LegendreP

Notations

Traditional name

Legendre polynomial

Traditional notation

$P_n(z)$

Mathematica StandardForm notation

`LegendreP[n, z]`

Primary definition

05.03.02.0001.01

$$P_n(z) = \frac{1}{2^n} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} z^{n-2k} \;/;\; n \in \mathbb{N}$$

05.03.02.0002.01

$$P_n(z) = P_{-n-1}(z) \;/;\; n \in \mathbb{Z} \land n < 0$$

Specific values

Specialized values

For fixed *n*

05.03.03.0001.01

$$P_n(0) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right) \Gamma\left(\frac{n}{2}+1\right)}$$

05.03.03.0002.01

$$P_n(1) = 1$$

05.03.03.0003.01

$$P_n(-1) = (-1)^n$$

05.03.03.0017.01

$$P_n^{(0,r)}(0) = \delta_{\sin\left(\frac{1}{2}(n-r)\pi\right)}\, r!\, (-1)^{\frac{n-r}{2}}\, 2^{-n} \binom{n+r}{r} \binom{n}{\frac{n+r}{2}} \;/;\; n \in \mathbb{N} \land r \in \mathbb{N}$$

05.03.03.0018.01

$$P_n^{(0,r)}(-1) = \frac{(-1)^{n+r}\,(n+r)!}{2^r\,r!\,(n-r)!}$$

For fixed z

05.03.03.0004.01

$$P_0(z) = 1$$

05.03.03.0005.01

$$P_1(z) = z$$

05.03.03.0006.01

$$P_2(z) = \frac{1}{2}\left(3z^2 - 1\right)$$

05.03.03.0007.01

$$P_3(z) = \frac{1}{2}\left(5z^3 - 3z\right)$$

05.03.03.0008.01

$$P_4(z) = \frac{1}{8}\left(35z^4 - 30z^2 + 3\right)$$

05.03.03.0009.01

$$P_5(z) = \frac{1}{8}\,z\left(63z^4 - 70z^2 + 15\right)$$

05.03.03.0010.01

$$P_6(z) = \frac{1}{16}\left(231z^6 - 315z^4 + 105z^2 - 5\right)$$

05.03.03.0011.01

$$P_7(z) = \frac{1}{16}\,z\left(429z^6 - 693z^4 + 315z^2 - 35\right)$$

05.03.03.0012.01

$$P_8(z) = \frac{1}{128}\left(6435z^8 - 12\,012z^6 + 6930z^4 - 1260z^2 + 35\right)$$

05.03.03.0013.01

$$P_9(z) = \frac{1}{128}\,z\left(12\,155z^8 - 25\,740z^6 + 18\,018z^4 - 4620z^2 + 315\right)$$

05.03.03.0014.01

$$P_{10}(z) = \frac{1}{256}\left(46\,189z^{10} - 109\,395z^8 + 90\,090z^6 - 30\,030z^4 + 3465z^2 - 63\right)$$

Values at infinities

05.03.03.0015.01

$$P_n(\infty) = \infty \;/;\; n > 0$$

05.03.03.0016.01

$$P_n(-\infty) = (-1)^n\,\infty \;/;\; n > 0$$

General characteristics

Domain and analyticity

The function $P_n(z)$is defined over $\mathbb{N} \otimes \mathbb{C}$. For fixed n, the function $P_n(z)$ is a polynomial in z of degree n.

05.03.04.0001.01

$$(n * z) \longrightarrow P_n(z) :: (\mathbb{N} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

05.03.04.0002.01

$$P_n(-z) = (-1)^n P_n(z)$$

Mirror symmetry

05.03.04.0003.01

$$P_n(\bar{z}) = \overline{P_n(z)}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

The function $P_n(z)$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.03.04.0004.01

$$\mathcal{S}ing_z(P_n(z)) = \{\{\tilde{\infty}, n\}\}$$

Branch points

With respect to z

The function $P_n(z)$ does not have branch points.

05.03.04.0005.01

$$\mathcal{BP}_z(P_n(z)) = \{\}$$

Branch cuts

With respect to z

The function $P_n(z)$ does not have branch cuts.

05.03.04.0006.01

$$\mathcal{BC}_z(P_n(z)) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

05.03.06.0023.01

$$P_n(z) \propto P_n(z_0) - \frac{P_n^1(z_0)}{\sqrt{1-z_0^2}}(z-z_0) + \frac{P_n^2(z_0)}{2\left(1-z_0^2\right)}(z-z_0)^2 + \ldots /; (z \to z_0)$$

05.03.06.0024.01

$$P_n(z) \propto P_n(z_0) - \frac{P_n^1(z_0)}{\sqrt{1-z_0^2}}(z-z_0) + \frac{P_n^2(z_0)}{2\left(1-z_0^2\right)}(z-z_0)^2 + O\left((z-z_0)^3\right)$$

05.03.06.0025.01

$$P_n(z) = \sum_{k=0}^{n} \frac{(-1)^k P_n^k(z_0)}{\left(1-z_0^2\right)^{k/2} k!}(z-z_0)^k$$

05.03.06.0026.01

$$P_n(z) = \sum_{k=0}^{n} \frac{2^k \left(\frac{1}{2}\right)_k}{k!} C_{n-k}^{k+\frac{1}{2}}(z_0)(z-z_0)^k$$

05.03.06.0027.01

$$P_n(z) = \sum_{k=0}^{n} \frac{2^{-k}\,\Gamma(n+k+1)}{(k!)^2\,\Gamma(n-k+1)}\, {}_2F_1\left(k-n, k+n+1; k+1; \frac{1-z_0}{2}\right)(z-z_0)^k$$

05.03.06.0028.01

$$P_n(z) = \sum_{k=0}^{n} (2k-1)!! \sum_{i_1=0}^{n-k} \ldots \sum_{i_{2k+1}=0}^{n-k} \delta_{\sum_{j=1}^{2k+1} i_j, n-k} \prod_{j=1}^{2k+1} P_{i_j}(z)\,(z-z_0)^k$$

