SphericalHarmonicY

Notations

Traditional name

Spherical harmonic

Traditional notation

$Y_n^m(\vartheta, \varphi)$

Mathematica StandardForm notation

```
SphericalHarmonicY[n, m, ϑ, φ]
```

Primary definition

05.10.02.0001.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}\, e^{i\varphi m} P_n^m(\cos(\vartheta)) /; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

05.10.02.0002.01

$$Y_n^0(k\pi, \varphi) = (-1)^{n\left(2\left\lfloor \frac{k}{2} \right\rfloor - k\right)} \sqrt{\frac{2n+1}{4\pi}} /; k \in \mathbb{Z}$$

05.10.02.0003.01

$$Y_n^m(\vartheta, \varphi) = 0 /; n \in \mathbb{N} \land m \in \mathbb{Z} \land n < |m|$$

05.10.02.0004.01

$$Y_n^m(\vartheta, \varphi) = Y_{-n-1}^m(\vartheta, \varphi) /; -n \in \mathbb{N}^+ \bigwedge m \in \mathbb{Z}$$

The following restrictions apply to all formulas of this function.

05.10.02.0005.01

$$n \in \mathbb{N} \land m \in \mathbb{Z} \land -n \leq m \leq n$$

Specific values

Specialized values

For fixed n, m, ϑ

05.10.03.0001.01

$$Y_n^m(\vartheta, 0) = e^{-i\varphi m}\, Y_n^m(\vartheta, \varphi)$$

For fixed *n*, *m*, *φ*

05.10.03.0002.01

$$Y_n^m(0, \varphi) == 0 \,/;\, m \neq 0$$

05.10.03.0003.01

$$Y_n^m(\pi, \varphi) == 0 \,/;\, m \neq 0$$

05.10.03.0004.01

$$Y_n^m(k\,\pi, \varphi) == 0 \,/;\, m \neq 0 \bigwedge k \in \mathbb{Z}$$

05.10.03.0005.01

$$Y_n^m\left(\frac{\pi}{2}, \varphi\right) == \frac{((n+m+1) \bmod 2)}{2} (-1)^{\frac{n+m}{2}} e^{i\,m\,\varphi} \sqrt{\frac{(2\,n+1)\,(n+m-1)!!\,(n-m-1)!!}{\pi\,(n+m)!!\,(n-m)!!}}$$

05.10.03.0006.01

$$Y_n^m\left(-\frac{\pi}{2}, \varphi\right) == \frac{((n+m+1) \bmod 2)}{2} (-1)^{\frac{n+m}{2}} e^{i\,m\,\varphi} \sqrt{\frac{(2\,n+1)\,(n+m-1)!!\,(n-m-1)!!}{\pi\,(n+m)!!\,(n-m)!!}}$$

For fixed *n*, *ϑ*, *φ*

05.10.03.0007.01

$$Y_n^0(\vartheta, \varphi) == \sqrt{\frac{2\,n+1}{4\,\pi}}\; P_n(\cos(\vartheta))$$

05.10.03.0008.01

$$Y_n^1(\vartheta, \varphi) == \frac{e^{i\,\varphi} \sqrt{(2\,n+1)\,n}}{4\sqrt{\pi}\,\sqrt{n+1}\,\sqrt{\cos^2\left(\frac{\vartheta}{2}\right)}\,\sqrt{\sin^2\left(\frac{\vartheta}{2}\right)}} (\cos(\vartheta)\,P_n(\cos(\vartheta)) - P_{n-1}(\cos(\vartheta)))$$

05.10.03.0009.01

$$Y_n^n(\vartheta, \varphi) == \frac{(-1)^n \sqrt{(2\,n+1)!}\; e^{i\,\varphi\,n}}{2^{n+1}\sqrt{\pi}\; n!} \sin^2(\vartheta)^{n/2}$$

05.10.03.0010.01

$$Y_n^{n+1}(\vartheta, \varphi) == 0$$

05.10.03.0011.01

$$Y_n^{-n}(\vartheta, \varphi) == \frac{\sqrt{(2\,n+1)!}\; e^{-i\,\varphi\,n}}{2^{n+1}\, n!\sqrt{\pi}} \sin^2(\vartheta)^{n/2}$$

05.10.03.0012.01

$$Y_n^{-n-1}(\vartheta, \varphi) == 0$$

05.10.03.0013.01

$$Y_n^{-n-m}(\vartheta, \varphi) == 0 \,/;\, m \in \mathbb{N}^+$$

05.10.03.0014.01

$$Y_n^{n+m}(\vartheta, \varphi) == 0 \,/;\, n \in \mathbb{N} \bigwedge m \in \mathbb{N}^+$$

For fixed m, *ϑ*, *φ*

05.10.03.0015.01

$$Y_0^m(\vartheta, \varphi) = \frac{e^{i m \varphi} \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2}}{2 \sqrt{\pi} \sqrt{\Gamma(1-m)} \sqrt{\Gamma(m+1)}}$$

05.10.03.0016.01

$$Y_1^m(\vartheta, \varphi) = \frac{\sqrt{3}\, e^{i m \varphi} (\cos(\vartheta) - m) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2}}{2 \sqrt{\pi} \sqrt{\Gamma(2-m)} \sqrt{\Gamma(m+2)}}$$

05.10.03.0017.01

$$Y_2^m(\vartheta, \varphi) = \frac{\sqrt{5}\, e^{i m \varphi} \left(3 \cos^2(\vartheta) - 3 m \cos(\vartheta) + m^2 - 1\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2}}{2 \sqrt{\pi} \sqrt{\Gamma(3-m)} \sqrt{\Gamma(m+3)}}$$

05.10.03.0018.01

$$Y_3^m(\vartheta, \varphi) = \frac{\sqrt{7}\, e^{i m \varphi} \left(15 \cos^3(\vartheta) - 15 m \cos^2(\vartheta) + \left(6 m^2 - 9\right) \cos(\vartheta) - m \left(m^2 - 4\right)\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2}}{\sqrt{4 \pi} \sqrt{\Gamma(4-m)} \sqrt{\Gamma(4+m)}}$$

05.10.03.0019.01

$$Y_4^m(\vartheta, \varphi) = \left(3 e^{i m \varphi} \left(105 \cos^4(\vartheta) - 105 m \cos^3(\vartheta) + 45 \left(m^2 - 2\right) \cos^2(\vartheta) - 5 m \left(2 m^2 - 11\right) \cos(\vartheta) + m^4 - 10 m^2 + 9\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2}\right) \Big/ \left(2 \sqrt{\pi} \sqrt{\Gamma(5-m)} \sqrt{\Gamma(5+m)}\right)$$

05.10.03.0020.01

$$Y_5^m(\vartheta, \varphi) = \left(\sqrt{11}\, e^{i m \varphi} \left(945 \cos^5(\vartheta) - 945 m \cos^4(\vartheta) + 210 \left(2 m^2 - 5\right) \cos^3(\vartheta) - 105 m \left(m^2 - 7\right) \cos^2(\vartheta) + 15 \left(m^4 - 13 m^2 + 15\right) \cos(\vartheta) - m \left(m^4 - 20 m^2 + 64\right)\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2}\right) \Big/ \left(2 \sqrt{\pi} \sqrt{\Gamma(6-m)} \sqrt{\Gamma(6+m)}\right)$$

05.10.03.0021.01

$$Y_6^m(\vartheta, \varphi) = \left(\sqrt{13}\, e^{i m \varphi} \left(10\,395 \cos^6(\vartheta) - 10\,395 m \cos^5(\vartheta) + 4725 \left(m^2 - 3\right) \cos^4(\vartheta) - 630 \left(2 m^2 - 17\right) m \cos^3(\vartheta) + 105 \left(2 m^4 - 32 m^2 + 45\right) \cos^2(\vartheta) - 21 \left(m^4 - 25 m^2 + 99\right) m \cos(\vartheta) + m^6 - 35 m^4 + 259 m^2 - 225\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}}\right) \Big/ \left(2 \sqrt{\pi} \sqrt{\Gamma(7-m)} \sqrt{\Gamma(m+7)}\right)$$

05.10.03.0022.01

$$Y_7^m(\vartheta, \varphi) = \left(\sqrt{15}\, e^{i m \varphi} \left(135\,135 \cos^7(\vartheta) - 135\,135 m \cos^6(\vartheta) + 31\,185 \left(2 m^2 - 7\right) \cos^5(\vartheta) - 17\,325 m \left(m^2 - 10\right) \cos^4(\vartheta) + 1575 \left(2 m^4 - 38 m^2 + 63\right) \cos^3(\vartheta) - 189 m \left(2 m^4 - 60 m^2 + 283\right) \cos^2(\vartheta) + 7 \left(4 m^6 - 170 m^4 + 1516 m^2 - 1575\right) \cos(\vartheta) - m \left(m^6 - 56 m^4 + 784 m^2 - 2304\right)\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2}\right) \Big/ \left(2 \sqrt{\pi} \sqrt{\Gamma(8-m)} \sqrt{\Gamma(m+8)}\right)$$

05.10.03.0023.01

$$Y_8^m(\vartheta, \varphi) = \left(\sqrt{17}\, e^{i m \varphi} \left(2027025 \cos^8(\vartheta) - 2027025\, m \cos^7(\vartheta) + 945945 \left(m^2 - 4\right) \cos^6(\vartheta) - 135135 \left(2 m^2 - 23\right) m \cos^5(\vartheta) + 51975 \left(m^4 - 22 m^2 + 42\right) \cos^4(\vartheta) - 3465 \left(2 m^4 - 70 m^2 + 383\right) m \cos^3(\vartheta) + 315 \left(2 m^6 - 100 m^4 + 1043 m^2 - 1260\right) \cos^2(\vartheta) - 9 \left(4 m^6 - 266 m^4 + 4396 m^2 - 15159\right) m \cos(\vartheta) + m^8 - 84 m^6 + 1974 m^4 - 12916 m^2 + 11025\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2}\right) \Big/ \left(2 \sqrt{\pi} \sqrt{\Gamma(9-m)} \sqrt{\Gamma(m+9)}\right)$$

05.10.03.0024.01

$$Y_9^m(\vartheta, \varphi) = \left(\sqrt{19}\, e^{i m \varphi} \left(34459425 \cos^9(\vartheta) - 34459425\, m \cos^8(\vartheta) + 8108100 \left(2 m^2 - 9\right) \cos^7(\vartheta) - 4729725\, m \left(m^2 - 13\right) \cos^6(\vartheta) + 945945 \left(m^4 - 25 m^2 + 54\right) \cos^5(\vartheta) - 135135\, m \left(m^4 - 40 m^2 + 249\right) \cos^4(\vartheta) + 6930 \left(2 m^6 - 115 m^4 + 1373 m^2 - 1890\right) \cos^3(\vartheta) - 495\, m \left(2 m^6 - 154 m^4 + 2933 m^2 - 11601\right) \cos^2(\vartheta) + 45 \left(m^8 - 98 m^6 + 2674 m^4 - 20217 m^2 + 19845\right) \cos(\vartheta) - m \left(m^8 - 120 m^6 + 4368 m^4 - 52480 m^2 + 147456\right)\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2}\right) \Big/ \left(2 \sqrt{\pi} \sqrt{\Gamma(10-m)} \sqrt{\Gamma(m+10)}\right)$$

05.10.03.0025.01

$$Y_{10}^m(\vartheta, \varphi) = \left(\sqrt{21}\, e^{i m \varphi} \left(654729075 \cos^{10}(\vartheta) - 654729075\, m \cos^9(\vartheta) + 310134825 \left(m^2 - 5\right) \cos^8(\vartheta) - 45945900 \left(2 m^2 - 29\right) m \cos^7(\vartheta) + 9459450 \left(2 m^4 - 56 m^2 + 135\right) \cos^6(\vartheta) - 2837835 \left(m^4 - 45 m^2 + 314\right) m \cos^5(\vartheta) + 315315 \left(m^6 - 65 m^4 + 874 m^2 - 1350\right) \cos^4(\vartheta) - 12870 \left(2 m^6 - 175 m^4 + 3773 m^2 - 16830\right) m \cos^3(\vartheta) + 1485 \left(m^8 - 112 m^6 + 3479 m^4 - 29828 m^2 + 33075\right) \cos^2(\vartheta) - 55 \left(m^8 - 138 m^6 + 5754 m^4 - 78877 m^2 + 251865\right) m \cos(\vartheta) + m^{10} - 165 m^8 + 8778 m^6 - 172810 m^4 + 1057221 m^2 - 893025\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2}\right) \Big/ \left(2 \sqrt{\pi} \sqrt{\Gamma(11-m)} \sqrt{\Gamma(m+11)}\right)$$

