frac{m\left(4\left(z^2-1\right)m^2 + \left(2-5z^2\right)m + z^2\right)\operatorname{sd}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)}{z^3 + m\left(z - z^3\right)}\Big)$$

09.43.20.0012.01

$$\frac{\partial^3 \operatorname{nc}^{-1}(z \mid m)}{\partial m^3} = \frac{1}{8(m-1)^3 m^3}$$

$$\Bigg((-23(m-1)m-8) E\left(\operatorname{am}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right) \mid m\right) - (m-1)(11m-7) F\left(\operatorname{am}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right) \mid m\right) + \frac{1}{\left((m-1)z^2 - m\right)^3}$$

$$\Bigg(\frac{1}{z}\Bigg(m\Bigg((1-m)z\sqrt{m\left(\frac{1}{z^2}-1\right)+1}\,\operatorname{cd}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)\left(-m z^2+z^2+m\right)^3 + \left(m\left(z^2-1\right)-z^2\right)\Big(23\left(z^2-1\right)^2$$

$$m^4 + \left(-59z^4+83z^2-24\right)m^3 + \left(54z^4-48z^2+9\right)m^2 + \left(11z^2-23z^4\right)m + 5z^4\Big)\Bigg)$$

$$\operatorname{sd}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)\Bigg) - 15(m-1)^3\left((m-1)z^2-m\right)^3 \operatorname{nc}^{-1}(z \mid m)\Bigg)\Bigg)$$

Symbolic differentiation

With respect to z

09.43.20.0013.01

$$\frac{\partial^n \operatorname{nc}^{-1}(z \mid m)}{\partial z^n} = \operatorname{nc}^{-1}(z \mid m)\,\delta_n +$$

$$\frac{\operatorname{ds}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)}{-m z^2+z^2+m} \sum_{j=0}^{n-1} \frac{(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{-2j+n-1}} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{j-k} (m-1)^{j-k}\left(z^2-1\right)^{-k}\left((1-m)z^2+m\right)^{k-j} /; n \in \mathbb{N}$$

09.43.20.0014.01

$$\frac{\partial^n \operatorname{nc}^{-1}(z \mid m)}{\partial z^n} = \operatorname{nc}^{-1}(z \mid m)\,\delta_n + \frac{\operatorname{ds}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)}{-m z^2+z^2+m}$$

$$\sum_{j=0}^{n-1} \frac{2^{2j-n+1}(m-1)^j z^{2j-n+1}\left((1-m)z^2+m\right)^{-j}\left(\frac{1}{2}\right)_j (1-n)_{2(n-j)-2}}{(n-j-1)!}\,{}_2F_1\left(\frac{1}{2}, -j; \frac{1}{2}-j; -\frac{-m z^2+z^2+m}{(m-1)\left(z^2-1\right)}\right) /; n \in \mathbb{N}$$

09.43.20.0015.01

$$\frac{\partial^n \operatorname{nc}^{-1}(z \mid m)}{\partial z^n} = \operatorname{nc}^{-1}(z \mid m)\,\delta_n + \frac{\sqrt{z^2-1}\ \operatorname{ds}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)}{\sqrt{-m z^2 + z^2 + m}}\ \frac{\partial^{n-1} \frac{1}{\sqrt{z^2-1}\ \sqrt{(1-m) z^2+m}}}{\partial z^{n-1}} \ /;\ n \in \mathbb{N}^+$$

09.43.20.0007.01

$$\frac{\partial^n \operatorname{nc}^{-1}(z \mid m)}{\partial z^n} = \frac{2^{n-1}\,\pi\,z^{n-1}\,(n-1)!\ \operatorname{ds}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)}{-m z^2 + z^2 + m} \sum_{j=0}^{n-1} \frac{(1-m)^{n-j-1}\left(z^2-1\right)^{-j}\left((1-m) z^2 + m\right)^{j-n+1}}{j!\,(n-j-1)!\,\Gamma\left(\frac{1}{2}-j\right)\Gamma\left(j-n+\frac{3}{2}\right)}$$
$$ {}_2F_1\left(\frac{1-j}{2}, -\frac{j}{2}; \frac{1}{2}-j; 1-\frac{1}{z^2}\right) {}_2F_1\left(\frac{j-n+2}{2}, \frac{j-n+1}{2}; j-n+\frac{3}{2}; \frac{m}{(1-m) z^2}+1\right) /;\ n \in \mathbb{N}^+$$