05.03.06.0029.01

$$P_n(z) \propto P_n(z_0)\,(1+O(z-z_0))$$

Expansions at $z == 0$

05.03.06.0030.01

$$P_n(z) \propto \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor}\,\Gamma\left(\frac{n+1}{2} + \left\lfloor \frac{n+1}{2} \right\rfloor - \frac{n}{2}\right)}{\sqrt{\pi}\,\left\lfloor \frac{n}{2} \right\rfloor!}(2z)^{2\left\lfloor \frac{n+1}{2} \right\rfloor - n}$$
$$\left(1 - \frac{2\left(2n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left\lfloor \frac{n}{2} \right\rfloor}{\left(n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(n - 2\left\lfloor \frac{n}{2} \right\rfloor + 2\right)} z^2 + \frac{4\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left\lfloor \frac{n}{2} \right\rfloor\left(2n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(2n - 2\left\lfloor \frac{n}{2} \right\rfloor + 3\right)}{\left(n - 2\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(n - 2\left\lfloor \frac{n}{2} \right\rfloor + 2\right)\left(n - 2\left\lfloor \frac{n}{2} \right\rfloor + 3\right)\left(n - 2\left\lfloor \frac{n}{2} \right\rfloor + 4\right)} z^4 + \ldots\right) /; (z \to 0)$$

05.03.06.0001.01

$$P_{2m}(z) \propto \frac{(-1)^m\,\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}\; m!}\left(1 - m(2m+1)z^2 + \frac{(m-1)m(1+2m)(3+2m)}{6} z^4 + \ldots\right) /; (z \to 0)$$

05.03.06.0002.01

$$P_{2m+1}(z) \propto (-1)^m\, 2^{-2m-1}\, 2(m+1)\binom{2m+1}{m} z\left(1 - \frac{m(2m+3)}{3} z^2 + \frac{(m-1)m(3+2m)(5+2m)}{30} z^4 + \ldots\right) /; (z \to 0)$$

05.03.06.0003.01

$$P_n(z) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-n)_{j+k}\,(n+1)_{j+k}\,(-z)^j}{(j+k)!\,j!\,k!\,2^{j+k}}$$

05.03.06.0004.01

$$P_n(z) = F_{1\times 0\times 0}^{2\times 0\times 0}\left(\begin{matrix} -n,\ 1+n;;;\\ 1;;;\end{matrix}\ \frac{1}{2},\ -\frac{z}{2}\right)$$

05.03.06.0005.01

$$P_n(z) = \frac{1}{2^n} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} z^{n-2k}$$

05.03.06.0006.01

$$P_n(z) \propto \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-n}{2}\right)\Gamma\left(\frac{n}{2}+1\right)} (1+O(z)) \;/;\; (z \to 0)$$

05.03.06.0007.01

$$P_n(z) \propto \frac{(-1)^{\left\lfloor \frac{n}{2} \right\rfloor}}{2^n} \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} (n+1)^{n-2\left\lfloor \frac{n}{2} \right\rfloor} z^{n-2\left\lfloor \frac{n}{2} \right\rfloor} \left(1+O\left(z^2\right)\right) \;/;\; (z \to 0)$$

Expansions at $z = 1$

05.03.06.0008.01

$$P_n(z) \propto 1 - \frac{-n\,(1+n)}{2}(z-1) + \frac{(-n)\,(1-n)\,(1+n)\,(2+n)}{16}(z-1)^2 - \ldots \;/;\; (z \to 1)$$

05.03.06.0009.01

$$P_n(z) = \sum_{k=0}^{n} \frac{(-n)_k\,(n+1)_k}{k!^2} \left(\frac{1-z}{2}\right)^k$$

05.03.06.0010.01

$$P_n(z) = {}_2F_1\left(-n,\ n+1;\ 1;\ \frac{1-z}{2}\right)$$

05.03.06.0011.01

$$P_n(z) \propto 1 + O(z-1) \;/;\; (z \to 1)$$

Expansions at $z = -1$

05.03.06.0012.01

$$P_n(z) \propto (-1)^n \left(1 - \frac{n\,(1+n)}{2}(z+1) + \frac{(n-1)\,n\,(1+n)\,(2+n)}{16}(z+1)^2 - \ldots\right) \;/;\; (z \to -1)$$

05.03.06.0013.01

$$P_n(z) = (-1)^n \sum_{k=0}^{n} \frac{(-n)_k\,(n+1)_k}{k!^2} \left(\frac{z+1}{2}\right)^k$$

05.03.06.0031.01

$$P_n(z) \propto (-1)^n\,{}_2F_1\left(-n,\ n+1;\ 1;\ \frac{z+1}{2}\right)$$

05.03.06.0014.01

$$P_n(z) \propto (-1)^n\,(1+O(z+1))$$

Expansions at $z == \infty$

For the function itself

Expansions in $1/z$

05.03.06.0015.02

$$P_n(z) \propto \frac{2^n \left(\frac{1}{2}\right)_n}{n!} z^n \left(1 - \frac{(n-1)\,n}{2\,(2\,n-1)\,z^2} + \frac{(n-3)\,(n-2)\,(n-1)\,n}{8\,(2\,n-3)\,(2\,n-1)\,z^4} + \ldots\right) /;\ (|z| \to \infty)$$

05.03.06.0032.01

$$P_n(z) \propto \frac{2^n \left(\frac{1}{2}\right)_n}{n!} z^n \left(1 - \frac{(n-1)\,n}{2\,(2\,n-1)\,z^2} + \frac{(n-3)\,(n-2)\,(n-1)\,n}{8\,(2\,n-3)\,(2\,n-1)\,z^4} + O\left(\frac{1}{z^6}\right)\right)$$

05.03.06.0033.01

$$P_n(z) == \frac{2^n \left(\frac{1}{2}\right)_n}{n!} z^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{\left(\frac{1-n}{2}\right)_k \left(-\frac{n}{2}\right)_k}{k! \left(\frac{1}{2}-n\right)_k z^{2\,k}}$$

05.03.06.0034.01

$$P_n(z) == \frac{2^n \left(\frac{1}{2}\right)_n}{n!} z^n\, {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; \frac{1}{2}-n; \frac{1}{z^2}\right)$$

05.03.06.0035.01

$$P_n(z) \propto \frac{2^n \left(\frac{1}{2}\right)_n}{n!} z^n \left(1 + O\left(\frac{1}{z^2}\right)\right)$$

Expansions in $1/(1-z)$

05.03.06.0036.01

$$P_n(z) \propto \frac{2^n}{n!} \left(\frac{1}{2}\right)_n (z-1)^n \left(1 - \frac{n}{1-z} - \frac{n\,(1-n)^2}{(1-2\,n)\,(1-z)^2} + \frac{(n-2)^2\,(n-1)\,n}{3\,(2\,n-1)\,(z-1)^3} + \ldots\right) /;\ (|z| \to \infty)$$

