LogIntegral

Notations

Traditional name

Logarithmic integral

Traditional notation

$\mathrm{li}(z)$

Mathematica StandardForm notation

`LogIntegral[z]`

Primary definition

06.36.02.0001.01

$$\mathrm{li}(z) = \int_0^z \frac{1}{\log(t)}\, dt$$

Specific values

Values at fixed points

06.36.03.0001.01

$$\mathrm{li}(0) = 0$$

06.36.03.0002.01

$$\mathrm{li}(1) = -\infty$$

Values at infinities

06.36.03.0003.01

$$\mathrm{li}(\infty) = \infty$$

06.36.03.0004.01

$$\mathrm{li}(\tilde{\infty}) = \tilde{\infty}$$

General characteristics

Domain and analyticity

$\mathrm{li}(z)$ is an analytical function of z which is defined over the whole complex z-plane.

06.36.04.0001.01

$z \longrightarrow \text{li}(z) :: \mathbb{C} \longrightarrow \mathbb{C}$

Symmetries and periodicities

Mirror symmetry

06.36.04.0002.01

$\text{li}(\bar{z}) = \overline{\text{li}(z)} \,/;\, z \notin (-\infty, 0)$

Periodicity

No periodicity

Poles and essential singularities

The function $\text{li}(z)$ does not have poles and essential singularities.

06.36.04.0003.01

$\mathcal{Sing}_z(\text{li}(z)) = \{\}$

Branch points

The function $\text{li}(z)$ has three branch points: $z = 0$, $z = 1$and $z = \tilde{\infty}$.

06.36.04.0004.01

$\mathcal{BP}_z(\text{li}(z)) = \{0, 1, \tilde{\infty}\}$

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$\mathcal{R}_z(\text{li}(z), 0) = \log$

06.36.04.0006.01

$\mathcal{R}_z(\text{li}(z), 1) = \log$

06.36.04.0007.01

$\mathcal{R}_z(\text{li}(z), \tilde{\infty}) = \log$

Branch cuts

The function $\text{li}(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 1)$. It is continuous from above along the interval $(-\infty, 0)$ and it has discontinuities from both sides along the interval $(0, 1)$.

06.36.04.0008.01

$\mathcal{BC}_z(\text{li}(z)) = \{\{(-\infty, 0), -i\}, \{(0, 1), \{\}\}\}$

06.36.04.0009.01

$\lim_{\epsilon \to +0} \text{li}(x + i\,\epsilon) = \text{li}(x) \,/;\, x < 0$

06.36.04.0010.01

$\lim_{\epsilon \to +0} \text{li}(x - i\,\epsilon) = \text{Ei}(\log(-x) - i\,\pi) \,/;\, x < 0$

06.36.04.0011.01

$\lim_{\epsilon \to +0} \text{li}(x + i\,\epsilon) = \text{li}(x) + \pi\, i \,/;\, 0 < x < 1$

06.36.04.0012.01

$$\lim_{\epsilon\to+0} \operatorname{li}(x - i\,\epsilon) == \operatorname{li}(x) - \pi\, i \;/;\; 0 < x < 1$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

06.36.06.0010.01

$$\operatorname{li}(z) \propto \operatorname{li}(z_0) + \frac{z - z_0}{\log(z_0)} - \frac{(z - z_0)^2}{\log^2(z_0)\, z_0} + \frac{\log(z_0) + 2}{\log^3(z_0)\, z_0^2} (z - z_0)^3 + \ldots \;/;\; (z \to z_0)$$

06.36.06.0011.01

$$\operatorname{li}(z) \propto \operatorname{li}(z_0) + \frac{z - z_0}{\log(z_0)} - \frac{(z - z_0)^2}{\log^2(z_0)\, z_0} + \frac{\log(z_0) + 2}{\log^3(z_0)\, z_0^2} (z - z_0)^3 + O\!\left((z - z_0)^4\right)$$

