Pochhammer

Notations

Traditional name

Pochhammer symbol

Traditional notation

$(a)_n$

Mathematica StandardForm notation

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Pochhammer[a, n]
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Primary definition

06.10.02.0001.01

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \text{ /; } (\neg\, (-a \in \mathbb{Z} \land -a \geq 0 \land n \in \mathbb{Z} \land n \leq -a))$$

06.10.02.0002.01

$$(a)_n = \prod_{k=0}^{n-1} (a+k) \text{ /; } n \in \mathbb{N}^+$$

For $\alpha = a$, $\nu = n$ integers with $a \leq 0$, $n \leq -a$, the Pochhammer symbol $(\alpha)_\nu$ can not be uniquely defined by a limiting procedure based on the above definition because the two variables α, ν can approach these integers a, n with $a \leq 0$, $n \leq -a$ at different speeds. For such integers with $a \leq 0$, $n \leq -a$ we define:

06.10.02.0003.01

$$(a)_n = \frac{(-1)^n \, (-a)!}{(-a-n)!} \text{ /; } -a \in \mathbb{N} \land n \in \mathbb{Z} \land n \leq -a$$

Specific values

Specialized values

For fixed *a*

06.10.03.0001.01

$$(a)_n = \prod_{k=0}^{n-1} (a+k) \text{ /; } n \in \mathbb{N}^+$$

06.10.03.0002.01

$$(a)_0 == 1$$

06.10.03.0003.01

$$(a)_1 == a$$

06.10.03.0004.01

$$(a)_2 == a\,(a+1)$$

06.10.03.0005.01

$$(a)_3 == a\,(a+1)\,(a+2)$$

06.10.03.0006.01

$$(a)_4 == a\,(a+1)\,(a+2)\,(a+3)$$

06.10.03.0007.01

$$(a)_5 == a\,(a+1)\,(a+2)\,(a+3)\,(a+4)$$

06.10.03.0008.01

$$(a)_{-n} == \prod_{k=1}^{n} \frac{1}{a-k} \;/;\; n \in \mathbb{N}^+$$

06.10.03.0009.01

$$(a)_{-5} == \frac{1}{(a-1)\,(a-2)\,(a-3)\,(a-4)\,(a-5)}$$

06.10.03.0010.01

$$(a)_{-4} == \frac{1}{(a-1)\,(a-2)\,(a-3)\,(a-4)}$$

06.10.03.0011.01

$$(a)_{-3} == \frac{1}{(a-1)\,(a-2)\,(a-3)}$$

06.10.03.0012.01

$$(a)_{-2} == \frac{1}{(a-1)\,(a-2)}$$

06.10.03.0013.01

$$(a)_{-1} == \frac{1}{a-1}$$

For fixed *n*

06.10.03.0014.01

$$(0)_{-n} == \frac{(-1)^n}{n!} \;/;\; n \in \mathbb{N}^+$$

06.10.03.0015.01

$$(0)_n == 0 \;/;\; n \in \mathbb{N}^+$$

06.10.03.0016.01

$$\left(-\frac{1}{2}\right)_n == -\frac{(2\,n-2)!}{2^{2\,n-1}\,(n-1)!}$$

06.10.03.0017.01

$$\left(\frac{1}{2}\right)_n = \frac{(2n-1)!}{2^{2n-1}(n-1)!}$$

06.10.03.0018.01

$$(1)_n = n!$$

06.10.03.0019.01

$$\left(\frac{3}{2}\right)_n = \frac{(2n+1)!}{4^n n!}$$

Values at fixed points

06.10.03.0020.01

$$(0)_0 = 1$$

General characteristics

Domain and analyticity

$(a)_n$ is an analytical function of a and n which is defined over $\mathbb{C}^2$.