With respect to m

09.43.20.0008.02

$$\frac{\partial^n \operatorname{nc}^{-1}(z \mid m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} \left(1-\frac{1}{z^2}\right)^{n+\frac{1}{2}} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; 1-\frac{1}{z^2}, m\left(1-\frac{1}{z^2}\right)\right) /;\ n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

09.43.20.0009.01

$$\frac{\partial^\alpha \operatorname{nc}^{-1}(z \mid m)}{\partial z^\alpha} = \frac{i\,z^{-\alpha}}{\sqrt{m}\ \Gamma(1-\alpha)} K\left(1-\frac{1}{m}\right) - \frac{i\,z^{1-\alpha}\sqrt{\pi}}{\sqrt{m}} \tilde{F}_{2\times 0\times 0}^{2\times 1\times 1}\left(\begin{matrix}\frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2}; \\ \frac{3-\alpha}{2}, 1-\frac{\alpha}{2};;;\end{matrix}\ z^2, \left(1-\frac{1}{m}\right) z^2\right) /;\ -1 < z < 1 \land m \in \mathbb{R}$$

With respect to m

09.43.20.0010.01

$$\frac{\partial^\alpha \operatorname{nc}^{-1}(z \mid m)}{\partial m^\alpha} = \frac{m^{-\alpha}\sqrt{\pi}}{2} \sqrt{1-\frac{1}{z^2}}\ \tilde{F}_{1\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix}\frac{1}{2}; \frac{1}{2}; \frac{1}{2}, 1; \\ \frac{3}{2};; 1-\alpha;\end{matrix}\ 1-\frac{1}{z^2}, m\left(1-\frac{1}{z^2}\right)\right) /;\ z > 1 \land m \in \mathbb{R}$$

Integration

Indefinite integration

Involving only one direct function

09.43.21.0001.01

$$\int \operatorname{nc}^{-1}(z \mid m)\,dz = \operatorname{nc}^{-1}(z \mid m)\,z - \frac{1}{\sqrt{1-m}} \log\left(\frac{\operatorname{dc}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)}{\sqrt{1-m}} + \operatorname{sc}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)\right)$$

Involving only one direct function with respect to m

09.43.21.0002.01

$$\int \operatorname{nc}^{-1}(z \mid m)\, dm = -2\left(\frac{z - z\sqrt{m - m z^2 + z^2}}{\sqrt{z^2-1}} + \sqrt{1-m}\left(E\left(i \sinh^{-1}\left(\frac{\sqrt{1-m}}{\sqrt{m}}\right) \middle| \frac{m}{m-1}\right) - E\left(i \sinh^{-1}\left(\frac{\sqrt{1-m}\, z}{\sqrt{m}}\right) \middle| \frac{m}{m-1}\right) - F\left(i \sinh^{-1}\left(\frac{\sqrt{1-m}}{\sqrt{m}}\right) \middle| \frac{m}{m-1}\right) + F\left(i \sinh^{-1}\left(\frac{\sqrt{1-m}\, z}{\sqrt{m}}\right) \middle| \frac{m}{m-1}\right)\right)\right) /; z > 1 \land m > 0$$

Representations through more general functions

Through hypergeometric functions of two variables

09.43.26.0001.01

$$\operatorname{nc}^{-1}(z \mid m) = K(m) - \frac{1}{z}\left(F_{2\times 0\times 0}^{1\times 2\times 2}\left(\begin{matrix}\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{1}{2}, 1; \\ \frac{3}{2}, 1;;;\end{matrix} -\frac{m}{z^2}, \frac{1}{z^2}\right) + \frac{m}{2} F_{1\times 1\times 1}^{2\times 1\times 1}\left(\begin{matrix}\frac{3}{2}, \frac{3}{2}; \frac{1}{2}; 1; \\ 2; \frac{3}{2}; \frac{3}{2};\end{matrix} -\frac{m}{z^2}, m\right)\right)$$