06.36.06.0012.01

$$\operatorname{li}(z) = \operatorname{li}(z_0) + \sum_{k=1}^{\infty} z_0^{1-k} \sum_{j=0}^{k-1} (-1)^j\, j!\, S_{k-1}^{(j)} \log^{-j-1}(z_0)\, (z - z_0)^k$$

06.36.06.0013.01

$$\operatorname{li}(z) \propto \operatorname{li}(z_0)\, (1 + O(z - z_0))$$

Expansions at $z == 1$

For the function itself

06.36.06.0001.02

$$\operatorname{li}(z) \propto \frac{1}{2}\left(\log(z - 1) - \log\!\left(\frac{1}{z - 1}\right)\right) + \gamma + \frac{z - 1}{2}\left(1 - \frac{z - 1}{12} + \frac{(z - 1)^2}{36} + \ldots\right) /;\; (z \to 1)$$

06.36.06.0014.01

$$\operatorname{li}(z) \propto \frac{1}{2}\left(\log(z - 1) - \log\!\left(\frac{1}{z - 1}\right)\right) + \gamma + \frac{z - 1}{2}\left(1 - \frac{z - 1}{12} + \frac{(z - 1)^2}{36} + O\!\left((z - 1)^3\right)\right)$$

06.36.06.0015.01

$$\operatorname{li}(z) == \frac{1}{2}\left(\log(z - 1) - \log\!\left(\frac{1}{z - 1}\right)\right) + \gamma + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + 1)!} \sum_{j=1}^{k+1} \frac{B_j\, S_k^{(j-1)}}{j} (1 - z)^{k+1}$$

Eric Weisstein

06.36.06.0016.01

$$\operatorname{li}(z) == \frac{1}{2}\left(\log(z-1) - \log\left(\frac{1}{z-1}\right)\right) + \gamma + \sum_{j=1}^{\infty} \frac{(-1)^j}{j\, j!} \sum_{h=0}^{\infty} p_{h,j}\, (z-1)^{h+j} + \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \sum_{k=0}^{\infty} q_{k,j}\, (z-1)^{j+k} \,/;$$

$$p_{h,j} == \frac{1}{h} \sum_{i=1}^{h} \frac{(-1)^i\, (-h+i+i\, j)}{i+1}\, p_{h-i,j} \bigwedge p_{0,j} == (-1)^j \bigwedge q_{k,j} == \frac{2}{k} \sum_{i=1}^{k} \frac{(-1)^i\, (j\, i+i-k)}{i+2}\, q_{k-i,j} \bigwedge q_{0,j} == \left(-\frac{1}{2}\right)^j$$

06.36.06.0004.02

$$\operatorname{li}(z) \propto \frac{1}{2}\left(\log(z-1) - \log\left(\frac{1}{z-1}\right)\right) + \gamma + \frac{z-1}{2}\, (1 + O(z-1))$$

Expansions on branch cuts

For the function itself

06.36.06.0017.01

$$\operatorname{li}(z) \propto \left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + 1\right) \operatorname{li}(x) + \operatorname{Ei}(\log(-x) - i\,\pi) \left(-\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor\right) + \frac{z-x}{\log(x)} - \frac{(z-x)^2}{x \log^2(x)} + \ldots \,/;\, (z \to x) \land x \in \mathbb{R} \land x < 0$$

06.36.06.0018.01

$$\operatorname{li}(z) = i\,\pi \left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor - \left\lfloor \frac{\arg(z-x)+\pi}{2\pi} \right\rfloor - \left\lfloor -\frac{\arg(z-x)}{2\pi} \right\rfloor\right) + \operatorname{li}(x) + \frac{z-x}{\log(x)} - \frac{(z-x)^2}{x \log^2(x)} + \ldots \,/;\, (z \to x) \land x \in \mathbb{R} \land 0 < x < 1$$

06.36.06.0019.01

$$\operatorname{li}(z) \propto \left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + 1\right) \operatorname{li}(x) + \operatorname{Ei}(\log(-x) - i\,\pi) \left(-\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor\right) + \frac{z-x}{\log(x)} - \frac{(z-x)^2}{x \log^2(x)} + O\left((z-x)^3\right) \,/;\, x \in \mathbb{R} \land x < 0$$

