SphericalHarmonicYGeneral

Notations

Traditional name

Spherical harmonic function

Traditional notation

$Y_{\lambda}^{\mu}(\vartheta, \varphi)$

Mathematica StandardForm notation

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SphericalHarmonicY[λ, μ, ϑ, φ]
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Primary definition

07.37.02.0001.01

$$Y_{\lambda}^{\mu}(\vartheta, \varphi) = \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} e^{i\varphi\mu} P_{\lambda}^{\mu}(\cos(\vartheta)) \,/;\, \neg(\lambda \in \mathbb{N} \bigwedge \mu \in \mathbb{Z} \bigwedge \lambda < |\mu|)$$

07.37.02.0002.01

$$Y_{n}^{m}(\vartheta, \varphi) = 0 \,/;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land n < |m|$$

Specific values

Specialized values

For fixed λ, μ, ϑ

07.37.03.0001.01

$$Y_{\lambda}^{\mu}(\vartheta, 0) = e^{-i\varphi\mu} Y_{\lambda}^{\mu}(\vartheta, \varphi)$$

For fixed λ, μ, φ

07.37.03.0002.01

$$Y_{\lambda}^{\mu}(0, \varphi) = 0 \,/;\, \mathrm{Re}(\mu) < 0 \bigwedge \mu \notin \mathbb{Z}$$

07.37.03.0003.01

$$Y_{n}^{m}(0, \varphi) = 0 \,/;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n \land m \neq 0$$

07.37.03.0004.01

$$Y_{\lambda}^{\mu}(\pi, \varphi) = \tilde{\infty} \,/;\, \mathrm{Re}(\mu) < 0 \bigwedge \mu \notin \mathbb{Z}$$

07.37.03.0005.01

$$Y_n^m(\pi, \varphi) = 0 \;/;\; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n \land m \neq 0$$

07.37.03.0006.01

$$Y_\lambda^\mu(-\pi, \varphi) = \tilde{\infty} \;/;\; \mathrm{Re}(\mu) < 0 \bigwedge \mu \notin \mathbb{Z}$$

07.37.03.0007.01

$$Y_n^m(k\,\pi, \varphi) = 0 \;/;\; n \in \mathbb{N} \bigwedge m \in \mathbb{Z} \bigwedge |m| \leq n \bigwedge m \neq 0 \bigwedge k \in \mathbb{Z}$$

07.37.03.0008.01

$$Y_n^m\left(\frac{\pi}{2}, \varphi\right) = \frac{((n+m+1) \bmod 2)}{2} (-1)^{\frac{n+m}{2}} e^{i\,m\,\varphi} \sqrt{\frac{(2\,n+1)\,(n+m-1)!!\,(n-m-1)!!}{\pi\,(n+m)!!\,(n-m)!!}} \;/;\; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.03.0009.01

$$Y_n^m\left(-\frac{\pi}{2}, \varphi\right) = \frac{((n+m+1) \bmod 2)}{2} (-1)^{\frac{n+m}{2}} e^{i\,m\,\varphi} \sqrt{\frac{(2\,n+1)\,(n+m-1)!!\,(n-m-1)!!}{\pi\,(n+m)!!\,(n-m)!!}} \;/;\; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

For fixed λ, ϑ, φ

07.37.03.0010.01

$$Y_\lambda^0(\vartheta, \varphi) = \sqrt{\frac{2\,\lambda+1}{4\,\pi}}\; P_\lambda(\cos(\vartheta))$$

07.37.03.0011.01

$$Y_\lambda^1(\vartheta, \varphi) = \frac{e^{i\,\varphi}\,\lambda\,\sqrt{2\,\lambda+1}\,\sqrt{\Gamma(\lambda)}}{4\,\sqrt{\pi}\,\sqrt{\Gamma(\lambda+2)}\,\sqrt{\cos^2\left(\frac{\vartheta}{2}\right)}\,\sqrt{\sin^2\left(\frac{\vartheta}{2}\right)}} (\cos(\vartheta)\,P_\lambda(\cos(\vartheta)) - P_{\lambda-1}(\cos(\vartheta)))$$