Through other functions

Involving some hypergeometric-type functions

09.43.26.0002.01

$$\operatorname{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}} K\left(1 - \frac{1}{m}\right) - \frac{i z}{\sqrt{m}} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, \left(1 - \frac{1}{m}\right) z^2\right) /; -1 < z < 1 \land m \in \mathbb{R}$$

09.43.26.0003.01

$$\operatorname{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}}\left(z F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; z^2, \frac{(m-1) z^2}{m}\right) - K\left(\frac{m-1}{m}\right)\right) /; (z > 1 \lor z < -1) \bigwedge (1-m) z^2 + m > 0$$

Representations through equivalent functions

With inverse function

09.43.27.0001.01

$$\operatorname{nc}\left(\operatorname{nc}^{-1}(z \mid m) \middle| m\right) = z$$

With related functions

Involving cd^{-1}

09.43.27.0002.01

$$\operatorname{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}} \operatorname{cd}^{-1}\left(z \middle| \frac{m-1}{m}\right) /; -1 < z < 1 \land m > 0$$

Involving cn^{-1}

09.43.27.0003.01

$$\operatorname{nc}^{-1}(z \mid m) = i \operatorname{cn}^{-1}(z \mid 1-m) /; -1 < z < 1 \land m \in \mathbb{R}$$

09.43.27.0004.01

$$\text{nc}^{-1}(z \mid m) = \text{cn}^{-1}\left(\frac{1}{z} \,\middle|\, m\right) /; z > 1 \land m \in \mathbb{R}$$

Involving cs^{-1}

09.43.27.0005.01

$$\text{nc}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) - i\, \text{cs}^{-1}\left(i\, z \,\middle|\, \frac{1}{1-m}\right)\right) /; z > 1 \land m < 1$$

Involving dc^{-1}

09.43.27.0006.01

$$\text{nc}^{-1}(z \mid m) = -\frac{1}{\sqrt{1-m}}\, \text{dc}^{-1}\left(z \,\middle|\, \frac{m}{m-1}\right) /; 0 < z < 1 \land m > 1$$

Involving dn^{-1}

09.43.27.0007.01

$$\text{nc}^{-1}(z \mid m) = -\frac{1}{\sqrt{m-1}}\, \text{dn}^{-1}\left(z \,\middle|\, \frac{1}{1-m}\right) /; -1 < z < 1 \land 0 < m < 1$$

Involving ds^{-1}

09.43.27.0008.01

$$\text{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}}\, K\left(\frac{m-1}{m}\right) - i\, \text{ds}^{-1}\left(\frac{\sqrt{m}}{z} \,\middle|\, 1-m\right) /; z > 0 \land m > 0$$

Involving nd^{-1}

09.43.27.0009.01

$$\text{nc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}}\, \text{nd}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) /; -1 < z < 1 \land m > 0$$

Involving ns^{-1}

09.43.27.0010.01

$$\text{nc}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \left(K\left(\frac{m}{m-1}\right) - \text{ns}^{-1}\left(z \,\middle|\, \frac{m}{m-1}\right)\right) /; z > 1 \land m < 1$$

Involving sc^{-1}

09.43.27.0011.01

$$\text{nc}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} \left(i\, K\left(\frac{m-1}{m}\right) - \text{sc}^{-1}\left(-i\, z \,\middle|\, \frac{1}{m}\right)\right) /; -1 < z < 1 \land m > 0$$

Involving sd^{-1}

09.43.27.0012.01

$$\text{nc}^{-1}(z \mid m) = \frac{i}{\sqrt{m}}\, K\left(\frac{m-1}{m}\right) - \text{sd}^{-1}\left(\frac{i\, z}{\sqrt{m}} \,\middle|\, m\right) /; -1 < z < 1 \land m > 0$$