05.10.03.0026.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}} e^{i \varphi m} \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-m/2} \sum_{k=0}^{n} \frac{(-n)_k\, (n+1)_k}{\Gamma(k-m+1)\, k!} \sin^{2k}\left(\frac{\vartheta}{2}\right)$$

05.10.03.0027.01

$$Y_n^m(\vartheta, \varphi) = 0 \,/;\, m > 0 \land n < m$$

05.10.03.0028.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^m\, 2^{n-1}\, e^{i \varphi m}}{\pi} \sqrt{\frac{(2n+1)\,(n-m)!}{(n+m)!}} \sin^2(\vartheta)^{m/2} \cos^{-m}(\vartheta) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k\, 2^{-2k}\, \Gamma\left(n-k+\frac{1}{2}\right) \cos^{n-2k}(\vartheta)}{k!\, \Gamma(n-m-2k+1)} \,/;\, 0 \le m \le n$$

For fixed n, φ

05.10.03.0029.01

$$Y_n^0(0, \varphi) = \frac{\sqrt{2n+1}}{2\sqrt{\pi}}$$

05.10.03.0030.01

$$Y_n^0(k\pi, \varphi) = (-1)^{n\left(2\left\lfloor \frac{k}{2} \right\rfloor - k\right)} \sqrt{\frac{2n+1}{4\pi}} \;/;\; k \in \mathbb{Z}$$

For fixed *n*, *φ*

05.10.03.0031.01

$$Y_0^0(\vartheta, \varphi) = \frac{1}{2\sqrt{\pi}}$$

05.10.03.0032.01

$$Y_1^{-1}(\vartheta, \varphi) = \frac{1}{2} e^{-i\varphi} \sqrt{\frac{3}{2\pi}} \sqrt{\sin^2(\vartheta)}$$

05.10.03.0033.01

$$Y_1^0(\vartheta, \varphi) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos(\vartheta)$$

05.10.03.0034.01

$$Y_1^1(\vartheta, \varphi) = -\frac{1}{2} e^{i\varphi} \sqrt{\frac{3}{2\pi}} \sqrt{\sin^2(\vartheta)}$$

05.10.03.0035.01

$$Y_2^{-2}(\vartheta, \varphi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} e^{-2i\varphi} \sin^2(\vartheta)$$

05.10.03.0036.01

$$Y_2^{-1}(\vartheta, \varphi) = \frac{1}{2} e^{-i\varphi} \sqrt{\frac{15}{2\pi}} \cos(\vartheta) \sqrt{\sin^2(\vartheta)}$$

05.10.03.0037.01

$$Y_2^0(\vartheta, \varphi) = \frac{1}{8} \sqrt{\frac{5}{\pi}} (3\cos(2\vartheta) + 1)$$

05.10.03.0038.01

$$Y_2^1(\vartheta, \varphi) = -\frac{1}{2} e^{i\varphi} \sqrt{\frac{15}{2\pi}} \cos(\vartheta) \sqrt{\sin^2(\vartheta)}$$

05.10.03.0039.01

$$Y_2^2(\vartheta, \varphi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} e^{2i\varphi} \sin^2(\vartheta)$$

05.10.03.0040.01

$$Y_3^{-3}(\vartheta, \varphi) == \frac{1}{8} e^{-3 i \varphi} \sqrt{\frac{35}{\pi}} \sin^2(\vartheta)^{3/2}$$

05.10.03.0041.01

$$Y_3^{-2}(\vartheta, \varphi) == \frac{1}{4} \sqrt{\frac{105}{2\pi}} e^{-2 i \varphi} \cos(\vartheta) \sin^2(\vartheta)$$

05.10.03.0042.01

$$Y_3^{-1}(\vartheta, \varphi) == \frac{1}{16} e^{-i \varphi} \sqrt{\frac{21}{\pi}} (5 \cos(2 \vartheta) + 3) \sqrt{\sin^2(\vartheta)}$$

05.10.03.0043.01

$$Y_3^{0}(\vartheta, \varphi) == \frac{1}{16} \sqrt{\frac{7}{\pi}} (3 \cos(\vartheta) + 5 \cos(3 \vartheta))$$

05.10.03.0044.01

$$Y_3^{1}(\vartheta, \varphi) == -\frac{1}{16} e^{i \varphi} \sqrt{\frac{21}{\pi}} (5 \cos(2 \vartheta) + 3) \sqrt{\sin^2(\vartheta)}$$

05.10.03.0045.01

$$Y_3^{2}(\vartheta, \varphi) == \frac{1}{4} \sqrt{\frac{105}{2\pi}} e^{2 i \varphi} \cos(\vartheta) \sin^2(\vartheta)$$

05.10.03.0046.01

$$Y_3^{3}(\vartheta, \varphi) == -\frac{1}{8} e^{3 i \varphi} \sqrt{\frac{35}{\pi}} \sin^2(\vartheta)^{3/2}$$

05.10.03.0047.01

$$Y_4^{-4}(\vartheta, \varphi) == \frac{3}{16} \sqrt{\frac{35}{2\pi}} e^{-4 i \varphi} \sin^4(\vartheta)$$

05.10.03.0048.01

$$Y_4^{-3}(\vartheta, \varphi) == \frac{3}{8} \sqrt{\frac{35}{\pi}} e^{-3 i \varphi} \cos(\vartheta) \sin^2(\vartheta)^{3/2}$$

05.10.03.0049.01

$$Y_4^{-2}(\vartheta, \varphi) == \frac{3}{8} \sqrt{\frac{5}{2\pi}} e^{-2 i \varphi} \left(7 \cos^2(\vartheta) - 1\right) \sin^2(\vartheta)$$

05.10.03.0050.01

$$Y_4^{-1}(\vartheta, \varphi) == \frac{3}{8} \sqrt{\frac{5}{\pi}} e^{-i \varphi} \cos(\vartheta) \left(7 \cos^2(\vartheta) - 3\right) \sqrt{\sin^2(\vartheta)}$$

05.10.03.0051.01

$$Y_4^{0}(\vartheta, \varphi) == \frac{3}{16 \sqrt{\pi}} \left(35 \cos^4(\vartheta) - 30 \cos^2(\vartheta) + 3\right)$$

05.10.03.0052.01

$$Y_4^1(\vartheta, \varphi) == -\frac{3}{8}\sqrt{\frac{5}{\pi}}\, e^{i\varphi} \cos(\vartheta) \left(7\cos^2(\vartheta) - 3\right) \sqrt{\sin^2(\vartheta)}$$

05.10.03.0053.01

$$Y_4^2(\vartheta, \varphi) == \frac{3}{8}\sqrt{\frac{5}{2\pi}}\, e^{2i\varphi} \left(7\cos^2(\vartheta) - 1\right) \sin^2(\vartheta)$$

05.10.03.0054.01

$$Y_4^3(\vartheta, \varphi) == -\frac{3}{8}\sqrt{\frac{35}{\pi}}\, e^{3i\varphi} \cos(\vartheta) \sin^2(\vartheta)^{3/2}$$

05.10.03.0055.01

$$Y_4^4(\vartheta, \varphi) == \frac{3}{16}\sqrt{\frac{35}{2\pi}}\, e^{4i\varphi} \sin^4(\vartheta)$$

05.10.03.0056.01

$$Y_5^{-5}(\vartheta, \varphi) == \frac{3}{32}\sqrt{\frac{77}{\pi}}\, e^{-5i\varphi} \sin^2(\vartheta)^{5/2}$$

05.10.03.0057.01

$$Y_5^{-4}(\vartheta, \varphi) == \frac{3}{16}\sqrt{\frac{385}{2\pi}}\, e^{-4i\varphi} \cos(\vartheta) \sin^4(\vartheta)$$

05.10.03.0058.01

$$Y_5^{-3}(\vartheta, \varphi) == \frac{1}{32}\sqrt{\frac{385}{\pi}}\, e^{-3i\varphi} \left(9\cos^2(\vartheta) - 1\right) \sin^2(\vartheta)^{3/2}$$

05.10.03.0059.01

$$Y_5^{-2}(\vartheta, \varphi) == \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\, e^{-2i\varphi} \cos(\vartheta) \left(3\cos^2(\vartheta) - 1\right) \sin^2(\vartheta)$$

05.10.03.0060.01

$$Y_5^0(\vartheta, \varphi) == \frac{1}{16}\sqrt{\frac{11}{\pi}} \left(63\cos^5(\vartheta) - 70\cos^3(\vartheta) + 15\cos(\vartheta)\right)$$

05.10.03.0061.01

$$Y_5^1(\vartheta, \varphi) == -\frac{1}{16}\sqrt{\frac{165}{2\pi}}\, e^{i\varphi} \left(21\cos^4(\vartheta) - 14\cos^2(\vartheta) + 1\right) \sqrt{\sin^2(\vartheta)}$$

05.10.03.0062.01

$$Y_5^2(\vartheta, \varphi) == \frac{1}{8}\sqrt{\frac{1155}{2\pi}}\, e^{2i\varphi} \cos(\vartheta) \left(3\cos^2(\vartheta) - 1\right) \sin^2(\vartheta)$$

05.10.03.0063.01

$$Y_5^3(\vartheta, \varphi) = -\frac{1}{32}\sqrt{\frac{385}{\pi}}\ e^{3 i \varphi}\left(9\cos^2(\vartheta) - 1\right)\sin^2(\vartheta)^{3/2}$$

05.10.03.0064.01

$$Y_5^4(\vartheta, \varphi) = \frac{3}{16}\sqrt{\frac{385}{2\pi}}\ e^{4 i \varphi}\cos(\vartheta)\sin^4(\vartheta)$$

05.10.03.0065.01

$$Y_5^5(\vartheta, \varphi) = -\frac{3}{32}\sqrt{\frac{77}{\pi}}\ e^{5 i \varphi}\sin^2(\vartheta)^{5/2}$$

General characteristics

Domain and analyticity

The function $Y_n^m(\vartheta, \varphi)$ is defined over $\mathbb{N} \otimes \mathbb{Z} \otimes \mathbb{C} \otimes \mathbb{C}$. For fixed n, m, the function $Y_n^m(\vartheta, \varphi)$ is a polynomial in $\sin^2\left(\frac{\vartheta}{2}\right)$ of degree n multiplied on function $\frac{\left(\text{coa}^2\left(\frac{\vartheta}{2}\right)\right)^{m/2}}{\left(\sin^2\left(\frac{\vartheta}{2}\right)\right)^{m/2}}$.

05.10.04.0001.01

$$(n * m * \vartheta * \varphi) \longrightarrow Y_n^m(\vartheta, \varphi) :: (\mathbb{N} \otimes \mathbb{Z} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

05.10.04.0002.01

$$Y_n^{-m}(\vartheta, \varphi) = (-1)^m e^{-2 i \varphi m}\, Y_n^m(\vartheta, \varphi)$$

05.10.04.0003.01

$$Y_n^m(-\vartheta, \varphi) = Y_n^m(\vartheta, \varphi)$$

05.10.04.0004.01

$$Y_n^m(\vartheta, -\varphi) = e^{-2 i m \varphi}\, Y_n^m(\vartheta, \varphi)$$

05.10.04.0005.01

$$Y_n^m(\vartheta, -\varphi) = (-1)^m\, Y_n^{-m}(\vartheta, \varphi)$$

05.10.04.0006.01

$$Y_n^m(-\vartheta, -\varphi) = (-1)^m\, Y_n^{-m}(\vartheta, \varphi)$$

Mirror symmetry

05.10.04.0007.01

$$Y_n^{-m}(\vartheta, \varphi) = (-1)^m\, \overline{Y_n^m(\vartheta, \varphi)} \;/;\; \vartheta \in \mathbb{R} \bigwedge \varphi \in \mathbb{R}$$

Periodicity

$Y_n^m(\vartheta, \varphi)$ is a periodic function with respect to ϑ and φ with periods 2π and $2\pi/m$ respectively.