06.36.06.0020.01

$$\operatorname{li}(z) = i\,\pi \left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor - \left\lfloor \frac{\arg(z-x)+\pi}{2\pi} \right\rfloor - \left\lfloor -\frac{\arg(z-x)}{2\pi} \right\rfloor\right) + \operatorname{li}(x) + \frac{z-x}{\log(x)} - \frac{(z-x)^2}{x \log^2(x)} + O\left((z-x)^3\right) \,/;\, x \in \mathbb{R} \land 0 < x < 1$$

06.36.06.0021.01

$$\operatorname{li}(z) = -2\, i\,\pi \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor \sum_{k=1}^{\infty} \frac{(z-x)^k\, x^{1-k}}{k!} \sum_{j=0}^{k-1} \frac{(-1)^j\, j!\, S_{k-1}^{(j)}}{\left(\log^2(-x)+\pi^2\right)^{j+1}} \sum_{h=0}^{\left\lfloor \frac{j+1}{2} \right\rfloor} \binom{j+1}{2h+1} \log^{j-2h}(-x)\, (i\,\pi)^{2h} -$$
$$\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor \operatorname{Ei}(\log(-x) - i\,\pi) + \left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + 1\right) \operatorname{li}(x) + \sum_{k=1}^{\infty} \frac{(z-x)^k\, x^{1-k}}{k!} \sum_{j=0}^{k-1} (-1)^j\, j!\, S_{k-1}^{(j)} \log^{-j-1}(x) \,/;\, x \in \mathbb{R} \land x < 0$$

Pavlyk 0. (2006)

06.36.06.0022.01

$$\operatorname{li}(z) = i\,\pi \left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor - \left\lfloor \frac{\arg(z-x)+\pi}{2\pi} \right\rfloor - \left\lfloor -\frac{\arg(z-x)}{2\pi} \right\rfloor\right) + \operatorname{li}(x) + \sum_{k=1}^{\infty} x^{1-k} \sum_{j=0}^{k-1} (-1)^j\, j!\, S_{k-1}^{(j)} \log^{-j-1}(x)\, (z-x)^k \,/;$$
$$x \in \mathbb{R} \land 0 < x < 1$$

06.36.06.0023.01

$$\operatorname{li}(z) \propto \left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor + 1\right) \operatorname{li}(x) + \operatorname{Ei}(\log(-x) - i\,\pi) \left(-\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor\right) + O(z-x) \,/;\, x \in \mathbb{R} \land x < 0$$

06.36.06.0024.01

$$\operatorname{li}(z) = i\,\pi\left(\left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor - \left\lfloor \frac{\arg(z-x)+\pi}{2\pi} \right\rfloor - \left\lfloor -\frac{\arg(z-x)}{2\pi} \right\rfloor\right) + \operatorname{li}(x) + O(z-x) \;/;\; x \in \mathbb{R} \land 0 < x < 1$$

Expansions at log(*z*) == 0

For the function itself

06.36.06.0002.01

$$\operatorname{li}(z) = \frac{1}{2}\left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) + \gamma + \sum_{k=1}^{\infty} \frac{\log^k(z)}{k\,k!}$$

06.36.06.0003.01

$$\operatorname{li}(z) = \log(z)\, {}_2F_2(1, 1; 2, 2; \log(z)) + \frac{1}{2}\left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) + \gamma$$

Asymptotic series expansions

06.36.06.0025.01

$$\operatorname{li}(z) \propto i\,\pi\left(\left\lfloor \frac{\arg(-z)+\pi}{2\pi} \right\rfloor + 2\left\lfloor -\frac{\arg(-z)}{2\pi} \right\rfloor + 1\right) + \frac{z}{\log(z)}\left(1 + \frac{1}{\log(z)} + \frac{2}{\log^2(z)} + O\left(\frac{1}{\log^3(z)}\right)\right) /;\; (|z| \to \infty \lor z \to 0)$$