07.37.03.0012.01

$$Y_\lambda^\lambda(\vartheta, \varphi) = \frac{\sqrt{2\,\lambda+1}}{2^{\lambda+1}\,\sqrt{\pi}\,\Gamma(-\lambda)\,\sqrt{\Gamma(2\,\lambda+1)}} e^{i\,\lambda\,\varphi} \sin^2(\vartheta)^{\lambda/2}\, \mathrm{B}_{\sin^2\left(\frac{\vartheta}{2}\right)}(-\lambda, -\lambda) \;/;\; \lambda \notin \mathbb{N}$$

07.37.03.0013.01

$$Y_n^n(\vartheta, \varphi) = \frac{(-1)^n\,\sqrt{(2\,n+1)!}\;e^{i\,\varphi\,n}}{2^{n+1}\,\sqrt{\pi}\;n!} \sin^2(\vartheta)^{n/2} \;/;\; n \in \mathbb{N}$$

07.37.03.0014.01

$$Y_n^n(\vartheta, \varphi) = 0 \;/;\; -n \in \mathbb{N}^+$$

07.37.03.0015.01

$$Y_\lambda^{\lambda+1}(\vartheta, \varphi) = \tilde{\infty} \;/;\; \lambda \notin \mathbb{Z} \bigwedge -\lambda - \frac{1}{2} \notin \mathbb{N}$$

07.37.03.0016.01

$$Y_n^{n+1}(\vartheta, \varphi) = 0 \;/;\; n \in \mathbb{N}$$

07.37.03.0017.01

$$Y_\lambda^{-\lambda}(\vartheta, \varphi) = \sqrt{\frac{2\,\lambda+1}{\pi}}\; \frac{\sqrt{\Gamma(2\,\lambda+1)}\;e^{-i\,\varphi\,\lambda}}{2^{\lambda+1}\,\Gamma(\lambda+1)} \sin^2(\vartheta)^{\lambda/2}$$

07.37.03.0018.01

$$Y_n^{-n}(\vartheta, \varphi) = \frac{\sqrt{(2n+1)!}\ e^{-i\varphi\lambda}}{2^{n+1}\, n!\, \sqrt{\pi}} \sin^2(\vartheta)^{n/2} /; n \in \mathbb{N}$$

07.37.03.0019.01

$$Y_\lambda^{-\lambda-1}(\vartheta, \varphi) = 0 /; -2\lambda \notin \mathbb{N}^+$$

07.37.03.0020.01

$$Y_\lambda^{-\lambda-m}(\vartheta, \varphi) = 0 /; m \in \mathbb{N}^+ \land -m-1-2\lambda \notin \mathbb{N}$$

07.37.03.0021.01

$$Y_n^{-n}(\vartheta, \varphi) = 0 /; -n \in \mathbb{N}^+$$

07.37.03.0022.01

$$Y_n^{m-n}(\vartheta, \varphi) = 0 /; -n \in \mathbb{N}^+ \land m \in \mathbb{N}^+$$

07.37.03.0023.01

$$Y_\lambda^{\lambda+m}(\vartheta, \varphi) = \tilde{\infty} /; m \in \mathbb{N}^+ \bigwedge \lambda \notin \mathbb{Z} \bigwedge -\frac{m+2\lambda+1}{2} \notin \mathbb{N}$$

07.37.03.0024.01

$$Y_n^{n+m}(\vartheta, \varphi) = 0 /; n \in \mathbb{N} \land m \in \mathbb{N}^+$$

07.37.03.0025.01

$$Y_n^{n-m}(\vartheta, \varphi) = 0 /; -n \in \mathbb{N}^+ \land m \in \mathbb{N}^+$$