05.10.04.0008.01

$$Y_n^m(\vartheta + 2\pi k, \varphi) == Y_n^m(\vartheta, \varphi) /; k \in \mathbb{Z}$$

05.10.04.0009.01

$$Y_n^m\left(\vartheta, \varphi + \frac{2\pi k}{m}\right) == Y_n^m(\vartheta, \varphi) /; k \in \mathbb{Z}$$

Phase shifts

05.10.04.0010.01

$$Y_n^m(\vartheta, \varphi + \pi) == (-1)^m Y_n^m(\vartheta, \varphi)$$

05.10.04.0011.01

$$Y_n^m(\pi - \vartheta, \varphi) == (-1)^{n+m} Y_n^m(\vartheta, \varphi)$$

05.10.04.0012.01

$$Y_n^m(\pi - \vartheta, \varphi + \pi) == (-1)^n Y_n^m(\vartheta, \varphi)$$

Poles and essential singularities

With respect to φ

For fixed n,m,ϑ, the function $Y_n^m(\vartheta, \varphi)$ has only one singular point at $\varphi = \tilde{\infty}$. It is an essential singular point.

05.10.04.0013.01

$$\mathcal{S}ing_{\varphi}\left(Y_n^m(\vartheta, \varphi)\right) == \{\{\tilde{\infty}, \infty\}\}$$

With respect to ϑ

For fixed n, φ, m /; $\frac{m}{2} \notin \mathbb{Z}$, the function $Y_n^m(\vartheta, \varphi)$ does not have poles and essential singularities.

05.10.04.0014.01

$$\mathcal{S}ing_{\vartheta}\left(Y_n^m(\vartheta, \varphi)\right) == \{\} /; \frac{m}{2} \notin \mathbb{Z}$$

For integer $\frac{m}{2}$, the function $Y_n^m(\vartheta, \varphi)$ is polynomial and has pole of order n at $\cos(\vartheta) = \tilde{\infty}$.

05.10.04.0015.01

$$\mathcal{S}ing_{\vartheta}\left(Y_n^m(\vartheta, \varphi)\right) == \{\{\tilde{\infty}, n\}\} /; \frac{m}{2} \in \mathbb{Z}$$

Branch points

With respect to φ

For fixed n,m,ϑ, the function $Y_n^m(\vartheta, \varphi)$ does not have branch points.

05.10.04.0016.01

$$\mathcal{BP}_{\varphi}\left(Y_n^m(\vartheta, \varphi)\right) == \{\}$$

With respect to ϑ

For fixed generic n, φ, m /; $\frac{m}{2} \notin \mathbb{Z}$, the function $Y_n^m(\vartheta, \varphi)$ has the set of branch points where: $\cos(\vartheta) == -1$, $\cos(\vartheta) == 1$and $\cos(\vartheta) == \tilde{\infty}$. For fixed φ and integers $\frac{m}{2}$, the function $Y_n^m(\vartheta, \varphi)$ does not have branch points.

Branch cuts

With respect to φ

For fixed n,m,ϑ, the function $Y_n^m(\vartheta, \varphi)$ does not have branch cuts.

05.10.04.0017.01

$$\mathcal{BC}_\varphi\left(Y_n^m(\vartheta, \varphi)\right) = \{\}$$

With respect to ϑ

For fixed generic n, φ, m /; $\frac{m}{2} \notin \mathbb{Z}$, the function $Y_n^m(\vartheta, \varphi)$ is a single-valued function on the ϑ-plane cut along the intervals $-\infty < \cos(\vartheta) < -1$ and $1 < \cos(\vartheta) < \infty$.

For fixed φ and integers $\frac{m}{2}$, the function $Y_n^m(\vartheta, \varphi)$ is a polynomial and does not have branch cuts.

Series representations

Generalized power series

Expansions at $\sin\left(\frac{\vartheta}{2}\right) == 0$

05.10.06.0001.02

$$Y_n^m(\vartheta, \varphi) \propto \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}\, e^{i\varphi m}}{\sqrt{\Gamma(n+m+1)}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \left(\frac{1}{\Gamma(1-m)} - \frac{(1-m)m + 2n(n+1)}{2\Gamma(2-m)} \sin^2\left(\frac{\vartheta}{2}\right) + \frac{\left(m^4 - 5m^3 - 4\left(n^2+n-2\right)m^2 + (8n(n+1)-4)m + 4(n-1)n(n+1)(n+2)\right)}{8\Gamma(3-m)} \sin^4\left(\frac{\vartheta}{2}\right) - \ldots\right) /; \left(\sin\left(\frac{\vartheta}{2}\right) \to 0\right)$$

05.10.06.0002.01

$$Y_n^m(\vartheta, \varphi) == \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}\, e^{i\varphi m}}{\sqrt{\Gamma(n+m+1)}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-n)_j (n+1)_j \left(-\frac{m}{2}\right)_{k-j}}{\Gamma(j-m+1)\, j!\, (k-j)!} \sin^{2k}\left(\frac{\vartheta}{2}\right) /; \left|\sin\left(\frac{\vartheta}{2}\right)\right| < 1$$

05.10.06.0003.01

$$Y_n^m(\vartheta, \varphi) == \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}\, e^{i\varphi m}}{\sqrt{\Gamma(n+m+1)}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} {}_1F_0\left(-\frac{m}{2};; \sin^2\left(\frac{\vartheta}{2}\right)\right) {}_2\tilde{F}_1\left(-n, n+1; 1-m; \sin^2\left(\frac{\vartheta}{2}\right)\right)$$

05.10.06.0004.02

$$Y_n^m(\vartheta, \varphi) \propto \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}\, e^{i\varphi m}}{\Gamma(1-m)\sqrt{\Gamma(n+m+1)}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \left(1 + O\left(\sin^2\left(\frac{\vartheta}{2}\right)\right)\right) /; \left(\sin\left(\frac{\vartheta}{2}\right) \to 0\right) \bigwedge -m \in \mathbb{N}$$

05.10.06.0005.01

$$Y_n^m(\vartheta, \varphi) == \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}\, e^{i\varphi m}}{\sqrt{\Gamma(n+m+1)}\, \sin^2\left(\frac{\vartheta}{2}\right)^{m/2}} \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{\Gamma(k-m+1)\, k!} \sin^{2k}\left(\frac{\vartheta}{2}\right)$$

05.10.06.0006.01

$$Y_n^m(\vartheta, \varphi) = (-1)^m \sqrt{\frac{(2n+1)(m+n)!}{4\pi(n-m)!}} e^{i\varphi m} \tan\left(\frac{\vartheta}{2}\right)^m \sin^2(\vartheta)^{m/2} \sin(\vartheta)^{-m} \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(k+m)!\,k!} \sin\left(\frac{\vartheta}{2}\right)^{2k}$$

05.10.06.0007.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{(2n+1)(n-m)!}{\pi(m+n)!}} e^{i\varphi m} 2^{-m-1} \sin^2(\vartheta)^{m/2} \sum_{k=0}^{n-m} \frac{(-n)_{k+m} (n+1)_{k+m}}{(k+m)!\,k!} \sin\left(\frac{\vartheta}{2}\right)^{2k}$$

05.10.06.0008.01

$$Y_n^m(\vartheta, \varphi) = (-1)^m \sqrt{\frac{(2n+1)(m+n)!}{\pi(n-m)!}} e^{i\varphi m} 2^{m-1} \sin^2(\vartheta)^{-\frac{m}{2}} \sum_{k=0}^{m+n} \frac{(-n)_{k-m} (n+1)_{k-m}}{(k-m)!\,k!} \sin\left(\frac{\vartheta}{2}\right)^{2k}$$

Expansions at $\cos\left(\frac{\vartheta}{2}\right) = 0$

05.10.06.0009.01

$$Y_n^m(\vartheta, \varphi) = (-1)^{m+n} \sqrt{\frac{(2n+1)(n-m)!}{4\pi(m+n)!}} e^{i\varphi m} \csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \tan^m\left(\frac{\vartheta}{2}\right) \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(k-m)!\,k!} \cos\left(\frac{\vartheta}{2}\right)^{2k}$$

05.10.06.0010.01

$$Y_n^m(\vartheta, \varphi) = (-1)^n \sqrt{\frac{(2n+1)(m+n)!}{4\pi(n-m)!}} e^{i\varphi m} \csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \cot^m\left(\frac{\vartheta}{2}\right) \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(k+m)!\,k!} \cos\left(\frac{\vartheta}{2}\right)^{2k}$$

05.10.06.0011.01

$$Y_n^m(\vartheta, \varphi) = (-1)^{m+n} \sqrt{\frac{(2n+1)(n-m)!}{4\pi(m+n)!}} e^{i\varphi m} \sin^m\left(\frac{\vartheta}{2}\right) \cos^m\left(\frac{\vartheta}{2}\right) \csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=0}^{n-m} \frac{(-n)_{k+m} (n+1)_{k+m}}{(k+m)!\,k!} \cos\left(\frac{\vartheta}{2}\right)^{2k}$$

05.10.06.0012.01

$$Y_n^m(\vartheta, \varphi) = (-1)^n \sqrt{\frac{(2n+1)(m+n)!}{4\pi(n-m)!}} e^{i\varphi m} \csc^m\left(\frac{\vartheta}{2}\right) \sec^m\left(\frac{\vartheta}{2}\right) \csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=0}^{m+n} \frac{(-n)_{k-m} (n+1)_{k-m}}{(k-m)!\,k!} \cos\left(\frac{\vartheta}{2}\right)^{2k}$$

Expansions at $\tan\left(\frac{\vartheta}{2}\right) = 0$

05.10.06.0013.01

$$Y_n^m(\vartheta, \varphi) = (-1)^{\frac{m}{2}(\operatorname{sgn}(m)+1)} e^{i\varphi m} \sqrt{\frac{2n+1}{4\pi}} \sqrt{(m+n)!\,(n-m)!}$$
$$n!\cos^{2n}\left(\frac{\vartheta}{2}\right) \csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=0}^{n-|m|} \frac{(-1)^k}{k!\,(k+|m|)!\,(n-k)!\,(n-|m|-k)!} \tan^{2k+|m|}\left(\frac{\vartheta}{2}\right)$$

Expansions at $\cot\left(\frac{\vartheta}{2}\right) = 0$

05.10.06.0014.01

$$Y_n^m(\vartheta, \varphi) = (-1)^{\frac{m}{2}(\operatorname{sgn}(m)-1)+n} e^{i m \varphi} \sqrt{\frac{2n+1}{4\pi}} \sqrt{(n+m)!\,(n-m)!}$$
$$n!\,\sin^{2n}\left(\frac{\vartheta}{2}\right)\csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}}\sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}}\sum_{k=0}^{n-|m|}\frac{(-1)^k}{k!\,(k+|m|)!\,(n-k)!\,(n-|m|-k)!}\cot^{2k+|m|}\left(\frac{\vartheta}{2}\right)$$

Expansions at sin(ϑ) == 0

05.10.06.0015.01

$$Y_n^m(\vartheta, \varphi)\, Y_n^m(\vartheta, -\varphi) = \frac{2n+1}{4\pi}\sum_{k=|m|}^{n}(-1)^{k+m}\frac{(k+n)!}{(-k+n)!}\frac{(2k-1)!!}{(2k)!!}\frac{\sin^{2k}(\vartheta)}{(k-m)!\,(k+m)!}$$

Expansions at cos(ϑ) == 0

05.10.06.0016.01

$$Y_n^m(\vartheta, \varphi) = e^{i\varphi m}\sqrt{\frac{(2n+1)\,(n-m)!}{(4\pi)\,(n+m)!}}\sin^m(\vartheta)\csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}}\sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}}\sum_{k=\frac{(n-m)\bmod 2}{2}}^{\frac{n-m}{2}}(-1)^{\frac{n+m-2k}{2}}\frac{(n+m+2k-1)!!}{(n-m-2k)!!}\frac{\cos^{2k}(\vartheta)}{(2k)!}$$

Expansions at tan(ϑ) == 0

05.10.06.0017.01

$$Y_n^m(\vartheta, \varphi) = e^{i\varphi m}\sqrt{\frac{(2n+1)\,(n+m)!\,(n-m)!}{4\pi}}\cos^n(\vartheta)\csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}}\sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}}\sum_{k=\frac{|m|}{2}}^{\frac{n-(n-m)\bmod 2}{2}}\frac{(-1)^{\frac{2k+m}{2}}}{(2k+m)!!\,(2k-m)!!}\frac{\tan^{2k}(\vartheta)}{(n-2k)!}$$

Expansions at cot(ϑ) == 0

05.10.06.0018.01

$$Y_n^m(\vartheta, \varphi) = e^{i\varphi m}\sqrt{\frac{(2n+1)\,(n+m)!\,(n-m)!}{4\pi}}\sin^n(\vartheta)\csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}}\sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}}\sum_{k=\frac{(n-m)\bmod 2}{2}}^{\frac{n-|m|}{2}}\frac{(-1)^{\frac{n+m-2k}{2}}}{(n+m-2k)!!\,(n-m-2k)!!}\frac{\cot^{2k}(\vartheta)}{(2k)!}$$