Involving sn^{-1}

09.43.27.0013.01

$$\text{nc}^{-1}(z \mid m) == \frac{i}{\sqrt{m}} \left(K\left(\frac{m-1}{m}\right) - \text{sn}^{-1}\left(z \,\middle|\, \frac{m-1}{m}\right)\right) /; 0 < z < 1 \land m > 0$$

Involving elliptic integrals

09.43.27.0014.01

$$\text{nc}^{-1}(z \mid m) == \frac{i}{\sqrt{m}} \left(F\left(\sin^{-1}(z) \,\middle|\, \frac{m-1}{m}\right) - K\left(\frac{m-1}{m}\right)\right) /; z > 1 \land m > 0$$

09.43.27.0016.01

$$\text{nc}^{-1}(z \mid m) = \frac{\sqrt{z^2-1}\ \text{ds}\left(\text{nc}^{-1}(z \mid m) \mid m\right)}{\sqrt{-m z^2 + z^2 + m}} \left(\frac{\sqrt{1-z^2}}{\sqrt{z^2-1}\ \sqrt{(1-m) z^2 + m}} \sqrt{\left(\frac{1}{m} - 1\right) z^2 + 1}\ F\left(\sin^{-1}(z) \,\middle|\, \frac{m-1}{m}\right) - i \sqrt{\frac{1}{m}}\ K\left(\frac{m-1}{m}\right)\right) /;$$

$$\neg \exists_{\tau,\{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\text{Im}\left(((z-1)\tau+1)^2 - 1\right) = 0 \bigwedge ((z-1)\tau+1)^2 - 1 < 0 \bigwedge \text{Im}\left((1-m)((z-1)\tau+1)^2 + m\right) = 0 \bigwedge (1-m)((z-1)\tau+1)^2 + m < 0\right)$$

09.43.27.0017.01

$$\text{nc}^{-1}(z \mid m) = \text{nc}^{-1}(z_0 \mid m) + \frac{\sqrt{z^2-1}\ \text{ds}\left(\text{nc}^{-1}(z \mid m) \mid m\right)}{\sqrt{-m z^2 + z^2 + m}} \left(\frac{\sqrt{1-z^2}\ \sqrt{\left(\frac{1}{m}-1\right) z^2 + 1}\ F\left(\sin^{-1}(z) \,\middle|\, \frac{m-1}{m}\right)}{\sqrt{z^2-1}\ \sqrt{(1-m) z^2 + m}} - \frac{\sqrt{1-z_0^2}\ \sqrt{\left(\frac{1}{m}-1\right) z_0^2 + 1}\ F\left(\sin^{-1}(z_0) \,\middle|\, \frac{m-1}{m}\right)}{\sqrt{z_0^2-1}\ \sqrt{(1-m) z_0^2 + m}}\right) /;$$

$$\neg \exists_{\tau,\{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\text{Im}\left((\tau(z-z_0)+z_0)^2 - 1\right) = 0 \bigwedge (\tau(z-z_0)+z_0)^2 - 1 < 0 \bigwedge \text{Im}\left((1-m)(\tau(z-z_0)+z_0)^2 + m\right) = 0 \bigwedge (1-m)(\tau(z-z_0)+z_0)^2 + m < 0\right)$$

Involving other related functions

09.43.27.0015.01

$$\text{nc}^{-1}(z \mid m) == \frac{1}{\sqrt{1-m}} \left(\frac{\sqrt{z_2^2}}{z_2} \text{elog}(z_1, z_2; a, b) - K\left(\frac{m}{m-1}\right)\right) /;$$

$$\{a, b, z_1\} == \left\{\frac{2m-1}{1-m}, \frac{m}{m-1}, z^2\right\} \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 == 0 \bigwedge 0 < z < 1 \bigwedge m > 1$$

History

– N. H. Abel (1826)
– A. G. Greenhill (1892)
– L. M. Milne–Thompson (1948)

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