05.03.06.0016.01

$$P_n(z) == \frac{2^n \left(\frac{1}{2}\right)_n}{n!} (z-1)^n \sum_{k=0}^{n} \frac{(-n)_k^{\,2}}{k!\,(-2\,n)_k} \left(\frac{2}{1-z}\right)^k$$

05.03.06.0017.01

$$P_n(z) == \frac{2^n \left(\frac{1}{2}\right)_n}{n!} (z-1)^n\, {}_2F_1\left(-n, -n; -2\,n; \frac{2}{1-z}\right)$$

05.03.06.0018.01

$$P_n(z) \propto \frac{2^n \left(\frac{1}{2}\right)_n}{n!} z^n \left(1 + O\left(\frac{1}{z}\right)\right)$$

Expansions at $n == \infty$

05.03.06.0037.01

$$P_n(z) \propto \frac{1}{\sqrt[4]{1-z^2}\,\sqrt{n}}\sqrt{\frac{2}{\pi}}\left(\cos\left(\frac{\pi}{4}-\left(n+\frac{1}{2}\right)\cos^{-1}(z)\right)-\frac{1}{128\,n^2\,(1-z^2)}\right.$$
$$\left(6\sqrt{1-z^2}\,\cos\left(\left(n-\frac{1}{2}\right)\cos^{-1}(z)+\frac{\pi}{4}\right)+9\cos\left(\frac{1}{4}\left((6-4\,n)\cos^{-1}(z)+\pi\right)\right)+\left(z^2-1\right)\cos\left(\frac{\pi}{4}-\left(n+\frac{1}{2}\right)\cos^{-1}(z)\right)\right)-$$
$$\left.\frac{\cos\left(\left(n-\frac{1}{2}\right)\cos^{-1}(z)+\frac{\pi}{4}\right)+\sqrt{1-z^2}\,\cos\left(\frac{\pi}{4}-\left(n+\frac{1}{2}\right)\cos^{-1}(z)\right)}{8\,n\sqrt{1-z^2}}+\ldots\right)/;\ (n\to\infty)$$

05.03.06.0038.01

$$P_n(z) \propto \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{n}}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k}\,2^{-j}\,(1-z^2)^{-\frac{2j+1}{4}}}{j!\,(k-j)!}\cos\left(\left(n-j+\frac{1}{2}\right)\cos^{-1}(z)-\frac{(2\,j+1)\,\pi}{4}\right)B_{k-j}^{\left(\frac{1}{2}-j\right)}\left(\frac{1}{2}-j\right)\left(\frac{1}{2}\right)_j\left(\frac{1}{2}\right)_k n^{-k}\ /;$$
$$(n\to\infty)$$

05.03.06.0039.01

$$P_n(z) \propto \sqrt{\frac{2}{\pi}}\,\frac{\cos\left(\frac{\pi}{4}-\left(n+\frac{1}{2}\right)\cos^{-1}(z)\right)}{\sqrt[4]{1-z^2}\,\sqrt{n}}\,(1+\ldots)\ /;\ (n\to\infty)$$

Other series representations

05.03.06.0019.01

$$P_n(z) = \sum_{k=0}^{n}\frac{(-1)^k\,(k+n)!}{(n-k)!\,k!^2\,2^{k+1}}\left((1-z)^k+(-1)^n\,(z+1)^k\right)$$

05.03.06.0020.01

$$P_n(z) = \left(\frac{z-1}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}^2\left(\frac{z+1}{z-1}\right)^k$$

05.03.06.0021.01

$$P_n(\cos(\theta)) = (-1)^n\sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{n-k}\cos((n-2\,k)\,\theta)$$

05.03.06.0022.01

$$P_n\left(x\,y-\sqrt{1-x^2}\,\sqrt{1-y^2}\,\cos(\alpha)\right) = P_n(x)\,P_n(y)+2\sum_{k=1}^{n}\frac{(-1)^k\cos(k\,\alpha)\,(n-k)!}{(k+n)!}P_n^k(x)\,P_n^k(y)$$

Integral representations

On the real axis

Of the direct function

05.03.07.0001.01

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi}\left(z-\sqrt{z^2-1}\,\cos(t)\right)^n dt$$

05.03.07.0002.01

$$P_n(z) == \frac{1}{\pi} \int_0^{\pi} \left(z + i \sqrt{1 - z^2} \cos(t)\right)^n \, dt$$

05.03.07.0003.01

$$P_n(z) == \frac{2^n}{\pi} \int_{-\infty}^{\infty} \frac{(i\, t + z)^n}{\left(t^2 + 1\right)^{n+1}} \, dt$$

Integral representations of negative integer order

Rodrigues-type formula.

05.03.07.0004.01

$$P_n(z) == \frac{(-1)^n}{2^n \, n!} \frac{\partial^n \left(1 - z^2\right)^n}{\partial z^n}$$

Generating functions

05.03.11.0001.01

$$P_n(z) == \left([t^n] \, \frac{1}{\sqrt{t^2 - 2\, z\, t + 1}}\right) /; -1 < z < 1$$

05.03.11.0002.01

$$P_n(z) == [t^n] \left(\frac{n!^2}{(a)_n \, (1 - a)_n} \, {}_2F_1\!\left(a, 1 - a; 1; \frac{1}{2}\left(-t - \sqrt{t^2 - 2\, z\, t + 1} + 1\right)\right) {}_2F_1\!\left(a, 1 - a; 1; \frac{1}{2}\left(t - \sqrt{t^2 - 2\, z\, t + 1} + 1\right)\right)\right) /;$$
$$-1 < z < -1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

05.03.13.0001.01

$$\left(1 - z^2\right) w''(z) - 2\, z\, w'(z) + (n + 1)\, n\, w(z) == 0 /; w(z) == c_1 \, P_n(z) + c_2 \, Q_n(z)$$

05.03.13.0002.02

$$W_z\left(P_n(z), Q_n(z)\right) == \frac{1}{\left(1 - z^2\right)}$$

05.03.13.0003.01

$$g'(z)\, w''(z) - \left(\frac{2\, g(z)\, g'(z)^2}{1 - g(z)^2} + g''(z)\right) w'(z) + \frac{n\, (n + 1)\, g'(z)^3}{1 - g(z)^2} \, w(z) = 0 /; w(z) = c_1 \, P_n(g(z)) + c_2 \, Q_n(g(z))$$