For fixed μ,ϑ,φ

07.37.03.0026.01

$$Y_{-\frac{1}{2}}^{\mu}(\vartheta, \varphi) = 0 /; \mu + \frac{1}{2} \notin \mathbb{N}^+$$

07.37.03.0027.01

$$Y_0^{\mu}(\vartheta, \varphi) = \frac{e^{i\mu\varphi} \cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\mu/2}}{2\sqrt{\pi}\ \sqrt{\Gamma(1-\mu)}\ \sqrt{\Gamma(\mu+1)}}$$

07.37.03.0028.01

$$Y_1^{\mu}(\vartheta, \varphi) = \frac{\sqrt{3}\ e^{i\mu\varphi} (\cos(\vartheta) - \mu) \cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\mu/2}}{2\sqrt{\pi}\ \sqrt{\Gamma(2-\mu)}\ \sqrt{\Gamma(\mu+2)}}$$

07.37.03.0029.01

$$Y_2^{\mu}(\vartheta, \varphi) = \frac{\sqrt{5}\ e^{i\mu\varphi} \left(3\cos^2(\vartheta) - 3\mu\cos(\vartheta) + \mu^2 - 1\right) \cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\mu/2}}{2\sqrt{\pi}\ \sqrt{\Gamma(3-\mu)}\ \sqrt{\Gamma(\mu+3)}}$$

07.37.03.0030.01

$$Y_3^{\mu}(\vartheta, \varphi) = \frac{\sqrt{7}\ e^{i\mu\varphi} \left(15\cos^3(\vartheta) - 15\mu\cos^2(\vartheta) + \left(6\mu^2 - 9\right)\cos(\vartheta) - \mu\left(\mu^2 - 4\right)\right) \cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\mu/2}}{\sqrt{4\pi}\ \sqrt{\Gamma(4-\mu)}\ \sqrt{\Gamma(4+\mu)}}$$

07.37.03.0031.01

$$Y_4^{\mu}(\vartheta, \varphi) = \left(3\, e^{i\mu\varphi} \left(105\cos^4(\vartheta) - 105\,\mu\cos^3(\vartheta) + 45\left(\mu^2-2\right)\cos^2(\vartheta) - 5\,\mu\left(2\,\mu^2-11\right)\cos(\vartheta) + \mu^4 - 10\,\mu^2 + 9\right)\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\mu/2}\right) \Big/ \left(2\sqrt{\pi}\,\sqrt{\Gamma(5-\mu)}\,\sqrt{\Gamma(5+\mu)}\right)$$

07.37.03.0032.01

$$Y_5^{\mu}(\vartheta, \varphi) = \left(\sqrt{11}\, e^{i\mu\varphi}\left(945\cos^5(\vartheta) - 945\,\mu\cos^4(\vartheta) + 210\left(2\,\mu^2-5\right)\cos^3(\vartheta) - 105\,\mu\left(\mu^2-7\right)\cos^2(\vartheta) + 15\left(\mu^4-13\,\mu^2+15\right)\cos(\vartheta) - \mu\left(\mu^4-20\,\mu^2+64\right)\right)\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\mu/2}\right) \Big/ \left(2\sqrt{\pi}\,\sqrt{\Gamma(6-\mu)}\,\sqrt{\Gamma(6+\mu)}\right)$$

07.37.03.0033.01

$$Y_6^{\mu}(\vartheta, \varphi) = \left(\sqrt{13}\, e^{i\mu\varphi} \left(10\,395\cos^6(\vartheta) - 10\,395\,\mu\cos^5(\vartheta) + 4725\left(\mu^2-3\right)\cos^4(\vartheta) - 630\left(2\,\mu^2-17\right)\mu\cos^3(\vartheta) + 105\left(2\,\mu^4-32\,\mu^2+45\right)\cos^2(\vartheta) - 21\left(\mu^4-25\,\mu^2+99\right)\mu\cos(\vartheta) + \mu^6 - 35\,\mu^4 + 259\,\mu^2 - 225\right)\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{\mu}{2}}\right) \Big/ \left(2\sqrt{\pi}\,\sqrt{\Gamma(7-\mu)}\,\sqrt{\Gamma(\mu+7)}\right)$$