06.10.04.0001.01

$$(a * n) \longrightarrow (a)_n :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.10.04.0002.01

$$(\bar{a})_{\bar{n}} = \overline{(a)_n}$$

Periodicity

No periodicity

Poles and essential singularities

With respect to *n*

For fixed a, the function $(a)_n$ has an infinite set of singular points:

a) $n = -a - k$ /; $k \in \mathbb{N}$ are the simple poles with residues $\frac{(-1)^k}{k!\,\Gamma(a)}$;

b) $n = \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

06.10.04.0003.01

$$\mathcal{Sing}_n((a)_n) = \{\{\{-a-k, 1\} /; k \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\}$$

06.10.04.0004.01

$$\operatorname{res}_n((a)_n)(-a-k) = \frac{(-1)^k}{k!\,\Gamma(a)} \;/;\; k \in \mathbb{N}$$

With respect to *a*

For fixed n, the function $(a)_n$ has an infinite set of singular points:

a) $a == -k - n$ /; $k \in \mathbb{N}$, are the simple poles with residues $\frac{(-1)^k}{k! \, \Gamma(-n-k)}$ /; $k + n \notin \mathbb{N}$;

b) $a == \tilde{\infty}$ is the point of convergence of poles, which is an essential singular point.

06.10.04.0005.01

$$\mathcal{Sing}_a((a)_n) == \{\{\{-k - n, 1\} /; k \in \mathbb{N}\}, \{\tilde{\infty}, \infty\}\}$$

06.10.04.0006.01

$$\text{res}_a((a)_n)(-k - n) == \frac{(-1)^k}{k! \, \Gamma(-n - k)} /; k \in \mathbb{N} \bigwedge k + n \notin \mathbb{N}$$

Branch points

With respect to *n*

The function $(a)_n$ does not have branch points with respect to n.

06.10.04.0007.01

$$\mathcal{BP}_n((a)_n) == \{\}$$

With respect to *a*

The function $(a)_n$ does not have branch points with respect to a.

06.10.04.0008.01

$$\mathcal{BP}_a((a)_n) == \{\}$$

Branch cuts

With respect to *n*

The function $(a)_n$ does not have branch cuts with respect to n.

06.10.04.0009.01

$$\mathcal{BC}_n((a)_n) == \{\}$$

With respect to *a*

The function $(a)_n$ does not have branch cuts with respect to a.

06.10.04.0010.01

$$\mathcal{BC}_a((a)_n) == \{\}$$

Series representations

Generalized power series

Expansions at $a == 0$

For the function itself

General case

06.10.06.0008.01

$$(a)_n \propto \Gamma(n)\, a + \Gamma(n)\,(\psi(n) + \gamma)\, a^2 + \Gamma(n)\left(\frac{1}{12}\left(6\,\gamma^2 - \pi^2\right) + \gamma\,\psi(n) + \frac{1}{2}\left(\psi(n)^2 + \psi^{(1)}(n)\right)\right) a^3 + \ldots \;/;\; (a \to 0)$$

06.10.06.0009.01

$$(a)_n \propto \Gamma(n)\, a + \Gamma(n)\,(\psi(n) + \gamma)\, a^2 + \Gamma(n)\left(\frac{1}{12}\left(6\,\gamma^2 - \pi^2\right) + \gamma\,\psi(n) + \frac{1}{2}\left(\psi(n)^2 + \psi^{(1)}(n)\right)\right) a^3 + O\left(a^4\right)$$

06.10.06.0010.01

$$(a)_n \propto a\,\Gamma(n) + O\left(a^2\right)$$

Special cases

06.10.06.0001.01

$$(a)_n = \sum_{k=0}^{n} (-1)^{k+n}\, S_n^{(k)}\, a^k \;/;\; n \in \mathbb{N}$$

06.10.06.0004.01

$$(a - n + 1)_n = \sum_{k=0}^{n} S_n^{(k)}\, a^k \;/;\; n \in \mathbb{N}$$

Expansions at $a = b$

For the function itself

General case

06.10.06.0011.01

$$(a)_n \propto (b)_n \left(1 + (\psi(b+n) - \psi(b))\,(a-b) + \frac{1}{2}\left(\psi(b)^2 - 2\,\psi(b+n)\,\psi(b) + \psi(b+n)^2 - \psi^{(1)}(b) + \psi^{(1)}(b+n)\right)(a-b)^2 + \ldots\right) /;\; \{a \to b\}$$