Expansions at ϑ == 0

05.10.06.0019.01

$$Y_n^m(\vartheta, \varphi) \propto \sqrt{\frac{2n+1}{4\pi}}\frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}}2^m e^{i\varphi m}\left(\vartheta^2\right)^{-m/2}$$
$$\left(\frac{1}{\Gamma(1-m)}+\frac{1}{12}\left(-\frac{3n(n+1)}{\Gamma(2-m)}+\frac{1}{\Gamma(-m)}\right)\vartheta^2+\frac{1}{1440}\left(\frac{30n(n+1)(m+1)}{\Gamma(2-m)}+\frac{45(n-1)n(n+1)(n+2)}{\Gamma(3-m)}+\frac{7-5m}{\Gamma(-m)}\right)\vartheta^4+\right.$$
$$\frac{1}{362880}\left(-\frac{63n(n+1)(4+m(3+5m))}{\Gamma(2-m)}-\frac{945(n-1)n(n+1)(n+2)(m+2)}{\Gamma(3-m)}-\right.$$
$$\left.\left.\frac{945(n-1)n(n+1)(n+2)\left(n^2+n-6\right)}{\Gamma(4-m)}+\frac{124+7m(5m-21)}{\Gamma(-m)}\right)\vartheta^6+O\left(\vartheta^8\right)\right)/;\ (\vartheta\to 0)$$

05.10.06.0020.01

$$Y_n^m(\vartheta, \varphi) \propto \frac{1}{\Gamma(1-m)}\sqrt{\frac{2n+1}{\pi}}\frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}}2^{m-1}e^{i\varphi m}\left(\vartheta^2\right)^{-m/2}\left(1+O\left(\vartheta^2\right)\right)/;\ (\vartheta\to 0)\bigwedge -m\in\mathbb{N}$$

Expansions at cos(ϑ) == ∞

05.10.06.0021.02

$$Y_n^m(\vartheta,\varphi)\propto\frac{\sqrt{2n+1}\,\sqrt{\Gamma(n-m+1)}\,e^{i\varphi m}}{2\pi\sqrt{\Gamma(n+m+1)}}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{m/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{m/2}}\left(\frac{2^n(\cos(\vartheta)-1)^n}{\Gamma(n-m+1)}\Gamma\left(n+\frac{1}{2}\right)\left(1+\frac{m-n}{1-\cos(\vartheta)}+\frac{(1-n)(m-n)(1+m-n)}{(1-2n)(1-\cos(\vartheta))^2}+\ldots\right)+\frac{2^{-1-n}(z-1)^{-1-n}}{\Gamma(-n-m)}\Gamma\left(-n-\frac{1}{2}\right)\left(1+\frac{1+m+n}{1-\cos(\vartheta)}+\frac{(2+n)(1+m+n)(2+m+n)}{(3+2n)(\cos(\vartheta)-1)^2}+\ldots\right)\right)/;\ |\cos(\vartheta)|\to\infty$$

05.10.06.0022.01

$$Y_n^m(\vartheta,\varphi)=\frac{2^{n-1}\sqrt{2n+1}\,\Gamma\left(n+\frac{1}{2}\right)e^{i\varphi m}}{\pi\sqrt{\Gamma(n-m+1)}\,\sqrt{\Gamma(n+m+1)}}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{m/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{m/2}}(\cos(\vartheta)-1)^n\sum_{k=0}^{n}\frac{(-n)_k\,(m-n)_k}{(-2n)_k\,k!}\left(\frac{2}{1-\cos(\vartheta)}\right)^k$$

05.10.06.0023.01

$$Y_n^m(\vartheta,\varphi)=\frac{2^{n-1}\sqrt{2n+1}\,\Gamma\left(n+\frac{1}{2}\right)e^{i\varphi m}}{\pi\sqrt{\Gamma(n-m+1)}\,\sqrt{\Gamma(n+m+1)}}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{m/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{m/2}}(\cos(\vartheta)-1)^n\,{}_2F_1\left(-n,m-n;-2n;\frac{2}{1-\cos(\vartheta)}\right)$$

05.10.06.0024.02

$$Y_n^m(\vartheta,\varphi)\propto\frac{2^{n-1}\sqrt{2n+1}\,\Gamma\left(n+\frac{1}{2}\right)e^{i\varphi m}}{\pi\sqrt{\Gamma(n-m+1)}\,\sqrt{\Gamma(n+m+1)}}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{m/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{m/2}}\cos^n(\vartheta)\left(1+O\left[\frac{1}{\mathrm{Cos}[\vartheta]}\right]\right)/;\ (|\cos(\vartheta)|\to\infty)$$

05.10.06.0025.01

$$Y_n^m(\vartheta,\varphi)=\frac{e^{i\varphi m}\,2^{n-1}\,\Gamma\left(n+\frac{1}{2}\right)\sqrt{2n+1}}{\pi\sqrt{\Gamma(n-m+1)}\,\sqrt{\Gamma(n+m+1)}}(\cos(\vartheta)-1)^n\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{m/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{m/2}}\sum_{k=0}^{n-m}\frac{(m-n)_k\,(-n)_k}{k!\,(-2n)_k}\left(\frac{2}{1-\cos(\vartheta)}\right)^k$$

05.10.06.0026.01

$$Y_n^m(\vartheta,\varphi)\propto\frac{e^{i\varphi m}\,2^{n-1}\,\Gamma\left(n+\frac{1}{2}\right)\sqrt{2n+1}}{\pi^{3/2}\sqrt{\Gamma(n-m+1)}\,\sqrt{\Gamma(n+m+1)}}\cos^n(\vartheta)\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{m/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{m/2}}\left(1+O\left(\frac{1}{\cos(\vartheta)}\right)\right)/;\ (|\cos(\vartheta)|\to\infty)$$

In Cartesian coordinates

05.10.06.0027.01

$$Y_n^m(\vartheta,\varphi)=\sqrt{x^2+y^2+z^2}^{\,-n}\sqrt{\frac{(2n+1)(m+n)!\,(n-m)!}{4\pi}}\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}\frac{\delta_{i+j+k,n}\,\delta_{i-j,m}}{i!\,j!\,k!\,2^i\,2^j}(x-iy)^j(-x-iy)^i z^k\ /;$$

$$n\in\mathbb{N}\land m\in\mathbb{Z}\land -n\le m\le n\land x=\cos(\varphi)\sin(\vartheta)\land y=\sin(\varphi)\sin(\vartheta)\land z=\cos(\vartheta)$$

Integral representations

On the real axis

Of the direct function

05.10.07.0001.01

$$Y_n^m(\vartheta, \varphi) = \frac{1}{4\pi^{3/2} i^{3m} n!} \sqrt{(2n+1)(n-m)!(n+m)!} \int_0^{2\pi} (\cos(\vartheta) + i\cos(t-\varphi)\sin(\vartheta))^n e^{imt} \, dt$$

05.10.07.0002.01

$$Y_n^m(\vartheta, \varphi) = (-1)^m e^{im\varphi} \sqrt{\frac{2n+1}{4\pi} \frac{(n+m)!}{(n-m)!}} \frac{\csc^m(\vartheta)}{(m-1)!} \int_{\cos(\vartheta)}^1 P_n(t)(t-\cos(\vartheta))^{m-1} \, dt \,/;\, 0 \leq m \leq n$$

05.10.07.0003.01

$$Y_n^m(\vartheta, \varphi) = \frac{i^m}{2\pi^{3/2}} e^{im\varphi} \sqrt{(2n+1)(n+m)!(n-m)!} \frac{1}{n!} \int_0^{\pi} (\cos(\vartheta) + i\sin(\vartheta)\cos(t))^n \cos(mt) \, dt$$

05.10.07.0004.01

$$Y_n^m(\vartheta, \varphi) = \frac{i^m}{4\pi^{3/2}} \sqrt{(2n+1)(n+m)!(n-m)!} \frac{1}{n!} \int_0^{2\pi} (\cos(\vartheta) + i\sin(\vartheta)\cos(t-\varphi))^n e^{imt} \, dt$$

05.10.07.0005.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^m}{2\pi^{3/2}} e^{im\varphi} \sqrt{(2n+1)\frac{(n+m)!}{(n-m)!}} \frac{\sin(\vartheta)^m}{(2m-1)!!} \int_0^{\pi} (\cos(\vartheta) + i\sin(\vartheta)\cos(t))^{n-m} \sin(t)^{2m} \, dt \,/;\, 0 \leq m \leq n$$

05.10.07.0006.01

$$Y_n^m(\vartheta, \varphi) = (-1)^m e^{im\varphi} \sqrt{\frac{2n+1}{4\pi} \frac{1}{(n+m)!(n-m)!}} \int_0^{\infty} e^{-t\cos(\vartheta)} J_m(t\sin(\vartheta)) t^n \, dt \,/;\, \cos(\vartheta) > 0$$

Involving the direct function

05.10.07.0007.01

$$\left|Y_n^m(\vartheta, \varphi)\right|^2 = \frac{2n+1}{4\pi} \int_0^{\infty} J_m\left(\frac{t}{2}\sin(\vartheta)\right)^2 J_{2n+1}(t) \, dt \,/;\, \vartheta \in \mathbb{R} \bigwedge \varphi \in \mathbb{R} \bigwedge 0 < \sin(\vartheta) < 1$$

Multiple integral representations

05.10.07.0008.01

$$Y_n^m(\vartheta, \varphi) = e^{im\varphi} \sqrt{\frac{2n+1}{4\pi} \frac{(n+m)!}{(n-m)!}} \csc(\vartheta)^m \underbrace{\int_1^{\cos(\vartheta)} \dots \int_1^{t_3} \int_1^{t_2}}_{m} P_n(t_1) \, dt_1 \, dt_2 \dots dt_m \,/;\, 0 \leq m \leq n$$

Integral representations of negative integer order

05.10.07.0009.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^{n+m} r^{n+1}}{(n-m)!} \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^m \frac{\partial^{n-m} \frac{1}{r}}{\partial z^{n-m}} \,/;$$

$$r = \sqrt{x^2+y^2+z^2} \bigwedge x = r\cos(\varphi)\sin(\vartheta) \bigwedge y = r\sin(\varphi)\sin(\vartheta) \bigwedge z = r\cos(\vartheta) \bigwedge 0 \leq m \leq n$$

05.10.07.0010.01

$$Y_n^m(\vartheta, \varphi) = \frac{1}{2^n\, n!} \sqrt{\frac{(2n+1)(n+m)!}{4\pi(n-m)!}}\, e^{i m \varphi} \sin^2(\vartheta)^{-\frac{m}{2}} \frac{\partial^{n-m}(z^2-1)^n}{\partial z^{n-m}} /; z = \cos(\vartheta)$$

05.10.07.0011.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^m}{2^n\, n!} \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}\, e^{i m \varphi} \sin^2(\vartheta)^{\frac{m}{2}} \frac{\partial^{n+m}(z^2-1)^n}{\partial z^{n+m}} /; z = \cos(\vartheta)$$

05.10.07.0012.01

$$Y_n^m(\vartheta, \varphi) = \frac{e^{i m \varphi}}{2^{n+1}\sqrt{\pi}} \sqrt{\frac{2n+1}{(n-m)!\,(n+m)!}} \cos^m\left(\frac{\vartheta}{2}\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \cot^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^{-m}\left(\frac{\vartheta}{2}\right) \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \frac{\partial^n\left((z+1)^{n-m}(z-1)^{n+m}\right)}{\partial z^n} /; z = \cos(\vartheta)$$

05.10.07.0013.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^m e^{i m \varphi}}{2^{n+1}\sqrt{\pi}} \sqrt{\frac{2n+1}{(n-m)!\,(n+m)!}} \cos^{-m}\left(\frac{\vartheta}{2}\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \cot^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^{m}\left(\frac{\vartheta}{2}\right) \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \frac{\partial^n\left((z+1)^{n+m}(z-1)^{n-m}\right)}{\partial z^n} /; z = \cos(\vartheta)$$

05.10.07.0014.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}} 2^{-n} e^{i m \varphi} \sin^m(\vartheta) \frac{\partial^m P_n(z)}{\partial z^m} /; z = \cos(\vartheta) \land m \geq 0$$

05.10.07.0015.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{(2n+1)(n+m)!}{4\pi(n-m)!}} \frac{e^{i m \varphi}}{2^n\, n!} \frac{1}{\sin^2(\vartheta)^{m/2}} \frac{\partial^{n-m}(z^2-1)^n}{\partial z^{n-m}} /; z = \cos(\vartheta)$$

