ModularLambda

Notations

Traditional name

Modular lambda function

Traditional notation

$\lambda(z)$

Mathematica StandardForm notation

`ModularLambda[z]`

Primary definition

09.51.02.0001.01

$$\lambda(z) == 16\, e^{i \pi z} \prod_{k=1}^{\infty} \left(\frac{1 + e^{2 k \pi i z}}{1 + e^{(2k-1) \pi i z}} \right)^8 \text{ /; } \operatorname{Im}(z) > 0$$

Specific values

Specialized values

09.51.03.0001.01

$$\lambda(i + 2m) == \frac{1}{2} \text{ /; } m \in \mathbb{Z}$$

Values at fixed points

09.51.03.0002.01

$$\lambda(i) == \frac{1}{2}$$

Values at infinities

09.51.03.0003.01

$$\lambda(i \infty) == 0$$

General characteristics

Domain and analyticity

$\lambda(z)$ is an analytical function of z which is defined over the upper half of the complex z-plane.

09.51.04.0001.01

$z \longrightarrow \lambda(z) :: \mathbb{C} \longrightarrow \mathbb{C}$

Symmetries and periodicities

Periodicity

$\lambda(z)$ is a periodic function with period 2.

09.51.04.0002.01

$\lambda(z + 2\,m) = \lambda(z) /; m \in \mathbb{Z}$

Poles and essential singularities

On the boundary of analyticity the function $\lambda(z)$ has a dense set of poles.

09.51.04.0003.01

$\mathcal{S}ing_z(\lambda(z)) = \{\} /; \mathrm{Im}(z) > 0$

Branch points

The function $\lambda(z)$ does not have branch points.

09.51.04.0004.01

$\mathcal{BP}_z(\lambda(z)) = \{\}$

Branch cuts

The function $\lambda(z)$ does not have branch cuts.

09.51.04.0005.01

$\mathcal{BC}_z(\lambda(z)) = \{\}$

Natural boundary of analyticity

The real axis $\mathrm{Im}(z) = 0$ is the natural boundary of the region of analyticity.

09.51.04.0006.01

$\mathcal{AB}_z(\lambda(z)) = \{(-\infty, \infty)\}$

Series representations

Generalized power series

Expansions at generic point $z = z_0$

09.51.06.0001.01

$$\lambda(z) \propto \lambda(z_0) - \frac{4\,i\,(\lambda(z_0)-1)\,\lambda(z_0)\;K(\lambda(z_0))^2}{\pi}(z-z_0) + \frac{8\,(\lambda(z_0)-1)\,\lambda(z_0)\,(E(\lambda(z_0))-K(\lambda(z_0))\,\lambda(z_0))\;K(\lambda(z_0))^3}{\pi^2}(z-z_0)^2 + \frac{1}{3\pi^3}16\,i\,(\lambda(z_0)-1)\,\lambda(z_0)\left(3\,E(\lambda(z_0))^2 - K(\lambda(z_0))^2 + 2\,K(\lambda(z_0))^2\,\lambda(z_0)^2 + K(\lambda(z_0))\,(K(\lambda(z_0)) - 6\,E(\lambda(z_0)))\,\lambda(z_0)\right) K(\lambda(z_0))^4\,(z-z_0)^3 + \frac{1}{3\pi^4}16\,(\lambda(z_0)-1)\,\lambda(z_0)\left(2\,K(\lambda(z_0))^3\,\lambda(z_0)^3 + 3\,K(\lambda(z_0))^2\,(K(\lambda(z_0)) - 4\,E(\lambda(z_0)))\,\lambda(z_0)^2 - K(\lambda(z_0))\left(-18\,E(\lambda(z_0))^2 + 6\,K(\lambda(z_0))\,E(\lambda(z_0)) + K(\lambda(z_0))^2\right)\lambda(z_0) - 2\left(3\,E(\lambda(z_0))^3 - 3\,K(\lambda(z_0))^2\,E(\lambda(z_0)) + K(\lambda(z_0))^3\right)\right) K(\lambda(z_0))^5\,(z-z_0)^4 - \frac{1}{15\pi^5}64\,i\,(\lambda(z_0)-1)\,\lambda(z_0)\left(15\,E(\lambda(z_0))^4 - 30\,K(\lambda(z_0))^2\,E(\lambda(z_0))^2 + 20\,K(\lambda(z_0))^3\,E(\lambda(z_0)) - 3\,K(\lambda(z_0))^4 + 2\,K(\lambda(z_0))^4\,\lambda(z_0)^4 + \left(6\,K(\lambda(z_0))^4 - 20\,E(\lambda(z_0))\,K(\lambda(z_0))^3\right)\lambda(z_0)^3 + K(\lambda(z_0))^2\left(60\,E(\lambda(z_0))^2 - 30\,K(\lambda(z_0))\,E(\lambda(z_0)) + K(\lambda(z_0))^2\right)\lambda(z_0)^2 + 2\,K(\lambda(z_0))\left(-30\,E(\lambda(z_0))^3 + 15\,K(\lambda(z_0))\,E(\lambda(z_0))^2 + 5\,K(\lambda(z_0))^2\,E(\lambda(z_0)) - 2\,K(\lambda(z_0))^3\right)\lambda(z_0)\right) K(\lambda(z_0))^6\,(z-z_0)^5 + \ldots \;/;\; (z \to z_0)$$