06.10.06.0012.01

$$(a)_n \propto (b)_n \left(1 + (\psi(b+n) - \psi(b))\,(a-b) + \frac{1}{2}\left(\psi(b)^2 - 2\,\psi(b+n)\,\psi(b) + \psi(b+n)^2 - \psi^{(1)}(b) + \psi^{(1)}(b+n)\right)(a-b)^2 + O\left((a-b)^3\right)\right) /;\; \{a \to b\}$$

06.10.06.0013.01

$$(a)_n \propto (b)_n\,(1 + O(a-b))$$

Special cases

06.10.06.0002.01

$$(a)_n = \sum_{k=0}^{n} S_n^{(k)}\,(a + n - 1)^k \;/;\; n \in \mathbb{N}$$

06.10.06.0003.01

$$(a)_n = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k+n}\, S_n^{(k)} \binom{k}{j} b^j\,(a-b)^{k-j} \;/;\; n \in \mathbb{N}$$

Expansions of $(a+\epsilon)_n$ at $\epsilon == 0$ /; $a \neq -m$

General case

06.10.06.0014.01

$$(a+\epsilon)_n \propto (a)_n\,(1+O(\epsilon))\;/;\;\neg\,(a\in\mathbb{Z}\land a\leq 0)$$

06.10.06.0015.01

$$(a+\epsilon)_n \propto (a)_n\left(1+(\psi(a+n)-\psi(a))\,\epsilon+O\left(\epsilon^2\right)\right)\;/;\;\neg\,(a\in\mathbb{Z}\land a\leq 0)$$

06.10.06.0016.01

$$(a+\epsilon)_n \propto (a)_n\left(1+(\psi(a+n)-\psi(a))\,\epsilon+\frac{1}{2}\left(\psi(a)^2-2\,\psi(a+n)\,\psi(a)+\psi(a+n)^2-\psi^{(1)}(a)+\psi^{(1)}(a+n)\right)\epsilon^2+O\left(\epsilon^3\right)\right)\;/;\;\neg\,(a\in\mathbb{Z}\land a\leq 0)$$

06.10.06.0017.01

$$(a+\epsilon)_n \propto (a)_n\Bigg(1+(\psi(a+n)-\psi(a))\,\epsilon+\frac{1}{2}\left(\psi(a)^2-2\,\psi(a+n)\,\psi(a)+\psi(a+n)^2-\psi^{(1)}(a)+\psi^{(1)}(a+n)\right)\epsilon^2-\frac{1}{6}\Big(\psi^{(0)}(a)^3-3\,\psi^{(0)}(a+n)\,\psi^{(0)}(a)^2+3\left(\psi^{(0)}(a+n)^2-\psi^{(1)}(a)+\psi^{(1)}(a+n)\right)\psi^{(0)}(a)-\psi^{(0)}(a+n)^3+3\,\psi^{(0)}(a+n)\left(\psi^{(1)}(a)-\psi^{(1)}(a+n)\right)+\psi^{(2)}(a)-\psi^{(2)}(a+n)\Big)\,\epsilon^3+O\left(\epsilon^4\right)\Bigg)\;/;\;\neg\,(a\in\mathbb{Z}\land a\leq 0)$$