05.10.07.0016.01

$$Y_n^m(\vartheta, \varphi) = (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \frac{e^{i m \varphi}}{2^n\, n!} \sin^2(\vartheta)^{m/2} \frac{\partial^{n+m}(z^2-1)^n}{\partial z^{n+m}} /; z = \cos(\vartheta)$$

05.10.07.0017.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi(n+m)!\,(n-m)!}} \frac{e^{i m \varphi}}{2^n} \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \cot^m\left(\frac{\vartheta}{2}\right) \cot^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \frac{\partial^n\left((z+1)^{n-m}(z-1)^{n+m}\right)}{\partial z^n} /; z = \cos(\vartheta)$$

05.10.07.0018.01

$$Y_n^m(\vartheta, \varphi) = (-1)^m \sqrt{\frac{2n+1}{4\pi(n+m)!\,(n-m)!}} \frac{e^{i m \varphi}}{2^n} \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \cot^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \tan^m\left(\frac{\vartheta}{2}\right) \frac{\partial^n\left((z+1)^{n+m}(z-1)^{n-m}\right)}{\partial z^n} /; z = \cos(\vartheta)$$

05.10.07.0019.01

$$Y_n^m(\vartheta, \varphi) = (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}\, e^{i m \varphi} \sin^m(\vartheta) \frac{\partial^m P_n(z)}{\partial z^m} /; z = \cos(\vartheta) \land 0 \leq m \leq n$$

Generating functions

05.10.11.0001.01

$$\frac{1}{\left(w^2-2\cos(\vartheta)\,w+1\right)^{m+1/2}}=\frac{(-1)^m}{(2m-1)!!\,\sin(\vartheta)^m}\sum_{n=m}^{\infty}w^{n-m}\sqrt{\frac{4\pi}{2n+1}\,\frac{(n+m)!}{(n-m)!}}\;Y_n^m(\vartheta,0)\;/;\;m\geq 0\land\vartheta\in\mathbb{R}\land|w|<1$$

05.10.11.0002.01

$$\frac{1}{\left(w^2-2\cos(\vartheta)\,w+1\right)^{m+1/2}}=\frac{(-1)^m}{(2m-1)!!\,\sin(\vartheta)^m}\sum_{n=m}^{\infty}\frac{1}{w^{n+m+1}}\sqrt{\frac{4\pi}{2n+1}\,\frac{(n+m)!}{(n-m)!}}\;Y_n^m(\vartheta,0)\;/;\;m\geq 0\land\vartheta\in\mathbb{R}\land|w|>1$$

05.10.11.0003.01

$$\frac{\left(\left(1+\sqrt{1-2w\cos(\vartheta)+w^2}\right)^2-w^2\right)^{-m}}{\sqrt{1-2w\cos(\vartheta)+w^2}}=\frac{(-1)^m}{2^m\sin^m(\vartheta)}\sum_{n=m}^{\infty}w^{n-m}\,n!\sqrt{\frac{4\pi}{(2n+1)\,(n+m)!\,(n-m)!}}\;Y_n^m(\vartheta,0)\;/;$$

$$m\geq 0\land\vartheta\in\mathbb{R}\land|w|<1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

With respect to ϑ

05.10.13.0001.01

$$\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\left(\sin(\vartheta)\frac{\partial Y_n^m(\vartheta,\varphi)}{\partial\vartheta}\right)+\left(n(n+1)-\frac{m^2}{\sin^2(\vartheta)}\right)Y_n^m(\vartheta,\varphi)=0$$

With respect to φ

05.10.13.0002.01

$$i\,\frac{\partial Y_n^m(\vartheta,\varphi)}{\partial\varphi}+m\,Y_n^m(\vartheta,\varphi)=0$$

Partial differential equations

05.10.13.0003.01

$$e^{i\varphi}\left(\frac{\partial Y_n^m(\vartheta,\varphi)}{\partial\vartheta}+i\cot(\vartheta)\frac{\partial Y_n^m(\vartheta,\varphi)}{\partial\varphi}\right)=\sqrt{(n-m)(n+m+1)}\;Y_n^{m+1}(\vartheta,\varphi)$$

05.10.13.0004.01

$$\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\left(\sin(\vartheta)\frac{\partial Y_n^m(\vartheta,\varphi)}{\partial\vartheta}\right)+\frac{1}{\sin^2(\vartheta)}\frac{\partial^2 Y_n^m(\vartheta,\varphi)}{\partial\varphi^2}+n(n+1)\,Y_n^m(\vartheta,\varphi)=0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

05.10.16.0001.01

$$Y_n^{-m}(\vartheta, \varphi) = \overline{(-1)^m Y_n^m(\vartheta, \varphi)} \;/;\; \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

05.10.16.0002.01

$$Y_n^{-m}(\vartheta, \varphi) = (-1)^m e^{-2 i \varphi m} Y_n^m(\vartheta, \varphi)$$

05.10.16.0003.01

$$Y_n^m(-\vartheta, \varphi) = Y_n^m(\vartheta, \varphi)$$

05.10.16.0004.01

$$Y_n^m(\vartheta, -\varphi) = e^{-2 i m \varphi} Y_n^m(\vartheta, \varphi)$$

05.10.16.0005.01

$$Y_n^m(\vartheta, -\varphi) = (-1)^m Y_n^{-m}(\vartheta, \varphi)$$

05.10.16.0006.01

$$Y_n^m(-\vartheta, -\varphi) = (-1)^m Y_n^{-m}(\vartheta, \varphi)$$

Products, sums, and powers of the direct function

Products involving the direct function

Clebsch-Gordan series for product of two spherical harmonics

05.10.16.0007.01

$$Y_{n_1}^{m_1}(\vartheta, \varphi) Y_{n_2}^{m_2}(\vartheta, \varphi) = \sqrt{\frac{(2 n_1 + 1)(2 n_2 + 1)}{4 \pi}} \sum_{k=\max(|n_1 - n_2|, |m_1 + m_2|)}^{n_1 + n_2} \frac{1}{\sqrt{2 k + 1}} Y_k^{m_1 + m_2}(\vartheta, \varphi) \langle n_1\, n_2\, 0\, 0 \mid n_1\, n_2\, k\, 0 \rangle \langle n_1\, n_2\, m_1\, m_2 \mid n_1\, n_2\, k\, m_1 + m_2 \rangle \;/;$$
$$n_1 \in \mathbb{N} \land n_2 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land m_2 \in \mathbb{Z} \land |m_1| \leq n_1 \land |m_2| \leq n_2$$

Clebsch-Gordan double series for product of three spherical harmonics

05.10.16.0008.01

$$Y_{n_1}^{m_1}(\vartheta, \varphi) Y_{n_2}^{m_2}(\vartheta, \varphi) Y_{n_3}^{m_3}(\vartheta, \varphi) = \frac{\sqrt{(2 n_1 + 1)(2 n_2 + 1)(2 n_3 + 1)}}{4 \pi} \sum_{k_1=\max(|n_1 - n_2|, |m_1 + m_2|)}^{n_1 + n_2} \sum_{k_2=\max(|k_1 - n_3|, |m_1 + m_2 + m_3|)}^{k_1 + n_3} \frac{1}{\sqrt{2 k_2 + 1}} Y_{k_2}^{m_1 + m_2 + m_3}(\vartheta, \varphi)$$
$$\langle n_1\, n_2\, 0\, 0 \mid n_1\, n_2\, k_1\, 0 \rangle \langle k_1\, n_3\, 0\, 0 \mid k_1\, n_3\, k_2\, 0 \rangle \langle n_1\, n_2\, m_1\, m_2 \mid n_1\, n_2\, k_1\, m_1 + m_2 \rangle$$
$$\langle k_1\, n_3\, m_1 + m_2\, m_3 \mid k_1\, n_3\, k_2\, m_1 + m_2 + m_3 \rangle \;/;\; n_j \in \mathbb{N} \land m_j \in \mathbb{Z} \land |m_j| \leq n_j \land j \in \{1, 2, 3\}$$

Clebsch-Gordan multiple series for product of several spherical harmonics

05.10.16.0009.01

$$\prod_{j=1}^{p} Y_{n_j}^{m_j}(\vartheta, \varphi) == \frac{1}{(4\pi)^{\frac{p-1}{2}}} \prod_{j=1}^{p} \sqrt{2 n_j + 1} \sum_{k_1=\max(|n_1-n_2|,|M_2|)}^{n_1+n_2} \sum_{k_2=\max(|k_1-n_3|,|M_3|)}^{k_1+n_3} \cdots \sum_{k_{p-1}=\max(|k_{p-2}-n_p|,|M_p|)}^{k_{p-2}+n_p} \frac{1}{\sqrt{2 k_{p-1}+1}}$$

$$Y_{k_{p-1}}^{M_p}(\vartheta, \varphi) \prod_{j=1}^{p-1} \langle k_{j-1}\, n_{j+1}\, 0\, 0 \mid k_{j-1}\, n_{j+1}\, k_j\, 0\rangle \langle k_{j-1}\, n_{j+1}\, M_j\, m_{j+1} \mid k_{j-1}\, n_{j+1}\, k_j\, M_{j+1}\rangle \text{ /; }$$

$$p \in \mathbb{Z} \bigwedge p > 1 \bigwedge n_k \in \mathbb{N} \bigwedge m_k \in \mathbb{Z} \bigwedge |m_k| \leq n_k \bigwedge k_0 == n_1 \bigwedge M_0 == 0 \bigwedge M_j == \sum_{k=1}^{j} m_k$$

Identities

Recurrence identities

Consecutive neighbors

05.10.17.0001.01

$$Y_n^m(\vartheta, \varphi) == \sqrt{\frac{(2n+1)(2n+3)}{(n-m+1)(n+m+1)}} \cos(\vartheta)\, Y_{n+1}^m(\vartheta, \varphi) - \sqrt{\frac{(2n+1)(n-m+2)(n+m+2)}{(2n+5)(n+m+1)(n-m+1)}}\, Y_{n+2}^m(\vartheta, \varphi)$$

05.10.17.0002.01

$$Y_n^m(\vartheta, \varphi) == \sqrt{\frac{(2n-1)(2n+1)}{(n-m)(n+m)}} \cos(\vartheta)\, Y_{n-1}^m(\vartheta, \varphi) - \sqrt{\frac{(2n+1)(n-m-1)(n+m-1)}{(2n-3)(n-m)(n+m)}}\, Y_{n-2}^m(\vartheta, \varphi) \text{ /; } |m| < n$$

05.10.17.0003.01

$$Y_n^m(\vartheta, \varphi) ==$$

$$\frac{2(m+1)\sqrt{(n-m)(n+m+1)}}{(m(m+1)-n(n+1))} e^{-i\varphi} \cot(\vartheta)\, Y_n^{m+1}(\vartheta, \varphi) + \frac{\sqrt{(n-m)(n-m-1)(n+m+2)(n+m+1)}}{(m(m+1)-n(n+1))} e^{-2i\varphi}\, Y_n^{m+2}(\vartheta, \varphi)$$

05.10.17.0004.01

$$Y_n^m(\vartheta, \varphi) == \frac{2(1-m)}{\sqrt{(n+m)(n-m+1)}} e^{i\varphi} \cot(\vartheta)\, Y_n^{m-1}(\vartheta, \varphi) + \frac{((m-1)(m-2)-n(n+1))}{\sqrt{(n+m)(n+m-1)(n-m+2)(n-m+1)}} e^{2i\varphi}\, Y_n^{m-2}(\vartheta, \varphi) \text{ /; }$$

$$0 < m \leq n$$

Functional identities

Relations between contiguous functions

05.10.17.0005.01

$$Y_n^m(\vartheta, \varphi) == \sec(\vartheta)\left(\frac{(n+m)\sqrt{\Gamma(n+m)}\,\sqrt{\Gamma(n-m+1)}}{\sqrt{2n+1}\,\sqrt{2n-1}\,\sqrt{\Gamma(n-m)}\,\sqrt{\Gamma(n+m+1)}} Y_{n-1}^m(\vartheta, \varphi) + \right.$$

$$\left.\frac{(n-m+1)\sqrt{\Gamma(n-m+1)}\,\sqrt{\Gamma(n+m+2)}}{\sqrt{2n+1}\,\sqrt{2n+3}\,\sqrt{\Gamma(n+m+1)}\,\sqrt{\Gamma(n-m+2)}} Y_{n+1}^m(\vartheta, \varphi)\right)$$