09.51.06.0002.01

$$\lambda(z) \propto \lambda(z_0)\,(1 + O(z-z_0))$$

Expansions at $z == 0$

09.51.06.0003.01

$$\lambda(z) \propto 1 - 16\,e^{-\frac{i\pi}{z}} + 128\,e^{-\frac{2i\pi}{z}} - 704\,e^{-\frac{3i\pi}{z}} + 3072\,e^{-\frac{4i\pi}{z}} - 11\,488\,e^{-\frac{5i\pi}{z}} + 38\,400\,e^{-\frac{6i\pi}{z}} - 117\,632\,e^{-\frac{7i\pi}{z}} + 335\,872\,e^{-\frac{8i\pi}{z}} - 904\,784\,e^{-\frac{9i\pi}{z}} + 2\,320\,128\,e^{-\frac{10i\pi}{z}} - 5\,702\,208\,e^{-\frac{11i\pi}{z}} + 13\,504\,512\,e^{-\frac{12i\pi}{z}} - 30\,952\,544\,e^{-\frac{13i\pi}{z}} + 68\,901\,888\,e^{-\frac{14i\pi}{z}} - 149\,403\,264\,e^{-\frac{15i\pi}{z}} + 316\,342\,272\,e^{-\frac{16i\pi}{z}} - 655\,445\,792\,e^{-\frac{17i\pi}{z}} + 1\,331\,327\,616\,e^{-\frac{18i\pi}{z}} - 2\,655\,115\,712\,e^{-\frac{19i\pi}{z}} + 5\,206\,288\,384\,e^{-\frac{20i\pi}{z}} + O\left(e^{-\frac{21i\pi}{z}}\right) \;/;\; \mathrm{Im}(z) > 0 \land (z \to 0)$$

09.51.06.0004.01

$$\lambda(z) = \frac{\left(1 + 2\sum_{k=1}^{\infty}(-1)^k\,e^{-\frac{i k^2 \pi}{z}}\right)^4}{\left(2\sum_{k=1}^{\infty} e^{-\frac{i k^2 \pi}{z}} + 1\right)^4}$$

09.51.06.0005.01

$$\lambda(z) \propto 1 - 16\,e^{-\frac{i\pi}{z}} + O\left(e^{-\frac{2i\pi}{z}}\right) \;/;\; \mathrm{Im}(z) > 0 \land (z \to 0)$$