06.10.06.0018.01

$$(a+\epsilon)_n \propto (a)_n\Bigg(1+(\psi(a+n)-\psi(a))\,\epsilon+\frac{1}{2}\left(\psi(a)^2-2\,\psi(a+n)\,\psi(a)+\psi(a+n)^2-\psi^{(1)}(a)+\psi^{(1)}(a+n)\right)\epsilon^2-\frac{1}{6}\Big(\psi^{(0)}(a)^3-3\,\psi^{(0)}(a+n)\,\psi^{(0)}(a)^2+3\left(\psi^{(0)}(a+n)^2-\psi^{(1)}(a)+\psi^{(1)}(a+n)\right)\psi^{(0)}(a)-\psi^{(0)}(a+n)^3+3\,\psi^{(0)}(a+n)\left(\psi^{(1)}(a)-\psi^{(1)}(a+n)\right)+\psi^{(2)}(a)-\psi^{(2)}(a+n)\Big)\,\epsilon^3+\frac{1}{24}\Big(\psi(a)^4-4\,\psi(a+n)\,\psi(a)^3+6\left(\psi(a+n)^2-\psi^{(1)}(a)+\psi^{(1)}(a+n)\right)\psi(a)^2-4\left(\psi(a+n)^3-3\left(\psi^{(1)}(a)-\psi^{(1)}(a+n)\right)\psi(a+n)-\psi^{(2)}(a)+\psi^{(2)}(a+n)\right)\psi(a)+\psi(a+n)^4+3\,\psi^{(1)}(a)^2+3\,\psi^{(1)}(a+n)^2-6\,\psi(a+n)^2\left(\psi^{(1)}(a)-\psi^{(1)}(a+n)\right)-6\,\psi^{(1)}(a)\,\psi^{(1)}(a+n)-4\,\psi(a+n)\left(\psi^{(2)}(a)-\psi^{(2)}(a+n)\right)-\psi^{(3)}(a)+\psi^{(3)}(a+n)\Big)\,\epsilon^4+O\left(\epsilon^5\right)\Bigg)\;/;\;\neg\,(a\in\mathbb{Z}\land a\leq 0)$$

Expansions of $(-m+\epsilon)_n$ at $\epsilon == 0$ /; $m-n\notin\mathbb{N}\land m\in\mathbb{N}$

General case

06.10.06.0019.01

$$(-m+\epsilon)_n \propto (-1)^m\;m!\,\Gamma(n-m)\,\epsilon\,(1+O(\epsilon))\;/;\;m-n\notin\mathbb{N}\bigwedge m\in\mathbb{N}$$

06.10.06.0020.01

$$(-m+\epsilon)_n \propto (-1)^m\;m!\,\Gamma(n-m)\left(\epsilon+(\psi(n-m)-\psi(m+1))\,\epsilon^2+O\left(\epsilon^3\right)\right)\;/;\;m-n\notin\mathbb{N}\bigwedge m\in\mathbb{N}$$

06.10.06.0021.01

$$(-m+\epsilon)_n \propto (-1)^m\, m!\, \Gamma(n-m) \left(\epsilon + (\psi(n-m)-\psi(m+1))\,\epsilon^2 + \frac{1}{6}\left(3\,\psi(m+1)^2 - 6\,\psi(n-m)\,\psi(m+1) - \pi^2 + 3\,\psi(n-m)^2 + 3\,\psi^{(1)}(m+1) + 3\,\psi^{(1)}(n-m)\right) \epsilon^3 + O\!\left(\epsilon^4\right)\right) /;\ m-n \notin \mathbb{N} \bigwedge m \in \mathbb{N}$$

06.10.06.0022.01

$$(-m+\epsilon)_n \propto (-1)^m\, m!\, \Gamma(n-m) \Big(\epsilon + (\psi(n-m)-\psi(m+1))\,\epsilon^2 + \frac{1}{6}\left(3\,\psi(m+1)^2 - 6\,\psi(n-m)\,\psi(m+1) - \pi^2 + 3\,\psi(n-m)^2 + 3\,\psi^{(1)}(m+1) + 3\,\psi^{(1)}(n-m)\right) \epsilon^3 + \frac{1}{6}\Big(-\psi(m+1)^3 + 3\,\psi(n-m)\,\psi(m+1)^2 + \left(-3\,\psi(n-m)^2 + \pi^2 - 3\,\psi^{(1)}(m+1) - 3\,\psi^{(1)}(n-m)\right)\psi(m+1) + \psi(n-m)^3 + \psi(n-m)\left(3\,\psi^{(1)}(m+1) - \pi^2 + 3\,\psi^{(1)}(n-m)\right) - \psi^{(2)}(m+1) + \psi^{(2)}(n-m)\Big)\,\epsilon^4 + O\!\left(\epsilon^5\right)\Big) /;\ m-n \notin \mathbb{N} \bigwedge m \in \mathbb{N}$$