06.36.06.0026.01

$$\operatorname{li}(z) \propto i\,\pi\left(\left\lfloor \frac{\arg(-z)+\pi}{2\pi} \right\rfloor + 2\left\lfloor -\frac{\arg(-z)}{2\pi} \right\rfloor + 1\right) + z\sum_{k=0}^{\infty} k!\,\log^{-k-1}(z) \;/;\; (|z| \to \infty \lor z \to 0)$$

06.36.06.0005.01

$$\operatorname{li}(z) \propto \frac{z}{\log(z)}\, {}_2F_0\left(1, 1; ; \frac{1}{\log(z)}\right) + \frac{1}{2}\left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) - \log(-\log(z)) \;/;\; (|z| \to \infty \lor z \to 0)$$

06.36.06.0027.01

$$\operatorname{li}(z) \propto \frac{z}{\log(z)} \sum_{k=0}^{\infty} \frac{k!}{\log^k(z)} + \frac{1}{2}\left(-\log\left(\frac{1}{\log(z)}\right) + \log\left(-\frac{1}{\log(z)}\right) - \log(-\log(z)) + \log(\log(z))\right) /;\; (|z| \to \infty)$$

06.36.06.0006.01

$$\operatorname{li}(z) \propto \frac{1}{2}\left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) - \log(-\log(z)) + \frac{z}{\log(z)}\left(1 + O\left(\frac{1}{\log(z)}\right)\right) /;\; (|z| \to \infty \lor z \to 0)$$

06.36.06.0028.01

$$\operatorname{li}(z) \propto i\,\pi\left(\left\lfloor \frac{\arg(-z)+\pi}{2\pi} \right\rfloor + 2\left\lfloor -\frac{\arg(-z)}{2\pi} \right\rfloor + 1\right) + \frac{z}{\log(z)}\left(1 + O\left(\frac{1}{\log(z)}\right)\right) /;\; (|z| \to \infty \lor z \to 0)$$

Residue representations

06.36.06.0007.01

$$\operatorname{li}(z) = \gamma - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma(-s)^2\,(-\log(z))^{-s}}{\Gamma(1-s)^2}\,\Gamma(s+1)\right)(-j-1)$$

06.36.06.0008.02

$$\operatorname{li}(z) = -\log(-\log(z)) - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - \operatorname{res}_s\left((-\log(z))^{-s}\,\frac{\Gamma(s)}{s}\right)(0) - \sum_{j=1}^{\infty} \operatorname{res}_s\left(\frac{(-\log(z))^{-s}}{s}\,\Gamma(s)\right)(-j)$$

Other series representations

06.36.06.0009.01

$$\mathrm{li}(x) = -x \sum_{k=0}^{\infty} \frac{L_k(-\log(x))}{k+1} \;/;\; 0 < x < 1$$

Integral representations

On the real axis

Of the direct function

06.36.07.0001.01

$$\mathrm{li}(z) = \int_0^z \frac{1}{\log(t)} \, dt$$

Contour integral representations

06.36.07.0002.01

$$\mathrm{li}(z) = \gamma - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s+1)\,\Gamma(-s)^2}{\Gamma(1-s)^2} (-\log(z))^{-s} \, ds$$

06.36.07.0003.01

$$\mathrm{Ei}(z) = \gamma - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+1)\,\Gamma(-s)^2}{\Gamma(1-s)^2} (-\log(z))^{-s} \, ds \;/;\; -1 < \gamma < 0 \bigwedge |\arg(-\log(z))| < \frac{\pi}{2}$$

06.36.07.0004.01

$$\mathrm{Ei}(z) = -\log(-\log(z)) - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)^2}{\Gamma(s+1)} (-\log(z))^{-s} \, ds$$

06.36.07.0005.01

$$\mathrm{li}(z) = -\log(-\log(z)) - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)^2}{\Gamma(s+1)} (-\log(z))^{-s} \, ds \;/;\; 0 < \gamma \bigwedge |\arg(-\log(z))| < \frac{\pi}{2}$$