Expansions at $z == \infty$

09.51.06.0006.01

$$\lambda(z) \propto 16\,e^{i\pi z} - 128\,e^{2i\pi z} + 704\,e^{3i\pi z} - 3072\,e^{4i\pi z} + 11\,488\,e^{5i\pi z} - 38\,400\,e^{6i\pi z} + 117\,632\,e^{7i\pi z} - 335\,872\,e^{8i\pi z} + 904\,784\,e^{9i\pi z} - 2\,320\,128\,e^{10i\pi z} + 5\,702\,208\,e^{11i\pi z} - 13\,504\,512\,e^{12i\pi z} + 30\,952\,544\,e^{13i\pi z} - 68\,901\,888\,e^{14i\pi z} + 149\,403\,264\,e^{15i\pi z} - 316\,342\,272\,e^{16i\pi z} + 655\,445\,792\,e^{17i\pi z} - 1\,331\,327\,616\,e^{18i\pi z} + 2\,655\,115\,712\,e^{19i\pi z} - 5\,206\,288\,384\,e^{20i\pi z} + O\left(e^{21i\pi z}\right) \;/;\; \mathrm{Im}(z) > 0 \land (|z| \to \infty)$$

09.51.06.0007.01

$$\lambda(z) = 1 - \frac{\left(1 + 2\sum_{k=1}^{\infty}(-1)^k\,e^{k^2 i\pi z}\right)^4}{\left(1 + 2\sum_{k=1}^{\infty} e^{k^2 i\pi z}\right)^4}$$

09.51.06.0008.01

$$\lambda(z) \propto 16\, e^{i \pi z} + O\left(e^{2 i \pi z}\right) /; \operatorname{Im}(z) > 0 \land (|z| \to \infty)$$

Product representations

09.51.08.0001.01

$$\lambda(z) == 16\, e^{i \pi z} \prod_{k=1}^{\infty} \left(\frac{1 + e^{2 k \pi i z}}{1 + e^{(2k-1) \pi i z}} \right)^8 /; \operatorname{Im}(z) > 0$$

Differential equations

Ordinary nonlinear differential equations

09.51.13.0001.01

$$486\, (w(z) - 1)^4\, ((w(z) - 1)\, w(z) + 1)\, w'(z)\, w^{(3)}(z)\, w(z)^4 + 12\, (w(z) - 2)\, (w(z) - 1)\, (2\, w(z) - 1)\, (w(z) + 1)$$
$$-729\, (w(z) - 1)^4\, ((w(z) - 1)\, w(z) + 1)\, w''(z)^2\, w(z)^4 +$$
$$((w(z) - 1)\, (((7\, (w(z) - 2)\, w(z) + 1)\, w(z) + 6)\, w(z) + 21)\, w(z) + 7)\, w'(z)^2\, w''(z)\, w(z) +$$
$$(112 - (w(z) - 1)\, w(z)\, ((w(z) - 1)\, w(z)\, ((w(z) - 1)\, w(z)\, ((224\, (w(z) - 1)\, w(z) - 827)\, (w(z) - 1)\, w(z) + 410) - 1099) - 728))$$
$$w'(z)^4 == 0 /; w(z) == \lambda(z)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.51.16.0001.01

$$\lambda(z + 1) == \frac{\lambda(z)}{\lambda(z) - 1}$$

09.51.16.0002.01

$$\lambda(z + 2) == \lambda(z)$$

09.51.16.0003.01

$$\lambda\left(-\frac{1}{z}\right) == 1 - \lambda(z)$$

09.51.16.0004.01

$$\lambda\left(\frac{z}{1 - 2 z}\right) == \lambda(z)$$

Identities

Functional identities

09.51.17.0001.01

$$\lambda(z) == \lambda(z + 2)$$

09.51.17.0002.01

$$\lambda(z) = \lambda\left(\frac{z}{1-2z}\right)$$

09.51.17.0003.01

$$\lambda(z) = 1 - \lambda\left(-\frac{1}{z}\right)$$

Differentiation

Low-order differentiation

09.51.20.0001.02

$$\frac{\partial \lambda(z)}{\partial z} = -\frac{4\,i\,K(\lambda(z))^2\,(\lambda(z)-1)\,\lambda(z)}{\pi}$$