05.03.13.0004.01

$$W_z\left(P_n(g(z)), Q_n(g(z))\right) == \frac{g'(z)}{1 - g(z)^2}$$

05.03.13.0005.01

$$g'(z)\,h(z)^2\,w''(z) - \left(\left(\frac{2\,g(z)\,g'(z)^2}{1-g(z)^2} + g''(z)\right) h(z)^2 + 2\,g'(z)\,h'(z)\,h(z)\right) w'(z) + \left(\frac{n\,(n+1)\,h(z)^2\,g'(z)^3}{1-g(z)^2} + 2\,h'(z)^2\,g'(z) + h(z)\left(h'(z)\left(\frac{2\,g(z)\,g'(z)^2}{1-g(z)^2} + g''(z)\right) - g'(z)\,h''(z)\right)\right) w(z) = 0 \;/;\; w(z) = c_1\,h(z)\,P_n(g(z)) + c_2\,h(z)\,Q_n(g(z))$$

05.03.13.0006.01

$$W_z\big(h(z)\,P_n(g(z)),\,h(z)\,Q_n(g(z))\big) = \frac{h(z)^2\,g'(z)}{1-g(z)^2}$$

05.03.13.0007.01

$$z^2\,w''(z) - z\left(2\,s + \frac{r\left(a^2\,z^{2r}+1\right)}{1-a^2\,z^{2r}} - 1\right) w'(z) + \left(-\frac{a^2\,n\,(n+1)\,r^2\left(a^2\,z^{2r}-1\right) z^{2r}}{\left(1-a^2\,z^{2r}\right)^2} + s^2 + \frac{r\,s\left(a^2\,z^{2r}+1\right)}{1-a^2\,z^{2r}}\right) w(z) = 0 \;/;\; w(z) = c_1\,z^s\,P_n(a\,z^r) + c_2\,z^s\,Q_n(a\,z^r)$$

05.03.13.0008.01

$$W_z\big(z^s\,P_n(a\,z^r),\,z^s\,Q_n(a\,z^r)\big) = \frac{a\,r\,z^{r+2s-1}}{1-a^2\,z^{2r}}$$

05.03.13.0009.01

$$w''(z) - \frac{a^2\,(\log(r) - 2\log(s))\,r^{2z} + \log(r) + 2\log(s)}{1-a^2\,r^{2z}}\,w'(z) + \left(\frac{a^2\,n\,(n+1)\log^2(r)\,r^{2z}}{1-a^2\,r^{2z}} + \log^2(s) + \frac{\left(a^2\,r^{2z}+1\right)\log(r)\log(s)}{1-a^2\,r^{2z}}\right) w(z) = 0 \;/;\; w(z) = c_1\,s^z\,P_n(a\,r^z) + c_2\,s^z\,Q_n(a\,r^z)$$

05.03.13.0010.01

$$W_z\big(s^z\,P_n(a\,r^z),\,s^z\,Q_n(a\,r^z)\big) = \frac{a\,r^z\,s^{2z}\log(r)}{1-a^2\,r^{2z}}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.03.16.0001.01

$$P_n(-z) = (-1)^n\,P_n(z)$$

Products, sums, and powers of the direct function

Products of the direct function

05.03.16.0002.01

$$P_n(z)\,P_m(z) = \sum_{k=|m-n|}^{m+n} b(n,m,k)\,P_k(z) \;/;\; b(n,m,k) = \left(\delta_{0,\left(\frac{1}{2}(k+m+n)\right) \bmod 1}\,(2k+1)\,(k+m-n-1)!!\,(k-m+n-1)!!\,(m+n-k-1)!!\,(k+m+n)!!\right) \Big/ \left((k+m-n)!!\,(k-m+n)!!\,(m+n-k)!!\,(k+m+n+1)!!\right)$$

Identities

Recurrence identities

Consecutive neighbors

05.03.17.0001.01

$$P_n(z) = \frac{(2n+3)z}{n+1} P_{n+1}(z) - \frac{n+2}{n+1} P_{n+2}(z)$$

05.03.17.0002.01

$$P_n(z) = \frac{(2n-1)z}{n} P_{n-1}(z) - \frac{n-1}{n} P_{n-2}(z)$$

Distant neighbors

05.03.17.0006.01

$$P_n(z) = C_m(n, z) P_{m+n}(z) - \frac{m+n+1}{m+n} C_{m-1}(n, z) P_{m+n+1}(z) /;$$
$$C_0(n, z) = 1 \bigwedge C_1(n, z) = \frac{(2n+3)z}{n+1} \bigwedge C_m(n, z) = \frac{z(2m+2n+1)}{m+n} C_{m-1}(n, z) - \frac{m+n}{m+n-1} \bigwedge m \in \mathbb{N}^+$$

05.03.17.0007.01

$$P_n(z) = \frac{m-n}{n-m+1} C_{m-1}(n, z) P_{n-m-1}(z) + C_m(n, z) P_{n-m}(z) /;$$
$$C_0(n, z) = 1 \bigwedge C_1(n, z) = \frac{(2n-1)z}{n} \bigwedge C_m(n, z) = \frac{z(2m-2n-1)}{m-n-1} C_{m-1}(n, z) - \frac{n-m+1}{n-m+2} C_{m-2}(n, z) \bigwedge m \in \mathbb{N}^+$$

Functional identities

Relations between contiguous functions

Recurrence relations

05.03.17.0003.01

$$n P_{n-1}(z) + (n+1) P_{n+1}(z) = (2n+1) z P_n(z)$$

05.03.17.0004.01

$$P_n(z) = \frac{1}{(2n+1)z} (n P_{n-1}(z) + (n+1) P_{n+1}(z))$$

Normalized recurrence relation

05.03.17.0005.01

$$z\, p(n, z) = \frac{n^2}{4n^2-1} p(n-1, z) + p(n+1, z) /; p(n, z) = \frac{2^{-n} \sqrt{\pi}\, n!}{\Gamma\left(n+\frac{1}{2}\right)} P_n(z)$$

Complex characteristics

Real part

05.03.19.0001.01

$$\operatorname{Re}(P_n(x+i\,y)) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j}\, y^{2j}}{(2\,j)!} \left(\frac{1}{2}\right)_{2j} C_{n-2j}^{\left(2j+\frac{1}{2}\right)}(x) \;/;\; x \in \mathbb{R} \land y \in \mathbb{R}$$

