Khinchin

Notations

Traditional name

Khinchin constant

Traditional notation

K

Mathematica StandardForm notation

`Khinchin`

Primary definition

02.09.02.0001.01

$$K = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k\,(k+2)}\right)^{\log_2(k)}$$

Specific values

02.09.03.0001.01

$$K = 2.685452001065306445309714835481795693820382293994462953051152345557218859537152002801141174 \ldots$$

Above approximate numerical value of K shows 90 decimal digits.

General characteristics

The Khintchine's number K is a constant. It is a positive real number.

Series representations

Generalized power series

02.09.06.0001.01

$$K = \exp\left(\frac{1}{\log(2)} \sum_{k=2}^{\infty} \log\left(\frac{k}{k-1}\right) \log\left(\frac{k+1}{k}\right)\right)$$

02.09.06.0002.01

$$K = \exp\left(\log(2) + \frac{1}{2\log(2)} \sum_{k=1}^{\infty} \frac{1}{k} \left(\psi\left(k+\frac{1}{2}\right) - \psi(k)\right) (\zeta(2k)-1)\right)$$

02.09.06.0003.01

$$K = \exp\left(\frac{1}{\log(2)} \sum_{k=2}^{\infty} \frac{(-1)^k \left(2-2^k\right)}{k} \zeta'(k)\right)$$

02.09.06.0004.01

$$K = \exp\left(\frac{1}{\log(2)} \left(\log^2(2) + \mathrm{Li}_2\left(-\frac{1}{2}\right) + \frac{1}{2} \sum_{k=2}^{\infty} (-1)^k \, \mathrm{Li}_2\left(\frac{4}{k^2}\right)\right)\right)$$

02.09.06.0005.01

$$K = \exp\left(\frac{1}{\log(2)} \left(\frac{\pi^2}{6} - \frac{1}{2}\log^2(2) + \sum_{k=2}^{\infty} \mathrm{Li}_2\left(-\frac{1}{k^2-1}\right)\right)\right)$$

02.09.06.0006.01

$$K = \exp\left(\frac{1}{\log(2)} \left(\sum_{k=1}^{\infty} \frac{\zeta(2k, n+1)}{k} \left(\log(2) + \frac{1}{2}\left(\psi\left(k+\frac{1}{2}\right) - \psi(k)\right)\right) - \sum_{k=2}^{n} \log\left(1-\frac{1}{k}\right) \log\left(1+\frac{1}{k}\right)\right)\right) /; n \in \mathbb{N}^+$$

02.09.06.0007.01

$$K = \exp\left(\frac{2}{\log(2)} \sum_{k=0}^{\infty} (-1)^k \left(\frac{\left(2^{k+1}-1\right) \log(k+1) \, (k+1)^{-k-2}}{k+2} - \frac{(k+1)^{-k-3} \log(k+1)}{k+3} \left(2^{k+2} \, {}_2F_1\left(1, k+3; k+4; -\frac{2}{k+1}\right) - {}_2F_1\left(1, k+3; k+4; -\frac{1}{k+1}\right)\right) - \frac{2^{k+1}-1}{k+2} \zeta^{(1,0)}(k+2, k+2)\right)\right)$$

02.09.06.0008.01

$$K = \exp\left(\frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{k} \sum_{j=1}^{2k-1} \frac{(-1)^{j-1}}{j}\right)$$

Integral representations

On the real axis

Of the direct function

02.09.07.0001.01

$$K = \exp\left(\frac{\log^2(2)}{2} + \frac{1}{\log(2)} \int_0^{\pi} \frac{\log(t\,|\cot(t)|)}{t} \, dt + \frac{\pi^2}{12\log(2)}\right)$$

02.09.07.0002.01

$$K = \exp\left(\frac{1}{\log(2)} \int_1^{\infty} \frac{\log(\lfloor t \rfloor)}{t\,(t+1)} \, dt\right)$$