Expansions of $(-m+\epsilon)_n$ at $\epsilon == 0$ /; $n \in \mathbb{Z} \bigwedge n \leq m \bigwedge m \in \mathbb{N}$

General case

06.10.06.0023.01

$$(\epsilon-m)_n \propto \frac{(-1)^n\, m!}{(m-n)!}\,(1+O(\epsilon)) /;\ n \in \mathbb{Z} \land n \leq m \land m \in \mathbb{N}$$

06.10.06.0024.01

$$(\epsilon-m)_n \propto \frac{(-1)^n\, m!}{(m-n)!}\left(1 + (\psi(m-n+1)-\psi(m+1))\,\epsilon + O\!\left(\epsilon^2\right)\right) /;\ n \in \mathbb{Z} \land n \leq m \land m \in \mathbb{N}$$

06.10.06.0025.01

$$(\epsilon-m)_n \propto \frac{(-1)^n\, m!}{(m-n)!}\left(1 + (\psi(m-n+1)-\psi(m+1))\,\epsilon + \frac{1}{2}\left(\psi(m+1)^2 - 2\,\psi(m-n+1)\,\psi(m+1) + \psi(m-n+1)^2 + \psi^{(1)}(m+1) - \psi^{(1)}(m-n+1)\right)\epsilon^2 + O\!\left(\epsilon^3\right)\right) /;\ n \in \mathbb{Z} \land n \leq m \land m \in \mathbb{N}$$

06.10.06.0026.01

$$(\epsilon-m)_n \propto \frac{(-1)^n\, m!}{(m-n)!}\Big(1 + (\psi(m-n+1)-\psi(m+1))\,\epsilon + \frac{1}{2}\left(\psi(m+1)^2 - 2\,\psi(m-n+1)\,\psi(m+1) + \psi(m-n+1)^2 + \psi^{(1)}(m+1) - \psi^{(1)}(m-n+1)\right)\epsilon^2 - \frac{1}{6}\Big(\psi(m+1)^3 - 3\,\psi(m-n+1)\,\psi(m+1)^2 + 3\left(\psi(m-n+1)^2 + \psi^{(1)}(m+1) - \psi^{(1)}(m-n+1)\right)\psi(m+1) - \psi(m-n+1)^3 - 3\,\psi(m-n+1)\left(\psi^{(1)}(m+1) - \psi^{(1)}(m-n+1)\right) + \psi^{(2)}(m+1) - \psi^{(2)}(m-n+1)\Big)\,\epsilon^3 + O\!\left(\epsilon^4\right)\Big) /;\ n \in \mathbb{Z} \land n \leq m \land m \in \mathbb{N}$$