Imaginary part

05.03.19.0002.01

$$\operatorname{Im}(P_n(x+i\,y)) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j+1}\, y^{2j+1}}{(2\,j+1)!} \left(\frac{1}{2}\right)_{2j+1} C_{-2j+n-1}^{\left(2j+\frac{3}{2}\right)}(x) \;/;\; x \in \mathbb{R} \land y \in \mathbb{R}$$

Argument

05.03.19.0003.01

$$\arg(P_n(x+i\,y)) = \tan^{-1}\left(\sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j} \left(\frac{1}{2}\right)_{2j}}{(2\,j)!} C_{n-2j}^{2j+\frac{1}{2}}(x)\, y^{2j},\; \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j+1} \left(\frac{1}{2}\right)_{2j+1}}{(2\,j+1)!} C_{-2j+n-1}^{2j+\frac{3}{2}}(x)\, y^{2j+1}\right) /;\; x \in \mathbb{R} \land y \in \mathbb{R}$$

Conjugate value

05.03.19.0004.01

$$\overline{P_n(x+i\,y)} = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j} \left(\frac{1}{2}\right)_{2j}}{(2\,j)!} C_{n-2j}^{2j+\frac{1}{2}}(x)\, y^{2j} - i \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j\, 2^{2j+1} \left(\frac{1}{2}\right)_{2j+1}}{(2\,j+1)!} C_{-2j+n-1}^{2j+\frac{3}{2}}(x)\, y^{2j+1} \;/;\; x \in \mathbb{R} \land y \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to *z*

05.03.20.0001.01

$$\frac{\partial P_n(z)}{\partial z} = \frac{n}{z^2-1}\,(z\,P_n(z) - P_{n-1}(z))$$

05.03.20.0002.01

$$\frac{\partial^2 P_n(z)}{\partial z^2} = \frac{n}{\left(z^2-1\right)^2}\left(2\,z\,P_{n-1}(z) + \left((n-1)\,z^2 - n - 1\right) P_n(z)\right)$$

Symbolic differentiation

With respect to *z*

05.03.20.0008.01

$$\frac{\partial^m P_n(z)}{\partial z^m} = (-1)^m \left(1-z^2\right)^{-\frac{m}{2}} P_n^m(z) \;/;\; m \in \mathbb{N}$$

05.03.20.0003.02

$$\frac{\partial^m P_n(z)}{\partial z^m} = 2^m \left(\frac{1}{2}\right)_m C_{n-m}^{m+\frac{1}{2}}(z) \;/;\; m \in \mathbb{N}$$

05.03.20.0004.02

$$\frac{\partial^m P_n(z)}{\partial z^m} == (z-1)^{-m}\, {}_2\tilde{F}_1\left(-n, n+1; 1-m; \frac{1-z}{2}\right) /; m \in \mathbb{N}$$

05.03.20.0005.02

$$\frac{\partial^m P_n(z)}{\partial z^m} == \frac{2^{-m}\,\Gamma(m+n+1)}{m!\,\Gamma(n-m+1)}\, {}_2F_1\left(m-n, m+n+1; m+1; \frac{1-z}{2}\right) /; m \in \mathbb{N}$$

05.03.20.0007.02

$$\frac{\partial^m P_n(z)}{\partial z^m} == (2m-1)!! \sum_{i_1=0}^{n-m} \ldots \sum_{i_{2m+1}=0}^{n-m} \delta_{\sum_{j=1}^{2m+1} i_j, n-m} \prod_{j=1}^{2m+1} P_{i_j}(z) /; m \in \mathbb{N} \land n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

05.03.20.0006.01

$$\frac{\partial^\alpha P_n(z)}{\partial z^\alpha} == z^{-\alpha}\, F_{1\times 0\times 1}^{2\times 0\times 1}\left(\begin{matrix} -n, n+1;; 1; \\ 1;; 1-\alpha; \end{matrix} \frac{1}{2}, -\frac{z}{2}\right)$$

Integration

Indefinite integration

Involving only one direct function

05.03.21.0001.01

$$\int P_n(z)\, dz == \frac{P_{n+1}(z) - P_{n-1}(z)}{2n+1}$$

Involving one direct function and elementary functions

Involving power function

05.03.21.0002.01

$$\int z^{\alpha-1} P_n(z)\, dz == \frac{z^\alpha}{\alpha} F_{1\times 0\times 1}^{2\times 0\times 1}\left(\begin{matrix} -n, 1+n;; \alpha; \\ 1;; \alpha+1; \end{matrix} \frac{1}{2}, -\frac{z}{2}\right)$$

Involving algebraic functions

05.03.21.0003.01

$$\int \left(1-z^2\right)^{\frac{1}{2}(-n-3)} P_n(z)\, dz == \frac{\left(1-z^2\right)^{\frac{1}{2}(-n-1)}}{n+1} P_{n+1}(z)$$

05.03.21.0004.01

$$\int \left(1-z^2\right)^{\frac{n}{2}-1} P_n(z)\, dz == -\frac{\left(1-z^2\right)^{n/2}}{n} P_{n-1}(z)$$

Involving logarithm

05.03.21.0005.01

$$\int \log\left(\frac{1+z}{1-z}\right) P_n(z)\, dz = \frac{2\, P_n(z)}{n^2+n} + \frac{1}{2\,n+1} \log\left(\frac{1+z}{1-z}\right) (P_{n+1}(z) - P_{n-1}(z))$$

Definite integration

Involving the direct function

Orthogonality:

05.03.21.0006.01

$$\int_{-1}^{1} P_m(t)\, P_n(t)\, dt = \frac{2\, \delta_{n,m}}{2\,n+1}$$

05.03.21.0007.01

$$\int_0^{\pi} P_m(\cos(t))\, P_n(\cos(t))\, P_k(\cos(t)) \sin(t)\, dt = 2 \begin{pmatrix} m & n & k \\ 0 & 0 & 0 \end{pmatrix}^2$$

Integral transforms

Laplace transforms

05.03.22.0001.01

$$\mathcal{L}_t[P_n(t)](z) = \frac{1}{2^n} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{k} \binom{2\,n-2\,k}{n} \Gamma(n-2\,k+1)\, z^{2\,k-n-1}$$