02.09.07.0003.01

$$K = 2\exp\left(\frac{1}{\log(2)} \int_0^1 \frac{1}{t\,(t+1)} \log\left(\frac{\pi\, t \left(1-t^2\right)}{\sin(\pi\, t)}\right) dt\right)$$

02.09.07.0005.01

$$K = 2\exp\left(\frac{1}{\log(2)}\int_0^1 \frac{1}{t\,(t+1)}\log(\Gamma(2+t)\,\Gamma(2-t))\,dt\right)$$

02.09.07.0004.01

$$K = \exp\left(\frac{1}{\log(2)}\int_0^1 \frac{1}{t+1}\log\left(\left\lfloor \frac{1}{t}\right\rfloor\right)dt\right)$$

Limit representations

02.09.09.0001.01

$$K = \lim_{n\to\infty} e^{\frac{1}{\log(2)}\sum_{m=1}^{\lfloor n\log_4(10)\rfloor+1}\frac{1}{m}(\zeta(2m)-1)\sum_{k=1}^{2m-1}\frac{(-1)^{k+1}}{k}}$$

The above formula is used for the numerical computation of Khinchin's constant in *Mathematica*.

Complex characteristics

Real part

02.09.19.0001.01

$$\operatorname{Re}(K) = K$$

Imaginary part

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$$\operatorname{Im}(K) = 0$$

Absolute value

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$$|K| = K$$

Argument

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$$\arg(K) = 0$$

Conjugate value

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$$\overline{K} = K$$

Signum value

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$$\operatorname{sgn}(K) = 1$$

Differentiation

Low-order differentiation

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$$\frac{\partial K}{\partial z} == 0$$

Fractional integro-differentiation

02.09.20.0002.01

$$\frac{\partial^{\alpha} K}{\partial z^{\alpha}} == \frac{z^{-\alpha} K}{\Gamma(1-\alpha)}$$

Integration

Indefinite integration

02.09.21.0001.01

$$\int K \, dz == K z$$

02.09.21.0002.01

$$\int z^{\alpha-1} K \, dz == \frac{z^{\alpha} K}{\alpha}$$

Integral transforms

Fourier exp transforms

02.09.22.0001.01

$$\mathcal{F}_t[K](z) == \sqrt{2\pi} \, K \, \delta(z)$$

Inverse Fourier exp transforms

02.09.22.0002.01

$$\mathcal{F}_t^{-1}[K](z) == \sqrt{2\pi} \, K \, \delta(z)$$

Fourier cos transforms

02.09.22.0003.01

$$\mathcal{F}c_t[K](z) == \sqrt{\frac{\pi}{2}} \, K \, \delta(z)$$

Fourier sin transforms

02.09.22.0004.01

$$\mathcal{F}s_t[K](z) == \sqrt{\frac{2}{\pi}} \, \frac{K}{z}$$

Laplace transforms

02.09.22.0005.01

$$\mathcal{L}_t[K](z) == \frac{K}{z}$$

Inverse Laplace transforms

02.09.22.0006.01

$\mathcal{L}_t^{-1}[K](z) = K\,\delta(z)$

Inequalities

02.09.29.0001.01

$$\frac{53}{20} < K < \frac{27}{10}$$

Theorems

The continued fraction mean theorem

For almost all $x \in \mathbb{R}$:

$$\lim_{n \to \infty} \left(\prod_{k=0}^{n} q_k \right)^{1/n} = K \;/;\quad x = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \ldots}}} \land q_k \in \mathbb{N}^+.$$

This relation fails for $x = e$, rational numbers, solutions of quadratic equations with rational coefficients and quadratic irrationals, such as ϕ, $\sqrt{2}$, $\sqrt{3}$.

Numerical verifications show that this relation can be valid for $x = \pi$, $x = \gamma$, and $x = K$. But it was not accurately proved.

So, K is the limit of the geometric mean of the first n partial quotients of the simple continued fraction for almost any real number x, when n tends to infinity. This limit exists and is a Khinchin constant independent from x for almost all numbers.

History

– A. Khinchin (1934)

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