05.10.17.0006.01

$$Y_n^m(\vartheta, \varphi) == -\frac{\tan(\vartheta)}{2m}\left(\frac{(n(n+1)-m(m-1))}{\sqrt{(n+m)(n-m+1)}} e^{i\varphi}\, Y_n^{m-1}(\vartheta, \varphi) + \sqrt{(n-m)(n+m+1)}\; e^{-i\varphi}\, Y_n^{m+1}(\vartheta, \varphi)\right) \text{ /; } -n+1 \leq m \leq n \land m \neq 0$$

Additional relations between contiguous functions

Below relations are correct only under some restrictions on the parameters

05.10.17.0007.01

$$Y_n^m(\vartheta,\varphi) = \csc(\vartheta)\, e^{i\varphi}\left(\sqrt{\frac{(n-m+1)(n-m+2)}{(2n+1)(2n+3)}}\; Y_{n+1}^{m-1}(\vartheta,\varphi) - \sqrt{\frac{(n+m-1)(n+m)}{(2n-1)(2n+1)}}\; Y_{n-1}^{m-1}(\vartheta,\varphi)\right)$$

05.10.17.0008.01

$$Y_n^m(\vartheta,\varphi) = \csc(\vartheta)\, e^{-i\varphi}\left(\sqrt{\frac{(n-m-1)(n-m)}{(2n-1)(2n+1)}}\; Y_{n-1}^{m+1}(\vartheta,\varphi) - \sqrt{\frac{(n+m+1)(n+m+2)}{(2n+1)(2n+3)}}\; Y_{n+1}^{m+1}(\vartheta,\varphi)\right)$$

05.10.17.0009.01

$$Y_n^m(\vartheta,\varphi) = e^{i\varphi}\csc(\vartheta)\left(\sqrt{\frac{n-m+1}{n+m}}\cos(\vartheta)\, Y_n^{m-1}(\vartheta,\varphi) - \sqrt{\frac{(2n+1)(n+m-1)}{(2n-1)(n+m)}}\; Y_{n-1}^{m-1}(\vartheta,\varphi)\right) /;\ -n+1 \le m \le n \land m \ne 0$$

05.10.17.0010.01

$$Y_n^m(\vartheta,\varphi) = e^{i\varphi}\csc(\vartheta)\left(\sqrt{\frac{(2n+1)(n-m+2)}{(2n+3)(n-m+1)}}\; Y_{n+1}^{m-1}(\vartheta,\varphi) - \cos(\vartheta)\sqrt{\frac{n+m}{n-m+1}}\; Y_n^{m-1}(\vartheta,\varphi)\right)$$

05.10.17.0011.01

$$Y_n^m(\vartheta,\varphi) = e^{-i\varphi}\csc(\vartheta)\left(\sqrt{\frac{(2n+1)(n-m-1)}{(2n-1)(n-m)}}\; Y_{n-1}^{m+1}(\vartheta,\varphi) - \cos(\vartheta)\sqrt{\frac{n+m+1}{n-m}}\; Y_n^{m+1}(\vartheta,\varphi)\right) /;\ -n \le m \le n-1$$

05.10.17.0012.01

$$Y_n^m(\vartheta,\varphi) = e^{-i\varphi}\csc(\vartheta)\left(\cos(\vartheta)\sqrt{\frac{n-m}{n+m+1}}\; Y_n^{m+1}(\vartheta,\varphi) - \sqrt{\frac{(2n+1)(n+m+2)}{(2n+3)(n+m+1)}}\; Y_{n+1}^{m+1}(\vartheta,\varphi)\right)$$

Relations of special kind

05.10.17.0013.01

$$Y_n^m(\vartheta,0) = e^{-i\varphi m}\, Y_n^m(\vartheta,\varphi)$$

05.10.17.0014.01

$$Y_n^m(\vartheta,0) = Y_n^m\left(\vartheta, \frac{2\pi p}{m}\right) /;\ p \in \mathbb{Z}$$

Complex characteristics

Conjugate value

05.10.19.0001.01

$$\overline{Y_n^m(\vartheta,\varphi)} = (-1)^m\, Y_n^{-m}(\vartheta,\varphi) /;\ \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to ϑ

05.10.20.0001.01

$$\frac{\partial Y_n^m(\vartheta, \varphi)}{\partial \vartheta} = m \cot(\vartheta) Y_n^m(\vartheta, \varphi) + \sqrt{(n-m)(n+m+1)}\, e^{-i \varphi} Y_n^{m+1}(\vartheta, \varphi)$$

05.10.20.0002.01

$$\frac{\partial Y_n^m(\vartheta, \varphi)}{\partial \vartheta} = \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}} e^{i \varphi m} \csc(\vartheta) \left(n \cos(\vartheta) P_n^m(\cos(\vartheta)) - (n+m) P_{n-1}^m(\cos(\vartheta))\right)$$

05.10.20.0003.01

$$\frac{\partial^2 Y_n^m(\vartheta, \varphi)}{\partial \vartheta^2} = m \left(m \cot^2(\vartheta) - \csc^2(\vartheta)\right) Y_n^m(\vartheta, \varphi) + \sqrt{(n-m)(m+n+1)}\, (2m+1) e^{-i \varphi} \cot(\vartheta) Y_n^{m+1}(\vartheta, \varphi) + \sqrt{(n-m)(n-m-1)(m+n+2)(m+n+1)}\, e^{-2 i \varphi} Y_n^{m+2}(\vartheta, \varphi)$$

05.10.20.0004.01

$$\frac{\partial^2 Y_n^m(\vartheta, \varphi)}{\partial \vartheta^2} = \frac{1}{2} \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}} e^{i \varphi m} \csc^2(\vartheta) \left(2(n+m)\cos(\vartheta) P_{n-1}^m(\cos(\vartheta)) + \left(2m^2 - 2n - 2n^2 \sin^2(\vartheta)\right) P_n^m(\cos(\vartheta))\right)$$

With respect to φ

05.10.20.0005.01

$$\frac{\partial Y_n^m(\vartheta, \varphi)}{\partial \varphi} = i\, m\, Y_n^m(\vartheta, \varphi)$$

05.10.20.0006.01

$$\frac{\partial^2 Y_n^m(\vartheta, \varphi)}{\partial \varphi^2} = -m^2 Y_n^m(\vartheta, \varphi)$$

Symbolic differentiation

With respect to φ

05.10.20.0007.02

$$\frac{\partial^p Y_n^m(\vartheta, \varphi)}{\partial \varphi^p} = (i\, m)^p Y_n^m(\vartheta, \varphi) /;\ p \in \mathbb{N}$$

Fractional integro-differentiation

With respect to φ

05.10.20.0008.01

$$\frac{\partial^\alpha Y_n^m(\vartheta, \varphi)}{\partial \varphi^\alpha} = \varphi^{-\alpha} (i\, \varphi\, m)^\alpha Q(-\alpha, 0, i\, \varphi\, m) Y_n^m(\vartheta, \varphi)$$

Integration

Indefinite integration

Involving only one direct function with respect to φ

05.10.21.0001.01

$$\int Y_n^m(\vartheta, \varphi)\, d\varphi = -\frac{i}{m} Y_n^m(\vartheta, \varphi)$$

Involving one direct function and elementary functions with respect to φ

Involving power function

05.10.21.0002.01

$$\int \varphi^{\alpha-1} Y_n^m(\vartheta, \varphi)\, d\varphi = -\varphi^{\alpha} (-i\, m\, \varphi)^{-\alpha} e^{-i \varphi m} \Gamma(\alpha, -i\, m\, \varphi)\, Y_n^m(\vartheta, \varphi)$$

Involving functions of the direct function and elementary functions with respect to ϑ

Involving elementary functions of the direct function and elementary functions

Involving products of the direct function and trigonometric functions

05.10.21.0003.01

$$\int \left((n-k)(k+n+1) - \frac{m^2 - l^2}{\sin^2(\vartheta)}\right) \sin(\vartheta)\, Y_k^l(\vartheta, \varphi)\, Y_n^m(\vartheta, \varphi)\, d\vartheta = (l-m) \cos(\vartheta)\, Y_n^m(\vartheta, \varphi)\, Y_k^l(\vartheta, \varphi) +$$
$$e^{-i \varphi} \sin(\vartheta) \left(\frac{\sqrt{\Gamma(k-l+1)}\, \sqrt{\Gamma(k+l+2)}}{\sqrt{\Gamma(k+l+1)}\, \sqrt{\Gamma(k-l)}} Y_n^m(\vartheta, \varphi)\, Y_k^{l+1}(\vartheta, \varphi) - \frac{\sqrt{\Gamma(n-m+1)}\, \sqrt{\Gamma(n+m+2)}}{\sqrt{\Gamma(n+m+1)}\, \sqrt{\Gamma(n-m)}} Y_n^{m+1}(\vartheta, \varphi)\, Y_k^l(\vartheta, \varphi)\right)$$

Definite integration

Involving the direct function

05.10.21.0004.01

$$\int_0^{\pi} \sin(\vartheta)\, Y_n^m(\vartheta, \varphi)^2\, d\vartheta = \frac{e^{2 i m \varphi}}{2 \pi} \,/;\, 0 \leq m \leq n$$

05.10.21.0005.01

$$\int_0^{\pi} \sin(\vartheta)\, Y_k^m(\vartheta, \varphi)\, Y_n^m(\vartheta, \varphi)\, d\vartheta = \frac{e^{2 i m \varphi}}{2 \pi} \delta_{n,k} \,/;\, k \in \mathbb{N} \land m \geq 0$$

05.10.21.0006.01

$$\int_0^{\pi} \csc(\vartheta)\, Y_n^m(\vartheta, \varphi)\, \overline{Y_n^{m_1}(\vartheta, \varphi)}\, d\vartheta = \frac{2n+1}{4 \pi m} \delta_{m,m_1} \,/;\, n > 0 \land m_1 \in \mathbb{N} \land 0 \leq m \leq n \land m_1 \leq n$$

05.10.21.0007.01

$$\int_0^{\frac{\pi}{2}} \sin^{m+1}(\vartheta) \cos^p(\vartheta)\, Y_n^m(\vartheta, \varphi)\, d\vartheta = (-1)^m \sqrt{\frac{2n+1}{4\pi} \frac{(n+m)!}{(n-m)!}} \frac{p!\, e^{i m \varphi}}{(p+n+m+1)!!\, (p-n+m)!!} \,/;\, p \in \mathbb{N} \land m \geq 0$$

05.10.21.0008.01

$$\int_a^b \left((n_1 - n_2)(n_1 + n_2 + 1) - \frac{m_1^2 - m_2^2}{1 - z^2}\right) Y_{n_1}^{m_1}\left(\cos^{-1}(z), \varphi\right) Y_{n_2}^{m_2}\left(\cos^{-1}(z), \varphi\right) dz = \left(b\, Y_{n_1}^{m_1}\left(\cos^{-1}(b), \varphi\right) Y_{n_2}^{m_2}\left(\cos^{-1}(b), \varphi\right)(n_1 - n_2) + Y_{n_1-1}^{m_2}\left(\cos^{-1}(b), \varphi\right) Y_{n_1}^{m_1}\left(\cos^{-1}(b), \varphi\right)(n_2 + m_2)\right) - \left(a\, Y_{n_1}^{m_1}\left(\cos^{-1}(a), \varphi\right) Y_{n_2}^{m_2}\left(\cos^{-1}(a), \varphi\right)(n_1 - n_2) + Y_{n_1-1}^{m_2}\left(\cos^{-1}(a), \varphi\right) Y_{n_1}^{m_1}\left(\cos^{-1}(a), \varphi\right)(n_2 + m_2)\right) /; n_1 \in \mathbb{N} \land n_2 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land m_2 \in \mathbb{Z} \land a \in \mathbb{R} \land b \in \mathbb{R} \land |m_1| \le n_1 \land |m_2| \le n_2$$

Multiple integration

05.10.21.0009.01

$$\int_0^{\pi} \int_0^{2\pi} \sin(\vartheta)\, Y_n^m(\vartheta, \varphi)\, d\varphi\, d\vartheta = \delta_{n,0}\, \delta_{m,0}$$

05.10.21.0010.01

$$\int_0^{\pi} \int_0^{2\pi} \sin(\vartheta)\, Y_{n_1}^{m_1}(\vartheta, \varphi)\, Y_{n_2}^{m_2}(\vartheta, \varphi)\, Y_{n_3}^{m_3}(\vartheta, -\varphi)\, d\varphi\, d\vartheta = \sqrt{\frac{(2 n_1 + 1)(2 n_2 + 1)}{4 \pi (2 n_3 + 1)}}\, \langle n_1\, n_2\, 0\, 0 \mid n_1\, n_2\, n_3\, 0\rangle \langle n_1\, n_2\, m_1\, m_2 \mid n_1\, n_2\, n_3\, m_3\rangle$$