Summation

Finite summation

05.03.23.0001.01

$$\sum_{k=1}^{n} \cos\left(k \cos^{-1}(z)\right) P_{n-k}(z) = n\, P_n(z)$$

05.03.23.0010.01

$$\sum_{k=0}^{n} (2\,k+1)\, P_k(z_1)\, P_k(z_2) = \frac{n+1}{z_1 - z_2} (P_{n+1}(z_1)\, P_n(z_2) - P_n(z_1)\, P_{n+1}(z_2)) \;/;\; n \in \mathbb{N}$$

05.03.23.0011.01

$$\sum_{k=0}^{n} (2\,k+1)\, P_k(z_1)\, Q_k(z_2) = \frac{1}{z_1 - z_2} \left((n+1) \left(P_{n+1}(z_1)\, Q_n(z_2) - P_n(z_1)\, Q_{n+1}(z_2)\right) - 1\right) \;/;\; n \in \mathbb{N}$$

Infinite summation

05.03.23.0002.01

$$\sum_{n=0}^{\infty} P_n(z)\, w^n = \frac{1}{\sqrt{w^2 - 2\,z\,w + 1}} \;/;\; -1 < z < 1 \land |w| < 1$$

05.03.23.0003.01

$$\sum_{n=0}^{\infty} \frac{1}{n!^2} P_n(z)\, w^n = {}_0F_1\left(;1;\frac{1}{2}(z-1)\,w\right) {}_0F_1\left(;1;\frac{1}{2}(z+1)\,w\right) /; -1<z<1 \land |w|<1$$

05.03.23.0004.01

$$\sum_{n=0}^{\infty} \frac{1}{n!} P_n(z)\, w^n = e^{w z}\, {}_0F_1\left(;1;\frac{1}{4}\left(z^2-1\right) w^2\right) /; -1<z<1 \land |w|<1$$

05.03.23.0005.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n\, (1-\gamma)_n}{n!^2} P_n(z)\, w^n = {}_2F_1\left(\gamma, 1-\gamma; 1; \frac{1}{2}\left(1-\sqrt{w^2-2 z w+1}-w\right)\right) {}_2F_1\left(\gamma, 1-\gamma; 1; \frac{1}{2}\left(1-\sqrt{w^2-2 z w+1}+w\right)\right)$$

05.03.23.0006.01

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} P_n(z)\, w^n = (1-w z)^{-\gamma}\, {}_2F_1\left(\frac{\gamma}{2}, \frac{\gamma+1}{2}; 1; \frac{\left(z^2-1\right) w^2}{(1-w z)^2}\right) /; -1<z<1 \land |w|<1$$

05.03.23.0012.01

$$\sum_{k=0}^{\infty} \frac{(a)_k\, (1-a)_k}{k!\, k!} P_k(z)\, t^k = {}_2F_1\left(a, 1-a; 1; \frac{1}{2}\left(-t-\sqrt{t^2-2 z t+1}+1\right)\right) {}_2F_1\left(a, 1-a; 1; \frac{1}{2}\left(t-\sqrt{t^2-2 z t+1}+1\right)\right)$$

05.03.23.0007.01

$$\sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{\nu-k} - \frac{1}{k+\nu+1}\right) P_k(x) = \frac{\pi P_\nu(x)}{\sin(\nu \pi)} /; x \in \mathbb{R} \bigwedge -1<x\leq 1 \bigwedge \nu \notin \mathbb{Z}$$

05.03.23.0008.01

$$\sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{\nu-k} - \frac{1}{k+\nu+1}\right) P_k(x)\, P_k(y) = \frac{\pi}{\sin(\nu \pi)} P_\nu(x)\, P_\nu(y) /;$$

$$x \in \mathbb{R} \bigwedge -1<x\leq 1 \bigwedge y \in \mathbb{R} \bigwedge -1<y\leq 1 \bigwedge x+y>0 \bigwedge \nu \notin \mathbb{Z}$$

05.03.23.0009.01

$$\sum_{n=0}^{\infty} (2 n+1)\, P_n(x)\, P_n(y) = 2\, \delta(x-y) /; -1<x<1 \bigwedge -1<y<1$$

05.03.23.0013.01

$$\sum_{k=0}^{\infty} \left(k+\frac{1}{2}\right) P_k(x)\, P_k(y)\, P_k(z) = \frac{\theta\left(-x^2+2 y z x-y^2-z^2+1\right)}{\pi \sqrt{-x^2+2 y z x-y^2-z^2+1}} /;$$

$$x \in \mathbb{R} \bigwedge -1<x<1 \bigwedge y \in \mathbb{R} \bigwedge -1<y<1 \bigwedge z \in \mathbb{R} \bigwedge -1<z<1$$

Operations

Orthogonality, completeness, and Fourier expansions

The set of functions $P_n(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{2n+1}{2}$) system on the interval $(-1, 1)$.

05.03.25.0001.01

$$\sum_{n=0}^{\infty} \left(\sqrt{\frac{2 n+1}{2}}\, P_n(x)\right) \left(\sqrt{\frac{2 n+1}{2}}\, P_n(y)\right) = \delta(x-y) /; -1<x<1 \bigwedge -1<y<1$$

05.03.25.0002.01

$$\int_{-1}^{1}\left(\sqrt{\frac{2m+1}{2}}\,P_m(t)\right)\left(\sqrt{\frac{2n+1}{2}}\,P_n(t)\right)dt=\delta_{m,n}$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{P_n(x)\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

05.03.25.0003.01

$$f(x)=\sum_{n=0}^{\infty}c_n\,\psi_n(x)\,/;\,c_n=\int_{-1}^{1}\psi_n(t)\,f(t)\,dt\bigwedge\psi_n(x)=\sqrt{\frac{2n+1}{2}}\,P_n(x)\bigwedge-1<x<1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

05.03.26.0030.01

$$P_n(z)={}_2\tilde{F}_1\left(-n,\,n+1;\,1;\,\frac{1-z}{2}\right)$$

05.03.26.0031.01

$$P_n(z)=2^{-n}\,(z+1)^n\,{}_2\tilde{F}_1\left(-n,\,-n;\,1;\,\frac{z-1}{z+1}\right)$$

05.03.26.0032.01

$$P_n(z)=\pi\left(\frac{1}{\Gamma\left(\frac{1-n}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}\,{}_2\tilde{F}_1\left(\frac{n+1}{2},\,-\frac{n}{2};\,\frac{1}{2};\,z^2\right)-\frac{z}{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}\,{}_2\tilde{F}_1\left(\frac{1-n}{2},\,\frac{n+2}{2};\,\frac{3}{2};\,z^2\right)\right)$$