09.51.20.0002.02

$$\frac{\partial^2 \lambda(z)}{\partial z^2} = -\frac{16\,K(\lambda(z))^3\,(\lambda(z)-1)\,\lambda(z)\,(K(\lambda(z))\,\lambda(z)-E(\lambda(z)))}{\pi^2}$$

09.51.20.0003.01

$$\frac{\partial^3 \lambda(z)}{\partial z^3} = \frac{32\,i\,K(\lambda(z))^4\,(\lambda(z)-1)\,\lambda(z)}{\pi^3}\left(3\,E(\lambda(z))^2 - 6\,K(\lambda(z))\,\lambda(z)\,E(\lambda(z)) + K(\lambda(z))^2\,(\lambda(z)+1)\,(2\,\lambda(z)-1)\right)$$

09.51.20.0004.01

$$\frac{\partial^4 \lambda(z)}{\partial z^4} = \frac{128\,K\,(\lambda(z))^5\,(\lambda(z)-1)\,\lambda(z)}{\pi^4} \left(-6\,E(\lambda(z))^3 + 18\,K(\lambda(z))\,\lambda(z)\,E(\lambda(z))^2 - 6\,K(\lambda(z))^2\,(\lambda(z)+1)\,(2\,\lambda(z)-1)\,E(\lambda(z)) + K(\lambda(z))^3\,(\lambda(z)+1)\left(2\,\lambda(z)^2+\lambda(z)-2\right)\right)$$

09.51.20.0005.01

$$\frac{\partial^5 \lambda(z)}{\partial z^5} = -\frac{512\,i\,K(\lambda(z))^6\,(\lambda(z)-1)\,\lambda(z)}{\pi^5}\left(15\,E(\lambda(z))^4 - 60\,K(\lambda(z))\,\lambda(z)\,E(\lambda(z))^3 + 30\,K(\lambda(z))^2\,(\lambda(z)+1)\,(2\,\lambda(z)-1)\,E(\lambda(z))^2 - 10\,K(\lambda(z))^3\,(\lambda(z)+1)\left(2\,\lambda(z)^2+\lambda(z)-2\right)E(\lambda(z)) + K(\lambda(z))^4\left(2\,\lambda(z)^4+6\,\lambda(z)^3+\lambda(z)^2-4\,\lambda(z)-3\right)\right)$$

Operations

Limit operation

09.51.25.0001.01

$$\lim_{\epsilon\to+0} \lambda(i\,\epsilon) = 1$$

Representations through equivalent functions

With related functions

Involving Weierstrass functions

09.51.27.0001.01

$$\lambda(z) = \frac{e_2 - e_3}{e_1 - e_3} /; z = \frac{\omega_3}{\omega_1} \bigwedge \{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_2(g_2, g_3)\} \bigwedge \omega_2 = -\omega_1 - \omega_3 \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

Involving theta functions

09.51.27.0002.01

$$\lambda(z) = \frac{\vartheta_2(0, q)^4}{\vartheta_3(0, q)^4} /; q = e^{i \pi z} \bigwedge \mathrm{Im}(z) > 0$$

09.51.27.0003.01

$$\lambda(z+1) = -\frac{\vartheta_2(0, q)^4}{\vartheta_4(0, q)^4} /; q = e^{i \pi z} \bigwedge \mathrm{Im}(z) > 0$$

09.51.27.0004.01

$$\lambda\left(-\frac{1}{z}\right) = \frac{\vartheta_4(0, q)^4}{\vartheta_3(0, q)^4} /; q = e^{i \pi z} \bigwedge \mathrm{Im}(z) > 0$$

Involving other related functions

09.51.27.0005.01

$$\lambda(z) = q^{-1}\left(e^{i \pi z}\right) /; \mathrm{Im}(z) > 0$$

History

– N .H. Abel
– C. G. J. Jacobi
– C. Hermite

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