05.10.21.0011.01

$$\int_0^{\pi} \int_0^{2\pi} \sin(\vartheta)\, Y_{n_1}^{m_1}(\vartheta, \varphi)\, Y_{n_2}^{m_2}(\vartheta, \varphi)\, \overline{Y_{n_3}^{m_3}(\vartheta, \varphi)}\, d\varphi\, d\vartheta = \sqrt{\frac{(2 n_1 + 1)(2 n_2 + 1)}{4 \pi (2 n_3 + 1)}}\, \langle n_1\, n_2\, 0\, 0 \mid n_1\, n_2\, n_3\, 0\rangle \langle n_1\, n_2\, m_1\, m_2 \mid n_1\, n_2\, n_3\, m_3\rangle /; n_1 \in \mathbb{N} \land n_2 \in \mathbb{N} \land n_3 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land m_2 \in \mathbb{Z} \land m_3 \in \mathbb{Z} \land |m_1| \le n_1 \land |m_2| \le n_2 \land |m_3| \le n_3$$

05.10.21.0012.01

$$\int_0^{\pi} \int_0^{2\pi} \sin(\vartheta)\, Y_{n_1}^{m_1}(\vartheta, \varphi)\, Y_{n_2}^{m_2}(\vartheta, \varphi)\, Y_{n_3}^{m_3}(\vartheta, \varphi)\, d\varphi\, d\vartheta = \sqrt{\frac{(2 n_1 + 1)(2 n_2 + 1)(2 n_3 + 1)}{4 \pi}} \begin{pmatrix} n_1 & n_2 & n_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \end{pmatrix} /; n_1 \in \mathbb{N} \land n_2 \in \mathbb{N} \land n_3 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land m_2 \in \mathbb{Z} \land m_3 \in \mathbb{Z} \land |m_1| \le n_1 \land |m_2| \le n_2 \land |m_3| \le n_3$$

Orthonormality relations:

05.10.21.0013.01

$$\int_0^{\pi} \int_0^{2\pi} \sin(\vartheta)\, \overline{Y_{n_1}^{m_1}(\vartheta, \varphi)}\, Y_{n_2}^{m_2}(\vartheta, \varphi)\, d\varphi\, d\vartheta = \delta_{n_1,n_2}\, \delta_{m_1,m_2} /; n_1 \in \mathbb{Z} \land n_2 \in \mathbb{Z} \land m_1 \in \mathbb{Z} \land m_2 \in \mathbb{Z}$$

05.10.21.0014.01

$$\int_0^{\pi} \int_0^{2\pi} \sin(\vartheta)\, Y_{n_1}^{m_1}(\vartheta, \varphi)\, Y_{n_2}^{m_2}(\vartheta, -\varphi)\, d\varphi\, d\vartheta = (-1)^{m_2}\, \delta_{m_1,m_2}\, \delta_{n_1,n_2} /; n_1 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land |m_1| \le n_1$$

05.10.21.0015.01

$$\int_0^{\pi} \int_0^{2\pi} \sin(\vartheta)\, Y_{n_1}^{m_1}(\vartheta, \varphi)\, Y_{n_2}^{m_2}(\vartheta, \varphi)\, d\varphi\, d\vartheta = (-1)^{m_2}\, \delta_{m_1,-m_2}\, \delta_{n_1,n_2} /; n_1 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land |m_1| \le n_1$$

Summation

Finite summation

Involving the direct function

05.10.23.0001.01

$$\sum_{m=-n}^{n} \overline{Y_n^m(\vartheta_1, \varphi_1)}\, Y_n^m(\vartheta_2, \varphi_2) = \frac{2n+1}{4\pi} P_n(\cos(\vartheta_1)\cos(\vartheta_2) + \cos(\varphi_1 - \varphi_2)\sin(\vartheta_1)\sin(\vartheta_2)) \;/;\; \vartheta_k \in \mathbb{R} \land \varphi_k \in \mathbb{R} \land k \in \{1, 2\}$$

05.10.23.0002.01

$$\sum_{m=-n}^{n} \left|Y_n^m(\vartheta, \varphi)\right|^2 = \frac{2n+1}{4\pi} \;/;\; \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

05.10.23.0003.01

$$\sum_{m=-n}^{n} m \left|Y_n^m(\vartheta, \varphi)\right|^2 = 0 \;/;\; \varphi \in \mathbb{R}$$

05.10.23.0004.01

$$\sum_{m=-n}^{n} m^2 \left|Y_n^m(\vartheta, \varphi)\right|^2 = \frac{n(n+1)(2n+1)}{8\pi} \sin^2(\vartheta) \;/;\; \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

05.10.23.0005.01

$$\sum_{m=-n}^{n} \sqrt{\left(n^2 - m^2\right)\left((n+1)^2 - m^2\right)}\; Y_{n-1}^m(\vartheta, \varphi)\, \overline{Y_{n+1}^m(\vartheta, \varphi)} = \frac{n(n+1)}{8\pi} \sqrt{(2n-1)(2n+3)} \left(3\cos^2(\vartheta) - 1\right) \;/;\; \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

05.10.23.0006.01

$$\sum_{m=-n}^{n} \frac{i^m}{\sqrt{(n-m)!\,(n+m)!}} Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}} \frac{(\cos(\vartheta) + i\sin(\vartheta)\cos(\varphi))^n}{n!} \;/;\; \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

05.10.23.0007.01

$$\sum_{l=m}^{m+p} \frac{(-1)^l\, w^{p-l+m}}{(p-l+m)!\,\sqrt{2l+1}\,\sqrt{(l-m)!\,(l+m)!}} Y_l^m(\vartheta, \varphi) =$$

$$(-1)^p \frac{(2m-1)!!}{2\sqrt{\pi}\,(2m+p)!} \left(\sin(\vartheta)\, e^{i\varphi}\right)^m \left(1 - 2w\cos(\vartheta) + w^2\right)^{p/2} C_p^{m+1/2}\!\left(\frac{\cos(\vartheta) - w}{\sqrt{1 - 2w\cos(\vartheta) + w^2}}\right) \;/;\; m \geq 0$$

Involving Clebsch-Gordan functions

The inverse Clebsch-Gordan series:

05.10.23.0008.01

$$\sum_{k=\max(M-n_2,-n_1)}^{\min(M+n_2,n_1)} \langle n_1\, n_2\, k\, M-k \mid n_1\, n_2\, L\, M\rangle\, Y_{n_1}^k(\vartheta, \varphi)\, Y_{n_2}^{M-k}(\vartheta, \varphi) = \sqrt{\frac{(2n_1+1)(2n_2+1)}{4\pi(2L+1)}} \langle n_1\, n_2\, 0\, 0 \mid n_1\, n_2\, L\, 0\rangle\, Y_L^M(\vartheta, \varphi) \;/;$$

$$n_1 \in \mathbb{N} \land n_2 \in \mathbb{N} \land L \in \mathbb{Z} \land M \in \mathbb{Z} \land |n_1 - n_2| \leq L \leq n_1 + n_2 \land -L \leq M \leq L$$

Infinite summation

05.10.23.0009.01

$$\sum_{n=m}^{\infty} \frac{Y_n^m(\vartheta,\varphi)\, w^{n-m}}{n!\,\sqrt{(2n+1)(n-m)!\,(m+n)!}} = \frac{\left(-\sin(\vartheta)\, e^{i\varphi}\right)^m}{2^{m+1}\sqrt{\pi}\; m!^2}\, {}_0F_1\!\left(;m+1; w\cos^2\!\left(\frac{\vartheta}{2}\right)\right) {}_0F_1\!\left(;m+1; -w\sin^2\!\left(\frac{\vartheta}{2}\right)\right) /; m \geq 0 \land \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R} \land |w| < 1$$

05.10.23.0010.01

$$\sum_{n=m}^{\infty} \frac{(n-m+p)!}{\sqrt{(2n+1)(n+m)!\,(n-m)!}}\, Y_n^m(\vartheta,\varphi)\, w^{n-m} = \frac{p!}{2^{m+1}\sqrt{\pi}\; m!}\, \frac{\left(-\sin(\vartheta)\, e^{i\varphi}\right)^m}{(1-w\cos(\vartheta))^{p+1}}\, {}_2F_1\!\left(\frac{p+1}{2}, \frac{p}{2}+1; m+1; -\left(\frac{w\sin(\vartheta)}{1-w\cos(\vartheta)}\right)^2\right) /; m \geq 0 \land p \in \mathbb{N} \land \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R} \land |w| < 1$$

05.10.23.0011.01

$$\sum_{n=m}^{\infty} \frac{(n+m-p)!\,(n-m+p-1)!}{n!\,\sqrt{(2n+1)(n-m)!\,(n+m)!}}\, Y_n^m(\vartheta,\varphi)\, w^{n-m} = \frac{(2m-p)!\,(p-1)!}{2^{m+1}\sqrt{\pi}\; m!^2} \left(-\sin(\vartheta)\, e^{i\varphi}\right)^m {}_2F_1\!\left(p, 2m-p+1; m+1; \frac{1-w-\sqrt{1-2w\cos(\vartheta)+w^2}}{2}\right) {}_2F_1\!\left(p, 2m-p+1; m+1; \frac{1+w-\sqrt{1-2w\cos(\vartheta)+w^2}}{2}\right) /; m \geq 0 \land p \in \mathbb{N} \land p \leq 2m \land \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R} \land |w| < 1$$

05.10.23.0012.01

$$\sum_{n=m}^{\infty} \sqrt{\frac{(n-m)!}{(2n+1)(n+m)!}}\, L_{n-m}^{2m}(z)\, Y_n^m(\vartheta,\varphi)\, w^{n-m} = \frac{1}{2^{m+1}\sqrt{\pi}\; m!}\, \frac{\left(-\sin(\vartheta)\, e^{i\varphi}\right)^m}{\left(1-2w\cos(\vartheta)+w^2\right)^{m+1/2}} \exp\!\left(-\frac{z\,w\,(\cos(\vartheta)-w)}{1-2w\cos(\vartheta)+w^2}\right) {}_0F_1\!\left(;m+1; -\frac{z^2 w^2 \sin^2(\vartheta)}{4\left(1-2w\cos(\vartheta)+w^2\right)^2}\right) /; m \geq 0 \land \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R} \land |w| < 1$$

05.10.23.0013.01

$$\sum_{n=m}^{\infty} i^{n-m} \sqrt{\frac{(2n+1)(n+m)!}{(n-m)!}}\, J_{n+1/2}(w)\, Y_n^m(\vartheta,\varphi) = \frac{\sqrt{w}\,\left(-w\sin(\vartheta)\, e^{i\varphi}\right)^m}{\sqrt{2}\;\pi}\, e^{i w\cos(\vartheta)} /; m \geq 0 \land \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

05.10.23.0014.01

$$\sum_{\substack{n=m,\\ \Delta n=2}}^{\infty} i^{n-m} \sqrt{\frac{(2n+1)(n+m)!}{(n-m)!}}\, J_{n+1/2}(w)\, Y_n^m(\vartheta,\varphi) = \frac{\sqrt{w}}{\sqrt{2}\;\pi} \left(-w\sin(\vartheta)\, e^{i\varphi}\right)^m \cos(w\cos(\vartheta)) /; m \geq 0 \land \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

05.10.23.0015.01

$$\sum_{\substack{n=m+1,\\ \Delta n=2}}^{\infty} i^{n-m-1} \sqrt{\frac{(2n+1)(n+m)!}{(n-m)!}}\, J_{n+1/2}(w)\, Y_n^m(\vartheta,\varphi) = \frac{\sqrt{w}}{\sqrt{2}\;\pi} \left(-w\sin(\vartheta)\, e^{i\varphi}\right)^m \sin(w\cos(\vartheta)) /; m \geq 0 \land \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

05.10.23.0016.01

$$\sum_{n=m}^{\infty} i^{n-m} J_{n+1/2}(w) Y_n^m(\vartheta_1, \varphi_1) \overline{Y_n^m(\vartheta_2, \varphi_2)} = \frac{\sqrt{2w}}{4\pi^{3/2}} J_m(w \sin(\vartheta_1)\sin(\vartheta_2)) e^{i w \cos(\vartheta_1)\cos(\vartheta_2)} e^{i m (\varphi_1 - \varphi_2)} /; m \geq 0 \land \vartheta_k \in \mathbb{R} \land \varphi_k \in \mathbb{R} \land k \in \{1, 2\}$$