05.03.26.0033.01

$$P_n(z)=\frac{2^n\,(-1)^n\,\sqrt{\pi}\,z^n}{n!}\,{}_2\tilde{F}_1\left(\frac{1-n}{2},\,-\frac{n}{2};\,\frac{1}{2}-n;\,\frac{1}{z^2}\right)$$

Involving ${}_2F_1$

05.03.26.0001.01

$$P_n(z)={}_2F_1\left(-n,\,n+1;\,1;\,\frac{1-z}{2}\right)$$

05.03.26.0034.01

$$P_n(z)=\frac{(z+1)^n}{2^n}\,{}_2F_1\left(-n,\,-n;\,1;\,\frac{z-1}{z+1}\right)$$

05.03.26.0002.01

$$P_n(z)=\frac{2^n\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(n+1)}\,(z-1)^n\,{}_2F_1\left(-n,\,-n;\,-2n;\,\frac{2}{1-z}\right)$$

05.03.26.0035.01

$$P_n(z)=\sqrt{\pi}\left(\frac{1}{\Gamma\left(\frac{1-n}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}\,{}_2F_1\left(\frac{n+1}{2},\,-\frac{n}{2};\,\frac{1}{2};\,z^2\right)-\frac{2z}{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}\,{}_2F_1\left(\frac{1-n}{2},\,\frac{n+2}{2};\,\frac{3}{2};\,z^2\right)\right)$$

05.03.26.0036.01

$$P_n(z) = \frac{2^n z^n \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\, n!}\, {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; \frac{1}{2}-n; \frac{1}{z^2}\right)$$

Through hypergeometric functions of two variables

05.03.26.0003.01

$$P_n(z) = F_{1\times 0\times 0}^{2\times 0\times 0}\left(\begin{matrix} -n,\, 1+n;;; \\ 1;;; \end{matrix}\ \frac{1}{2}, -\frac{z}{2}\right)$$

Through Meijer *G*

Classical cases for the direct function itself

05.03.26.0004.01

$$P_n(z) = -\frac{1}{\pi} \lim_{\nu\to n} \sin(\pi\,\nu)\, G_{2,2}^{1,2}\left(\frac{z-1}{2} \,\middle|\, \begin{matrix} \nu+1, -\nu \\ 0, 0 \end{matrix}\right)$$

Classical cases involving algebraic functions

05.03.26.0005.01

$$(z+1)^{-n-1} P_n\left(\frac{1-z}{1+z}\right) = \frac{1}{\Gamma(n+1)^2}\, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} -n, -n \\ 0, 0 \end{matrix}\right) /; z \notin (-\infty, -1)$$

05.03.26.0006.01

$$(z+1)^{-n-1} P_n\left(\frac{z-1}{z+1}\right) = \frac{1}{\Gamma(n+1)^2}\, G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} -n, -n \\ 0, 0 \end{matrix}\right) /; z \notin (-1, 0)$$

05.03.26.0007.01

$$(z+1)^{-\frac{n+1}{2}} P_n\left(\frac{1}{\sqrt{z+1}}\right) = \frac{2^n}{\Gamma(n+1)\sqrt{\pi}}\, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ 0, 0 \end{matrix}\right)$$

05.03.26.0008.01

$$(z+1)^{-\frac{n+1}{2}} P_n\left(\sqrt{\frac{z}{z+1}}\right) = \frac{2^n}{\Gamma(n+1)\sqrt{\pi}}\, G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1-n}{2}, \frac{1-n}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /; z \notin (-1, 0)$$

05.03.26.0009.01

$$(z+1)^{-\frac{n+1}{2}} P_n\left(\frac{z+2}{2\sqrt{z+1}}\right) = \frac{1}{\Gamma(n+1)\sqrt{\pi}}\, G_{2,2}^{1,2}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, -n \\ 0, 0 \end{matrix}\right)$$

05.03.26.0010.01

$$(z+1)^{-\frac{n+1}{2}} P_n\left(\frac{2z+1}{2\sqrt{z}\sqrt{z+1}}\right) = \frac{1}{\Gamma(n+1)\sqrt{\pi}}\, G_{2,2}^{2,1}\left(z \,\middle|\, \begin{matrix} \frac{1-n}{2}, \frac{1-n}{2} \\ -\frac{n}{2}, \frac{n+1}{2} \end{matrix}\right) /; z \notin (-1, 0)$$

Classical cases involving unit step θ

05.03.26.0011.01

$$\theta(1-|z|)\, P_n(2z-1) = G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} n+1, -n \\ 0, 0 \end{matrix}\right) /; z \notin (-1, 0)$$

05.03.26.0012.01

$$\theta(|z|-1)\, P_n(2z-1) = G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} n+1, -n \\ 0, 0 \end{matrix}\right)$$

05.03.26.0013.01

$$\theta(1-|z|)\,P_n\left(\frac{2}{z}-1\right) = G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} 1,\, 1 \\ n+1,\, -n \end{matrix}\right)$$

05.03.26.0014.01

$$\theta(|z|-1)\,P_n\left(\frac{2}{z}-1\right) = G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} 1,\, 1 \\ n+1,\, -n \end{matrix}\right) /;\, z \notin (-\infty,\, -1)$$

05.03.26.0015.01

$$\theta(1-|z|)\,P_n\left(\frac{z+1}{2\sqrt{z}}\right) = G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1-n}{2},\, \frac{n}{2}+1 \\ \frac{n+1}{2},\, -\frac{n}{2} \end{matrix}\right) /;\, z \notin (-1,\, 0)$$

05.03.26.0016.01

$$\theta(|z|-1)\,P_n\left(\frac{z+1}{2\sqrt{z}}\right) = G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{n}{2}+1,\, \frac{1-n}{2} \\ \frac{n+1}{2},\, -\frac{n}{2} \end{matrix}\right)$$

05.03.26.0017.01

$$\theta(|z|-1)\left(\frac{z-1}{z}\right)^{\frac{1}{2}\left(n-2\left\lfloor\frac{n}{2}\right\rfloor-1\right)} P_n\left(\sqrt{\frac{z-1}{z}}\right) = \frac{(-1)^{\left\lfloor\frac{n}{2}\right\rfloor}}{\left\lfloor\frac{n}{2}\right\rfloor!}\,\Gamma\left(n-\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right) G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} 1,\, 1 \\ \left\lfloor\frac{n}{2}\right\rfloor+1,\, -n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2} \end{matrix}\right) /;\, n \in \mathbb{N}$$