06.10.06.0027.01

$$(\epsilon - m)_n \propto \frac{(-1)^n\, m!}{(m-n)!} \Big(1 + (\psi(m-n+1) - \psi(m+1))\,\epsilon + \frac{1}{2}\left(\psi(m+1)^2 - 2\,\psi(m-n+1)\,\psi(m+1) + \psi(m-n+1)^2 + \psi^{(1)}(m+1) - \psi^{(1)}(m-n+1)\right) \epsilon^2 - \frac{1}{6}\Big(\psi(m+1)^3 - 3\,\psi(m-n+1)\,\psi(m+1)^2 + 3\left(\psi(m-n+1)^2 + \psi^{(1)}(m+1) - \psi^{(1)}(m-n+1)\right)\psi(m+1) - \psi(m-n+1)^3 - 3\,\psi(m-n+1)\left(\psi^{(1)}(m+1) - \psi^{(1)}(m-n+1)\right) + \psi^{(2)}(m+1) - \psi^{(2)}(m-n+1)\Big)\,\epsilon^3 + \frac{1}{24}\Big(\psi(m+1)^4 - 4\,\psi(m-n+1)\,\psi(m+1)^3 + 6\left(\psi(m-n+1)^2 + \psi^{(1)}(m+1) - \psi^{(1)}(m-n+1)\right)\psi(m+1)^2 - 4\left(\psi(m-n+1)^3 + 3\left(\psi^{(1)}(m+1) - \psi^{(1)}(m-n+1)\right)\psi(m-n+1) - \psi^{(2)}(m+1) + \psi^{(2)}(m-n+1)\right)\psi(m+1) + \psi(m-n+1)^4 + 3\,\psi^{(1)}(m+1)^2 + 3\,\psi^{(1)}(m-n+1)^2 + 6\,\psi(m-n+1)^2\left(\psi^{(1)}(m+1) - \psi^{(1)}(m-n+1)\right) - 6\,\psi^{(1)}(m+1)\,\psi^{(1)}(m-n+1) - 4\,\psi(m-n+1)\left(\psi^{(2)}(m+1) - \psi^{(2)}(m-n+1)\right) + \psi^{(3)}(m+1) - \psi^{(3)}(m-n+1)\Big)\,\epsilon^4 + O\!\left(\epsilon^5\right)\Big) \;/;\; n \in \mathbb{Z} \land n \le m \land m \in \mathbb{N}$$

Asymptotic series expansions

Expansions at $a == \infty$

06.10.06.0005.01

$$(a)_n \propto a^n \sum_{k=0}^{\infty} \frac{(-1)^k\,(-n)_k}{k!}\, B_k^{(n+1)}(n)\, a^{-k} \;/;\; (|a| \to \infty) \bigwedge |\arg(a+n)| < \pi$$

06.10.06.0006.01

$$(a)_n \propto a^n \left(1 + \frac{(n-1)\,n}{2\,a} + O\!\left(\frac{1}{a^2}\right)\right) \;/;\; (|a| \to \infty) \bigwedge |\arg(a+n)| < \pi$$

Expansions at $n == \infty$

06.10.06.0007.01

$$(a)_n \propto \frac{\sqrt{2\,\pi}}{\Gamma(a)}\, e^{-n}\, n^{a+n-\frac{1}{2}} \Big(1 + \frac{6\,a^2 - 6\,a + 1}{12\,n} + \frac{36\,a^4 - 120\,a^3 + 120\,a^2 - 36\,a + 1}{288\,n^2} + \frac{1080\,a^6 - 7560\,a^5 + 18\,900\,a^4 - 20\,160\,a^3 + 8190\,a^2 - 450\,a - 139}{51\,840\,n^3} + \frac{1}{2\,488\,320\,n^4}\left(6480\,a^8 - 77\,760\,a^7 + 362\,880\,a^6 - 828\,576\,a^5 + 945\,000\,a^4 - 465\,840\,a^3 + 34\,464\,a^2 + 23\,352\,a - 571\right) + O\!\left(\frac{1}{n^5}\right)\Big) \;/;\; (|n| \to \infty) \bigwedge |\arg(a+n)| < \pi$$

Limit representations

06.10.09.0001.01

$$(a)_n == \lim_{m \to \infty} m^n \prod_{k=0}^{m-1} \frac{a+k}{a+k+n}$$

06.10.09.0002.01

$$(a)_n = n! \lim_{z \to \infty} (2z)^{-n} C_n^{(a)}(z)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.10.16.0001.01

$$(a)_{k+mn} = (a)_k \, m^{mn} \prod_{j=0}^{m-1} \left(\frac{a+j+k}{m} \right)_n \; /; \; m \in \mathbb{N}$$

06.10.16.0002.01

$$(a\,m+b)_n = m^n \prod_{k=0}^{m-1} \left(a + \frac{b+k}{m} \right)_{\frac{n}{m}} \; /; \; m \in \mathbb{N}^+$$