Differential equations

Ordinary nonlinear differential equations

06.36.13.0001.01

$$z\, w''(z) + w'(z)^2 = 0 \;/;\; w(z) = \mathrm{li}(z)$$

Complex characteristics

Real part

06.36.19.0001.01

$$\operatorname{Re}(\operatorname{li}(x + i\, y)) = \frac{1}{2} \log\left(\tan^{-1}(x, y)^2 + \frac{1}{4} \log^2(x^2 + y^2)\right) + \sum_{k=1}^{\infty} \frac{2^{-k}}{k\, k!} \cos\left(k \tan^{-1}\left(\frac{2 \tan^{-1}(x, y)}{\log(x^2 + y^2)}\right)\right) \left(\frac{4 \tan^{-1}(x, y)^2}{\log^2(x^2 + y^2)} + 1\right)^{k/2} \log^k(x^2 + y^2) + \gamma$$

06.36.19.0002.01

$$\operatorname{Re}(\operatorname{li}(x + i\, y)) = \frac{1}{2} \log\left(\tan^{-1}(x, y)^2 + \frac{1}{4} \log^2(x^2 + y^2)\right) + \sum_{k=1}^{\infty} \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{(-1)^j\, 2^{2 j - k}}{k\, k!} \binom{k}{2 j} \tan^{-1}(x, y)^{2 j} \log^{k - 2 j}(x^2 + y^2) + \gamma$$

06.36.19.0003.01

$$\operatorname{Re}(\operatorname{li}(x + i\, y)) = \frac{1}{2} \left(\operatorname{li}\left(x + x \sqrt{-\frac{y^2}{x^2}}\right) + \operatorname{li}\left(x - x \sqrt{-\frac{y^2}{x^2}}\right)\right)$$

Imaginary part

06.36.19.0004.01

$$\operatorname{Im}(\operatorname{li}(x + i\, y)) = \frac{1}{2} \left(\tan^{-1}\left(\frac{1}{2} \log(x^2 + y^2), \tan^{-1}(x, y)\right) - \tan^{-1}\left(\frac{1}{2} \log(x^2 + y^2), -\tan^{-1}(x, y)\right)\right) + \sum_{k=1}^{\infty} \frac{2^{-k}}{k\, k!} \left(\frac{4 \tan^{-1}(x, y)^2}{\log^2(x^2 + y^2)} + 1\right)^{k/2} \log^k(x^2 + y^2) \sin\left(k \tan^{-1}\left(\frac{2 \tan^{-1}(x, y)}{\log(x^2 + y^2)}\right)\right)$$

06.36.19.0005.01

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-1)^j\, 2^{2 j - k + 1} \tan^{-1}(x, y)^{2 j + 1} \log^{k - 2 j - 1}(x^2 + y^2)}{k\, (2 j + 1)!\, (k - 2 j - 1)!} + \frac{1}{2} \left(\tan^{-1}\left(\frac{1}{2} \log(x^2 + y^2), \tan^{-1}(x, y)\right) - \tan^{-1}\left(\frac{1}{2} \log(x^2 + y^2), -\tan^{-1}(x, y)\right)\right)$$

06.36.19.0006.01

$$\operatorname{Im}(\operatorname{li}(x + i\, y)) = \frac{x}{2 y} \sqrt{-\frac{y^2}{x^2}} \left(\operatorname{li}\left(x - x \sqrt{-\frac{y^2}{x^2}}\right) - \operatorname{li}\left(x + x \sqrt{-\frac{y^2}{x^2}}\right)\right)$$

Absolute value

06.36.19.0007.01

$$|\operatorname{li}(x + i\, y)| = \sqrt{\operatorname{li}\left(x - x \sqrt{-\frac{y^2}{x^2}}\right) \operatorname{li}\left(x + x \sqrt{-\frac{y^2}{x^2}}\right)}$$