Generalized cases involving algebraic functions

05.03.26.0018.01

$$\left(z^2+1\right)^{-\frac{n+1}{2}} P_n\left(\frac{z}{\sqrt{z^2+1}}\right) = \frac{2^n}{\Gamma(n+1)\sqrt{\pi}}\, G_{2,2}^{2,1}\left(z,\, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1-n}{2},\, \frac{1-n}{2} \\ 0,\, \frac{1}{2} \end{matrix}\right) /;\, \mathrm{Re}(z) > 0$$

05.03.26.0019.01

$$\left(z^2+1\right)^{-\frac{n+1}{2}} P_n\left(\frac{2z^2+1}{2z\sqrt{z^2+1}}\right) = \frac{1}{\Gamma(n+1)\sqrt{\pi}}\, G_{2,2}^{2,1}\left(z,\, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1-n}{2},\, \frac{1-n}{2} \\ -\frac{n}{2},\, \frac{n+1}{2} \end{matrix}\right) /;\, \mathrm{Re}(z) > 0$$

Generalized cases involving unit step θ

05.03.26.0020.01

$$\theta(1-|z|)\,P_n(z) = G_{2,2}^{2,0}\left(z,\, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{n}{2}+1,\, \frac{1-n}{2} \\ 0,\, \frac{1}{2} \end{matrix}\right)$$

05.03.26.0021.01

$$\theta(|z|-1)\,P_n(z) = G_{2,2}^{0,2}\left(z,\, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{n}{2}+1,\, \frac{1-n}{2} \\ 0,\, \frac{1}{2} \end{matrix}\right)$$

05.03.26.0022.01

$$\theta(1-|z|)\,P_n\left(\frac{1}{z}\right) = G_{2,2}^{2,0}\left(z,\, \frac{1}{2} \,\middle|\, \begin{matrix} 1,\, \frac{1}{2} \\ -\frac{n}{2},\, \frac{n+1}{2} \end{matrix}\right)$$

05.03.26.0023.01

$$\theta(|z|-1)\,P_n\left(\frac{1}{z}\right) = G_{2,2}^{0,2}\left(z,\, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{2},\, 1 \\ -\frac{n}{2},\, \frac{n+1}{2} \end{matrix}\right)$$

05.03.26.0024.01

$$\theta(1-|z|)\,P_n\left(\frac{z^2+1}{2\,z}\right) = G_{2,2}^{2,0}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{1-n}{2},\,\frac{n}{2}+1\\ \frac{n+1}{2},\,-\frac{n}{2}\end{matrix}\right)$$

05.03.26.0025.01

$$\theta(|z|-1)\,P_n\left(\frac{z^2+1}{2\,z}\right) = G_{2,2}^{0,2}\left(z,\,\frac{1}{2}\,\middle|\,\begin{matrix}\frac{n}{2}+1,\,\frac{1-n}{2}\\ \frac{n+1}{2},\,-\frac{n}{2}\end{matrix}\right)$$

Through other functions

Involving some hypergeometric-type functions

05.03.26.0026.01

$$P_n(z) = P_n^0(z)$$

05.03.26.0027.01

$$P_n(z) = \mathbb{P}_n^0(z)$$

05.03.26.0028.01

$$P_n(z) = P_n^{(0,0)}(z)$$

05.03.26.0029.01

$$P_n(z) = C_n^{\frac{1}{2}}(z)$$

Involving spheroidal functions

05.03.26.0037.01

$$P_n(z) = PS_{n,0}(0,\,z)$$

Representations through equivalent functions

With related functions

05.03.27.0001.01

$$\frac{Q_{n-\frac{1}{2}}(z)}{P_{n-\frac{1}{2}}(z)} - \frac{Q_{n+\frac{1}{2}}(z)}{P_{n+\frac{1}{2}}(z)} = \frac{1}{\left(n+\frac{1}{2}\right)P_{n-\frac{1}{2}}(z)\,P_{n+\frac{1}{2}}(z)}$$

Inequalities

05.03.29.0001.01

$$|P_n(\cos(\theta))| < \sqrt{\frac{2}{\pi\,n\,\sin(\theta)}}\ /;\ n\in\mathbb{N}^+$$

05.03.29.0002.01

$$|P_n(\cos(\theta))| < \frac{1}{\sqrt[8]{1+\frac{\pi^2}{16}\left(n+\frac{1}{2}\right)^4\sin^4(\theta)}}\ /;\ 0<\theta<\pi\bigwedge n>0$$

Brychkov Yu.A. (2006)

Zeros

05.03.30.0001.01

$$\frac{P_n(z)}{z-z_0} = \frac{1}{\left(1-z_0^2\right)\left(\frac{\partial P_n(x)}{\partial x}\,|_{z=z_0}\right)} \sum_{k=0}^{n-1} (2k+1)\, P_k(z)\, P_k(z_0) \;/;\; P_n(z_0) = 0$$

Theorems

Expansions in generalized Fourier series

$$f(x) = \sum_{k=0}^{\infty} c_k\, \psi_k(x) \;/;\quad c_k = \int_{-1}^{1} f(t)\, \psi_k(t)\, dt,\;\; \psi_k(x) = \sqrt{\frac{2n+1}{2}}\; P_k(x),\;\; k \in \mathbb{N}.$$

Gauss' numerical integration methods

$$\int_a^b f(x)\, dx = \frac{b-a}{2} \sum_{k=1}^{n} w_k\, f(y_k) + \frac{2^{2n+1}\, n!^4}{(2n+1)\,(2n)!^3}\, f^{(2n)}(\xi) \;/;$$

$$y_k = \frac{b-a}{2}\, x_k + \frac{b+a}{2} \bigwedge P_n(x_k) = 0 \bigwedge w_k = \frac{2}{1-x_k^2} \left(P_n'(x_k)\right)^{-2} \bigwedge n \in \mathbb{Z}^+,\, a,\, b \in \mathbb{R} \bigwedge a < \xi < b.$$

History

– D. Bernoulli (1748)
– A. M. Legendre (1782, 1785)
– E. Heine (1842)
– P. L. Chebyshev (1855)
– L. Schläfli (1881)
– I. Todhunter (1875) introduced the notation $P_n(z)$

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