06.10.16.0003.01

$$(1-b)_n = \sum_{k=0}^{n} \frac{(-1)^k \, n!^2}{(n-k)! \, k!^2} (b)_k \; /; \; n \in \mathbb{N}^+$$

Addition formulas

06.10.16.0004.01

$$(a+b)_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a+k)_{n-k} (-b)_k \; /; \; n \in \mathbb{N}$$

06.10.16.0005.01

$$(a+b)_n = n! \sum_{k=0}^{n} \frac{(a)_k (b)_{n-k}}{k! \, (n-k)!} \; /; \; n \in \mathbb{N}$$

06.10.16.0006.01

$$(a)_{m+n} = (a)_m (a+m)_n$$

Multiple arguments

06.10.16.0007.01

$$(a)_{2n} = 2^{2n} \left(\frac{a}{2} \right)_n \left(\frac{a+1}{2} \right)_n$$

06.10.16.0010.01

$$(a)_{3n} = 3^{3n} \left(\frac{a}{3} \right)_n \left(\frac{a+1}{3} \right)_n \left(\frac{a+2}{3} \right)_n$$

06.10.16.0011.01

$$(a)_{mn} = m^{mn} \prod_{j=0}^{m-1} \left(\frac{a+j}{m} \right)_n \; /; \; m \in \mathbb{N}^+$$

Products, sums, and powers of the direct function

Products of the direct function

06.10.16.0008.01

$$(a)_n \left(a+\frac{1}{2}\right)_n == \frac{1}{4^n} (2\,a)_{2\,n}$$

Sums of the direct function

06.10.16.0009.01

$$(a)_{n+1} - (b)_{n+1} == (a-b) \sum_{k=0}^{n} (a)_k\, (b+k+1)_{n-k} \,/;\, n \in \mathbb{N}$$

Identities

Recurrence identities

Consecutive neighbors

06.10.17.0001.02

$$(a)_n == \frac{a+n-1}{a-1} (a-1)_n$$

06.10.17.0002.02

$$(a)_n == \frac{a}{a+n} (a+1)_n$$

06.10.17.0008.01

$$(a)_n == (a+n-1)\,(a)_{n-1}$$

06.10.17.0009.01

$$(a)_n == \frac{1}{a+n} (a)_{n+1}$$

Distant neighbors

06.10.17.0003.02

$$(a)_n == \frac{\Gamma(a-m)\,\Gamma(a+n)}{\Gamma(a)\,\Gamma(a-m+n)} (a-m)_n$$

06.10.17.0004.02

$$(a)_n == \frac{\Gamma(a+m)\,\Gamma(a+n)}{\Gamma(a)\,\Gamma(a+m+n)} (a+m)_n$$

06.10.17.0010.01

$$(a)_n == \frac{\Gamma(a+n)}{\Gamma(a-m+n)} (a)_{n-m}$$

06.10.17.0011.01

$$(a)_n == \frac{\Gamma(a+n)}{\Gamma(a+m+n)} (a)_{n+m}$$

Functional identities

Relations of special kind

06.10.17.0005.01

$$(a)_n = \frac{(-1)^n}{(1-a)_{-n}} \;/;\; n \in \mathbb{Z}$$

06.10.17.0006.01

$$(a)_n = \frac{(a-m)_{m+n}}{(a-m)_m}$$

06.10.17.0007.01

$$(a)_n = (a)_m \, (a+m)_{n-m}$$

Differentiation

Low-order differentiation

With respect to a

06.10.20.0001.01

$$\frac{\partial (a)_n}{\partial a} = (a)_n \left(\psi(a+n) - \psi(a)\right)$$

06.10.20.0002.01

$$\frac{\partial^2 (a)_n}{\partial a^2} = (a)_n \left((\psi(a) - \psi(a+n))^2 - \psi^{(1)}(a) + \psi^{(1)}(a+n)\right)$$