Multiple infinite summation

Completeness relation:

05.10.23.0017.01

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \overline{Y_n^m(\vartheta_1, \varphi_1)} Y_n^m(\vartheta_2, \varphi_2) = \delta(\varphi_1 - \varphi_2)\,\delta(\cos(\vartheta_1) - \cos(\vartheta_2)) /; \vartheta_k \in \mathbb{R} \land \varphi_k \in \mathbb{R} \land k \in \{1, 2\}$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

05.10.26.0001.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}} e^{i \varphi m} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{m/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{m/2}} {}_2\tilde{F}_1\left(-n, n+1; 1-m; \sin^2\left(\frac{\vartheta}{2}\right)\right)$$

Involving ${}_2F_1$

05.10.26.0002.01

$$Y_n^m(\vartheta, \varphi) = \frac{1}{2\,\Gamma(1-m)} \sqrt{\frac{2n+1}{\pi}} \frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}} e^{i \varphi m} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{m/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{m/2}} {}_2F_1\left(-n, n+1; 1-m; \sin^2\left(\frac{\vartheta}{2}\right)\right) /; m \notin \mathbb{N}^+$$

05.10.26.0003.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^m}{2\,m!} \sqrt{\frac{(2n+1)(m+n)!}{\pi (n-m)!}} e^{i \varphi m} \tan^m\left(\frac{\vartheta}{2}\right) \csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} {}_2F_1\left(-n, n+1; m+1; \sin^2\left(\frac{\vartheta}{2}\right)\right) /; m \geq 0$$

05.10.26.0004.01

$$Y_n^m(\vartheta, \varphi) = \frac{2^{n-1} \sqrt{2n+1}\, \Gamma\left(n+\frac{1}{2}\right) e^{i \varphi m}}{\pi \sqrt{\Gamma(n-m+1)} \sqrt{\Gamma(n+m+1)}} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{m/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{m/2}} (\cos(\vartheta) - 1)^n {}_2F_1\left(-n, m-n; -2n; \frac{2}{1-\cos(\vartheta)}\right)$$

05.10.26.0005.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^{\frac{m}{2}(\operatorname{sgn}(m)+1)} e^{i m \varphi}}{|m|!\, 2^{|m|+1}} \sqrt{\frac{(2n+1)(n+|m|)!}{\pi (n-|m|)!}} \sin^2(\vartheta)^{\frac{|m|}{2}} {}_2F_1\left(|m|-n, n+|m|+1; |m|+1; \sin^2\left(\frac{\vartheta}{2}\right)\right)$$

05.10.26.0006.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^{n-m} (-1)^{\frac{m}{2}(\operatorname{sgn}(m)+1)} (2n)!\, e^{i m \varphi}}{2^{|m|+1} n!} \sqrt{\frac{2n+1}{\pi (n+m)! (n-m)!}} \sin^{2n-2|m|}\left(\frac{\vartheta}{2}\right) \sin^2(\vartheta)^{\frac{|m|}{2}} {}_2F_1\left(-n, |m|-n; -2n; \csc^2\left(\frac{\vartheta}{2}\right)\right) /; n > 0$$

05.10.26.0007.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^{n-m} (-1)^{\frac{m}{2}(\operatorname{sgn}(m)+1)} e^{i m \varphi}}{|m|!\, 2^{|m|+1}} \sqrt{\frac{(2n+1)(n+|m|)!}{\pi (n-|m|)!}} \sin^2(\vartheta)^{\frac{|m|}{2}} {}_2F_1\left(|m|-n, n+|m|+1; |m|+1; \cos^2\left(\frac{\vartheta}{2}\right)\right)$$

05.10.26.0008.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^{\frac{m}{2}(\operatorname{sgn}(m)+1)} (2n)!\, e^{i m \varphi}}{2^{|m|+1} n!} \sqrt{\frac{2n+1}{\pi (n+m)! (n-m)!}} \cos^{2n-2|m|}\left(\frac{\vartheta}{2}\right) \sin^2(\vartheta)^{\frac{|m|}{2}} {}_2F_1\left(-n, |m|-n; -2n; \sec^2\left(\frac{\vartheta}{2}\right)\right) /; n > 0$$

05.10.26.0009.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^{\frac{m}{2}(\operatorname{sgn}(m)+1)} e^{i m \varphi}}{2^{|m|+1} |m|!} \sqrt{\frac{(2n+1)(n+|m|)!}{\pi (n-|m|)!}} \cos^{2n-2|m|}\left(\frac{\vartheta}{2}\right) \sin^2(\vartheta)^{\frac{|m|}{2}} {}_2F_1\left(|m|-n, -n; |m|+1; -\tan^2\left(\frac{\vartheta}{2}\right)\right)$$

05.10.26.0010.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^{n-m} (-1)^{\frac{1}{2} m (\operatorname{sgn}(m)+1)} e^{i m \varphi}}{2^{|m|+1} |m|!} \sqrt{\frac{(2n+1)(n+|m|)!}{\pi (n-|m|)!}} \sin^{2n-2|m|}\left(\frac{\vartheta}{2}\right) \sin^2(\vartheta)^{\frac{|m|}{2}} {}_2F_1\left(|m|-n, -n; |m|+1; -\cot^2\left(\frac{\vartheta}{2}\right)\right)$$

Through Meijer *G*

Classical cases

05.10.26.0011.01

$$Y_n^m(\vartheta, \varphi) = -\frac{1}{\pi} \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}} e^{i \varphi m} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{m/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{m/2}} \lim_{\nu \to n} \sin(\pi \nu)\, G_{2,2}^{1,2}\left(-\sin^2\left(\frac{\vartheta}{2}\right) \middle| \begin{matrix} \nu+1, -\nu \\ 0, m \end{matrix}\right) /; n \in \mathbb{Z}$$

Through other functions

Involving Legendre functions

05.10.26.0012.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}} e^{i \varphi m} P_n^m(\cos(\vartheta))$$

05.10.26.0013.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-m+1)}}{\sqrt{\Gamma(n+m+1)}} e^{i \varphi m} (-\cos(\vartheta)-1)^{-\frac{m}{2}} (\cos(\vartheta)+1)^{m/2}\, \mathbb{P}_n^m(\cos(\vartheta))$$

Involving some hypergeometric-type functions

05.10.26.0014.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}} \frac{\Gamma(n+1)\, e^{i m \varphi}}{\sqrt{\Gamma(n-m+1)}\, \sqrt{\Gamma(n+m+1)}} \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \cot^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \tan^m\left(\frac{\vartheta}{2}\right) P_n^{(-m,m)}(\cos(\vartheta))$$

05.10.26.0015.01

$$Y_n^m(\vartheta,\varphi) = \frac{2^{-2m-1}\, e^{i m\varphi} \sqrt{2n+1}\, \Gamma\left(\frac{1}{2}-m\right) \sqrt{\Gamma(n+m+1)}}{\pi \sqrt{\Gamma(n-m+1)}} \csc^2\left(\frac{\vartheta}{2}\right)^{m/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \tan^m\left(\frac{\vartheta}{2}\right) C_{n+m}^{\frac{1}{2}-m}(\cos(\vartheta))$$

Representations through equivalent functions

With related functions

Involving Wigner-D functions

05.10.27.0001.01

$$Y_n^m(\vartheta,\varphi) = \sqrt{\frac{2n+1}{4\pi}}\, D_{0,-m}^n(0,\vartheta,\varphi)$$

05.10.27.0002.01

$$Y_n^m(\vartheta,\varphi) = (-1)^m \sqrt{\frac{2n+1}{4\pi}}\, \overline{D_{0,m}^n(0,\vartheta,\varphi)}$$

05.10.27.0003.01

$$Y_n^m(\vartheta,\varphi) = (-1)^m \sqrt{\frac{2n+1}{4\pi}}\, D_{-m,0}^n(\varphi,\vartheta,0)$$

05.10.27.0004.01

$$Y_n^m(\vartheta,\varphi) = \sqrt{\frac{2n+1}{4\pi}}\, \overline{D_{m,0}^n(\varphi,\vartheta,0)}$$

Zeros

When $Y_n^m(\vartheta,\varphi)$ is not identically zero, it possesses a finite number of zeros in the interval $0<\vartheta<\pi$, all of which are nondegenerate.

For integers m and n with $n \geq |m|$, the function $Y_n^m(\vartheta,\varphi)$ has $n-|m|$ zeros in the interval $0<\vartheta<\pi$. If $m \neq 0$, there are also two more zeros at $\vartheta = 0, \pi$. All of these zeros are symmetric about $\vartheta = \pi/2$.

Theorems

Eigenfunction to the angular part of the Laplace operator in spherical coordinates

The function $Y_n^m(\vartheta,\varphi)$ is an eigenfunction to the angular part L^2 of the Laplace operator in spherical coordinates $-L^2 = \frac{1}{\sin(\vartheta)^2}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\left(\sin(\vartheta)\frac{\partial}{\partial\vartheta}\right)$ with eigenvalue $n(n+1)$.

Eigenfunction of the z-component of the quantum mechanical angular momentum operator

The function $Y_n^m(\vartheta,\varphi)$ is an eigenfunction to the z-component of the quantum mechanical angular momentum operator $\hat{L}_z = -i\,\partial/\partial\varphi$ with eigenvalue m.

Eigenfunctions of the curl operator in spherical coordinates

The function $\boldsymbol{u}(r, \vartheta, \varphi) == \lambda\, \boldsymbol{r} \times \nabla \psi(r, \vartheta, \varphi) + \nabla \times (\boldsymbol{r} \times \nabla \psi(r, \vartheta, \varphi))$ with $\psi(r, \vartheta, \varphi) == g(r)\, Y_n^m(\vartheta, \varphi)$ and $g(r) == \frac{\sqrt{2}\, i}{\lambda \sqrt{n(n+1)}} \left(c_1 \frac{J_{n+1/2}(\lambda r)}{\sqrt{\lambda r}} + \frac{Y_{n+1/2}(\lambda r)}{\sqrt{\lambda r}} \right)$ are eigenfunctions of the curl operator in spherical coordinates $\nabla \times \boldsymbol{u}(r, \vartheta, \varphi) == \lambda\, \boldsymbol{u}(r, \vartheta, \varphi)$.

Multiple expansion theorem

Any function $f(\vartheta, \varphi)$ that is square integrable over $0 \le \vartheta \le \pi$, $0 \le \varphi \le 2\pi$ can be expanded in a series of spherical harmonics $Y_n^m(\vartheta, \varphi)$, with series coefficients $a_{n,m}$ called the multipole moments:

$$f(\vartheta, \varphi) == \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m}\, Y_n^m(\vartheta, \varphi) \;/;\; a_{n,m} == \int_0^{2\pi} \int_0^{\pi} \sin(\vartheta)\, \overline{Y_n^m(\vartheta, \varphi)}\, f(\vartheta, \varphi)\, d\vartheta\, d\varphi.$$

Expansion of two-point distances

The power $|\mathbf{r}_1 - \mathbf{r}_2|^n$ of the distance between two points $\mathbf{r}_1$ and $\mathbf{r}_2$ can be expanded in the following way (assuming $|\mathbf{r}_1| < |\mathbf{r}_2|$ and ω is the angle betwen $\mathbf{r}_1$ and $\mathbf{r}_2$) (Sack 1964):

$$|\mathbf{r}_1 - \mathbf{r}_2|^n = \sum_{l=0}^{\infty} r_2^n \frac{\left(-\frac{n}{2}\right)_l}{\left(\frac{1}{2}\right)_l} \left(\frac{r_1}{r_2}\right)^l {}_2F_1\left(l - \frac{n}{2}, -\frac{1}{2}(n+1); l + \frac{3}{2}; \left(\frac{r_1}{r_2}\right)^2\right) P_l(\cos(\omega))$$

History

A.M. Legendre (1785); P.S. Laplace (1785) gave the name “spherical harmonic”; K.F. Gauss (1828).

References

L.C. Biedenharn and J.D. louck, *Angular Momentum in Quantum Physics*, Addison-Wesley, Reading, 1981.

L.C. Biedenharn and J.D. louck, *The Racah-Wigner Algebra in Quantum Theory*, Addison-Wesley, Reading, 1981.

E.W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, Cambridge, 1955.

M.E. Rose, *Elementary Theory of Angular Momentum*, Dover, New York, 1995.

D.A. Varshalovich, A.N. Moskalev and V.K. Khersonskii, *Quantum Theory of Angular Momentum*, World Scientific, Singapore, 1988.

Copyright