Argument

06.36.19.0008.01

$$\arg(\operatorname{li}(x + i\, y)) = \tan^{-1}\left(\frac{1}{2} \left(\operatorname{li}\left(x + x \sqrt{-\frac{y^2}{x^2}}\right) + \operatorname{li}\left(x - x \sqrt{-\frac{y^2}{x^2}}\right)\right), \frac{x}{2 y} \sqrt{-\frac{y^2}{x^2}} \left(\operatorname{li}\left(x - x \sqrt{-\frac{y^2}{x^2}}\right) - \operatorname{li}\left(x + x \sqrt{-\frac{y^2}{x^2}}\right)\right)\right)$$

Conjugate value

06.36.19.0009.01

$$\overline{\text{li}(x+i\,y)} = \frac{1}{2}\left(\text{li}\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)+\text{li}\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)\right)-\frac{i\,x}{2\,y}\sqrt{-\frac{y^2}{x^2}}\left(\text{li}\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)-\text{li}\left(x+x\sqrt{-\frac{y^2}{x^2}}\right)\right)$$

Signum value

06.36.19.0010.01

$$\text{sgn}(\text{li}(x+i\,y)) = \left(\frac{i}{y}\sqrt{-\frac{y^2}{x^2}}\;x\left(\text{li}\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)-\text{li}\left(\sqrt{-\frac{y^2}{x^2}}\;x+x\right)\right)+\text{li}\left(\sqrt{-\frac{y^2}{x^2}}\;x+x\right)+\text{li}\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)\right)\Big/ \left(2\sqrt{\text{li}\left(x-x\sqrt{-\frac{y^2}{x^2}}\right)\text{li}\left(\sqrt{-\frac{y^2}{x^2}}\;x+x\right)}\right)$$

Differentiation

Low-order differentiation

06.36.20.0001.01

$$\frac{\partial\,\text{li}(z)}{\partial z} = \frac{1}{\log(z)}$$

06.36.20.0002.01

$$\frac{\partial^2\,\text{li}(z)}{\partial z^2} = -\frac{1}{z\log^2(z)}$$

Symbolic differentiation

06.36.20.0006.01

$$\frac{\partial^n\,\text{li}(z)}{\partial z^n} = \delta_n\,\text{li}(z)+z^{1-n}\sum_{k=0}^{n-1}(-1)^k\,k!\,S_{n-1}^{(k)}\log^{-k-1}(z)\;/;\;n\in\mathbb{N}$$

06.36.20.0003.01

$$\frac{\partial^n\,\text{li}(z)}{\partial z^n} = \delta_n\,\text{li}(z)+\text{Boole}\left(n\neq 0,\; z^{1-n}\sum_{k=0}^{n-1}(-1)^k\,k!\,S_{n-1}^{(k)}\log^{-k-1}(z)\right)/;\;n\in\mathbb{N}$$

06.36.20.0004.01

$$\frac{\partial^n\,\text{li}(z)}{\partial z^n} = z^{1-n}\sum_{k=0}^{n-1}(-1)^k\,k!\,S_{n-1}^{(k)}\log^{-k-1}(z)\;/;\;n\in\mathbb{N}^+$$

Fractional integro-differentiation

06.36.20.0005.01

$$\frac{\partial^{\alpha}\operatorname{li}(z)}{\partial z^{\alpha}} = \frac{\theta(\operatorname{Re}(1-\alpha))\, z^{-\alpha}}{\Gamma(1-\alpha)} \sum_{k=0}^{\infty} \frac{(\alpha)_k \operatorname{Ei}((k+1)\log(z))\, z^{-k}}{k!} + \frac{\theta(-\operatorname{Re}(1-\alpha))\, z^{-\alpha}}{\Gamma(\lfloor\alpha\rfloor-\alpha+1)\log(z)} \sum_{k=0}^{\infty} \frac{(\alpha-\lfloor\alpha\rfloor)_k\, z^{-k}}{k!} \sum_{m=0}^{\lfloor\alpha\rfloor+1} \binom{\lfloor\alpha\rfloor}{\lfloor\alpha\rfloor-m} (m-k-\alpha+1)_{\lfloor\alpha\rfloor-m} \sum_{p=0}^{m-1} \frac{1}{p!} \left(\left((-1)^p\, p! + (k+1)\log(z)\, {}_2\tilde{F}_2(1,1;2,1-p;(k+1)\log(z))\right) \left((-1)^{m+p+1}(m-1)! + \sum_{h=0}^{m} \log^{1-h}(z)\, S_m^{(h)}\, p! \sum_{j=0}^{p-1} \frac{(-1)^j}{j!\,(p-h-j+1)!} \right) \right)$$