With respect to n

06.10.20.0003.01

$$\frac{\partial (a)_n}{\partial n} = (a)_n \, \psi(a+n)$$

06.10.20.0004.01

$$\frac{\partial^2 (a)_n}{\partial n^2} = (a)_n \left(\psi(a+n)^2 + \psi^{(1)}(a+n)\right)$$

Symbolic differentiation

With respect to a

06.10.20.0005.02

$$\frac{\partial^m (a)_n}{\partial a^m} = \frac{(-1)^{m-1} \, m! \sin(\pi n)}{\pi} \sum_{k=0}^{\infty} \frac{\Gamma(k+n+1)}{k! \, (a+k+n)^{m+1}} \;/;\; m \in \mathbb{N} \bigwedge n \notin \mathbb{N}$$

06.10.20.0006.02

$$\frac{\partial^m (a)_n}{\partial a^m} = \frac{(-1)^m \, m! \, \Gamma(a+n)^{m+1}}{\Gamma(-n)} \, {}_{m+2}\tilde{F}_{m+1}(a_1, a_2, \ldots, a_{m+1}, n+1; a_1+1, a_2+1, \ldots, a_{m+1}+1; 1) \;/;$$
$$a_1 = a_2 = \ldots = a_{m+1} = a+n \bigwedge m \in \mathbb{N} \bigwedge n \notin \mathbb{N}$$

06.10.20.0007.01

$$\frac{\partial^m (a)_n}{\partial a^m} = \sum_{k=1}^{n} (-1)^{k+n} \, S_n^{(k)} \, (k-m+1)_m \, a^{k-m} \;/;\; m \in \mathbb{N} \land n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to a

06.10.20.0008.01

$$\frac{\partial^{\alpha} (a)_n}{\partial a^{\alpha}} == \sum_{k=0}^{n} \frac{(-1)^{k+n} S_n^{(k)} k! \, a^{k-\alpha}}{\Gamma(k-\alpha+1)} \;/;\; n \in \mathbb{N}$$

With respect to n

06.10.20.0009.01

$$\frac{\partial^{\alpha} (a)_n}{\partial n^{\alpha}} == \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(a) \, n^{k-\alpha}}{\Gamma(a) \, \Gamma(k-\alpha+1)}$$

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

06.10.26.0001.01

$$(a)_n == \Gamma(n+1) \, (a-1+n;\, a-1,\, n)$$

Representations through equivalent functions

With related functions

06.10.27.0001.01

$$(a)_n == \frac{\Gamma(a+n)}{\Gamma(a)}$$

06.10.27.0002.01

$$(a)_n == \frac{(-1)^n \, \Gamma(1-a)}{\Gamma(1-a-n)} \;/;\; n \in \mathbb{Z}$$

06.10.27.0003.01

$$(m)_n == \frac{(m+n-1)!}{(m-1)!} \;/;\; \neg\, (-m \in \mathbb{N} \land -m-n \in \mathbb{N})$$

06.10.27.0004.01

$$(-m)_n == \frac{(-1)^n \, m!}{(m-n)!} \;/;\; m \in \mathbb{N} \land n \in \mathbb{N}$$

06.10.27.0005.01

$$(a)_k == \Gamma(k+1) \binom{a+k-1}{k}$$

06.10.27.0006.01

$$(a)_k == \Gamma(k+1) \binom{a+k-1}{a-1}$$

06.10.27.0007.01

$$(a)_n = \frac{(n-1)!}{B(a, n)}$$

06.10.27.0008.01

$$(z+1)_z = C_z \, \Gamma(z+2)$$

Zeros

06.10.30.0001.01

$$(a)_n = 0 \; /; \; a + n = -k \bigwedge k \in \mathbb{N} \bigwedge -a \notin \mathbb{N}$$

History

– A. L. Crelle (1831) used a similar symbol
– L. A. Pochhammer (1890)
– P. E. Appell (1880) used the name "Pochhammer symbol"

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