Integration

Indefinite integration

Involving only one direct function

06.36.21.0001.01

$$\int \operatorname{li}(a z)\, dz = z \operatorname{li}(a z) - \frac{\operatorname{Ei}(2\log(a z))}{a}$$

06.36.21.0002.01

$$\int \operatorname{li}(z)\, dz = z \operatorname{li}(z) - \operatorname{Ei}(2\log(z))$$

Involving one direct function and elementary functions

Involving power function

Involving power

Linear argument

06.36.21.0003.01

$$\int z^{\alpha-1} \operatorname{li}(a z)\, dz = \frac{z^{\alpha}(a z)^{-\alpha}\left((a z)^{\alpha}\operatorname{li}(a z) - \operatorname{Ei}((\alpha+1)\log(a z))\right)}{\alpha}$$

06.36.21.0004.01

$$\int z^{\alpha-1} \operatorname{li}(z)\, dz = \frac{z^{\alpha}\operatorname{li}(z) - \operatorname{Ei}((\alpha+1)\log(z))}{\alpha}$$

Power arguments

06.36.21.0005.01

$$\int z^{\alpha-1} \operatorname{li}(a z^r)\, dz = \frac{z^{\alpha}}{\alpha}\left(\operatorname{li}(a z^r) - (a z^r)^{-\frac{\alpha}{r}} \operatorname{Ei}\left(\frac{(r+\alpha)\log(a z^r)}{r}\right)\right)$$

Involving logarithm

Involving log

06.36.21.0006.01

$$\int \log(b\, z)\, \text{li}(a\, z)\, dz = \frac{1}{2\, a} (2\, \text{Ei}(2 \log(a\, z)) (\log(a\, z) - \log(b\, z) + 1) - a\, z\, (a\, z - 2\, (\log(b\, z) - 1)\, \text{li}(a\, z)))$$

Involving logarithm and a power function

Involving log and power

06.36.21.0007.01

$$\int z^{\alpha-1} \log(b\, z)\, \text{li}(a\, z)\, dz = \frac{1}{\alpha^2\, (\alpha+1)} (z^\alpha\, (a\, z)^{-\alpha}\, (((\alpha+1)\, (\alpha \log(b\, z) - 1)\, \text{li}(a\, z) - a\, z\, \alpha)\, (a\, z)^\alpha + (\alpha+1)\, \text{Ei}((\alpha+1) \log(a\, z))\, (\alpha \log(a\, z) - \alpha \log(b\, z) + 1)))$$

06.36.21.0008.01

$$\int z \log(b\, z)\, \text{li}(a\, z)\, dz = \frac{1}{12\, a^2} \left(a^2\, ((6 \log(b\, z) - 3)\, \text{li}(a\, z) - 2\, a\, z)\, z^2 + \text{Ei}(3 \log(a\, z))\, (6 \log(a\, z) - 6 \log(b\, z) + 3)\right)$$

06.36.21.0009.01

$$\int z^2 \log(b\, z)\, \text{li}(a\, z)\, dz = \frac{1}{36\, a^3} \left(a^3\, (4\, (3 \log(b\, z) - 1)\, \text{li}(a\, z) - 3\, a\, z)\, z^3 + 4\, \text{Ei}(4 \log(a\, z))\, (3 \log(a\, z) - 3 \log(b\, z) + 1)\right)$$

06.36.21.0010.01

$$\int z^3 \log(b\, z)\, \text{li}(a\, z)\, dz = \frac{1}{80\, a^4} \left(a^4\, (5\, (4 \log(b\, z) - 1)\, \text{li}(a\, z) - 4\, a\, z)\, z^4 + 5\, \text{Ei}(5 \log(a\, z))\, (4 \log(a\, z) - 4 \log(b\, z) + 1)\right)$$

06.36.21.0011.01

$$\int z^{\alpha-1} \log^n(a\, z)\, \text{li}(a\, z)\, dz = \frac{(-\alpha)^{-n}\, z^\alpha\, (a\, z)^{-\alpha}\, \Gamma(n+1, -\alpha \log(a\, z))\, \text{li}(a\, z)}{\alpha} - (-\alpha)^{-n-1}\, z^\alpha\, (a\, z)^{-\alpha}\, n!\, \sum_{k=0}^{n} \frac{\Gamma(k, -(\alpha+1) \log(a\, z))\, \alpha^k\, (\alpha+1)^{-k}}{k!} \,/;\, n \in \mathbb{N}$$

Definite integration

Involving the direct function

06.36.21.0012.01

$$\int_1^\infty t^{\alpha-1}\, \text{li}(t)\, dt = \frac{\log(-\alpha-1)}{\alpha} \,/;\, \text{Re}(\alpha) < -1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_pF_q$

06.36.26.0001.01

$$\mathrm{li}(z) = \log(z)\, {}_2F_2(1, 1; 2, 2; \log(z)) + \frac{1}{2}\left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) + \gamma$$

Through Meijer *G*

Classical cases for the direct function itself

06.36.26.0002.01

$$\mathrm{li}(z) = \gamma - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - G_{2,3}^{1,2}\left(-\log(z) \,\middle|\, \begin{matrix} 1, 1 \\ 1, 0, 0 \end{matrix}\right)$$

06.36.26.0003.01

$$\mathrm{li}(z) = -\log(-\log(z)) - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - G_{1,2}^{2,0}\left(-\log(z) \,\middle|\, \begin{matrix} 1 \\ 0, 0 \end{matrix}\right)$$

Through other functions

06.36.26.0004.01

$$\mathrm{li}(z) = -\Gamma(0, -\log(z)) + \frac{1}{2}\left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) - \log(-\log(z))$$

06.36.26.0005.01

$$\mathrm{li}(e^z) = -\Gamma(0, -z) + \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right) - \log(-z) \;/;\; -\pi < \mathrm{Im}(z) \le \pi$$

06.36.26.0006.01

$$\mathrm{li}(z) = -E_1(-\log(z)) + \frac{1}{2}\left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) - \log(-\log(z))$$

06.36.26.0007.01

$$\mathrm{li}(e^z) = -E_1(-z) + \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right) - \log(-z) \;/;\; -\pi < \mathrm{Im}(z) \le \pi$$

Representations through equivalent functions

With related functions

06.36.27.0001.01

$$\mathrm{li}(z) = \mathrm{Ei}(\log(z))$$

06.36.27.0002.01

$$\mathrm{li}(e^z) = \mathrm{Ei}(z) \;/;\; -\pi < \mathrm{Im}(z) \le \pi$$

06.36.27.0003.02

$$\mathrm{li}(z) = \mathrm{Ci}(i \log(z)) - i\, \mathrm{Si}(i \log(z)) - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - \log(i \log(z))$$

06.36.27.0004.01

$$\mathrm{li}(z) = \mathrm{Chi}(\log(z)) + \mathrm{Shi}(\log(z)) - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) + \log(\log(z))\right)$$

Theorems

The prime number theorem

$$\pi(x) \underset{x\to\infty}{\sim} \mathrm{li}(x) + \sum_{k=2}^{n} \frac{\mu(k)}{k}\, \mathrm{li}\left(\sqrt[k]{x}\right).$$

History

– L. Euler (1768)
– L. Mascheroni (1790)
– T. Caluso (1805)
– J. von Soldner (1809) introduced the notation li
– C. A. Bretschneider (1837)
– A.de Morgan (1842)
– H. Amstein (1895) introduced branch cut for complex argument

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