07.37.03.0034.01

$$Y_7^{\mu}(\vartheta, \varphi) = \left(\sqrt{15}\, e^{i\mu\varphi}\left(135\,135\cos^7(\vartheta) - 135\,135\,\mu\cos^6(\vartheta) + 31\,185\left(2\,\mu^2-7\right)\cos^5(\vartheta) - 17\,325\,\mu\left(\mu^2-10\right)\cos^4(\vartheta) + 1575\left(2\,\mu^4-38\,\mu^2+63\right)\cos^3(\vartheta) - 189\,\mu\left(2\,\mu^4-60\,\mu^2+283\right)\cos^2(\vartheta) + 7\left(4\,\mu^6-170\,\mu^4+1516\,\mu^2-1575\right)\cos(\vartheta) - \mu\left(\mu^6-56\,\mu^4+784\,\mu^2-2304\right)\right)\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\mu/2}\right) \Big/ \left(2\sqrt{\pi}\,\sqrt{\Gamma(8-\mu)}\,\sqrt{\Gamma(\mu+8)}\right)$$

07.37.03.0035.01

$$Y_8^{\mu}(\vartheta, \varphi) = \left(\sqrt{17}\, e^{i\mu\varphi}\left(2\,027\,025\cos^8(\vartheta) - 2\,027\,025\,\mu\cos^7(\vartheta) + 945\,945\left(\mu^2-4\right)\cos^6(\vartheta) - 135\,135\left(2\,\mu^2-23\right)\mu\cos^5(\vartheta) + 51\,975\left(\mu^4-22\,\mu^2+42\right)\cos^4(\vartheta) - 3465\left(2\,\mu^4-70\,\mu^2+383\right)\mu\cos^3(\vartheta) + 315\left(2\,\mu^6-100\,\mu^4+1043\,\mu^2-1260\right)\cos^2(\vartheta) - 9\left(4\,\mu^6-266\,\mu^4+4396\,\mu^2-15\,159\right)\mu\cos(\vartheta) + \mu^8 - 84\,\mu^6 + 1974\,\mu^4 - 12\,916\,\mu^2 + 11\,025\right)\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\left(\frac{\vartheta}{2}\right)^{-\mu/2}\right) \Big/ \left(2\sqrt{\pi}\,\sqrt{\Gamma(9-\mu)}\,\sqrt{\Gamma(\mu+9)}\right)$$

07.37.03.0036.01

$$Y_9^\mu(\vartheta, \varphi) = \left(\sqrt{19}\, e^{i\mu\varphi} \left(34\,459\,425 \cos^9(\vartheta) - 34\,459\,425\, \mu \cos^8(\vartheta) + 8\,108\,100 \left(2\mu^2 - 9\right) \cos^7(\vartheta) - 4\,729\,725\, \mu \left(\mu^2 - 13\right) \cos^6(\vartheta) + 945\,945 \left(\mu^4 - 25\mu^2 + 54\right) \cos^5(\vartheta) - 135\,135\, \mu \left(\mu^4 - 40\mu^2 + 249\right) \cos^4(\vartheta) + 6930 \left(2\mu^6 - 115\mu^4 + 1373\mu^2 - 1890\right) \cos^3(\vartheta) - 495\, \mu \left(2\mu^6 - 154\mu^4 + 2933\mu^2 - 11\,601\right) \cos^2(\vartheta) + 45 \left(\mu^8 - 98\mu^6 + 2674\mu^4 - 20\,217\mu^2 + 19\,845\right) \cos(\vartheta) - \mu \left(\mu^8 - 120\mu^6 + 4368\mu^4 - 52\,480\mu^2 + 147\,456\right)\right) \cos^2\!\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\!\left(\frac{\vartheta}{2}\right)^{-\mu/2}\right) \Big/ \left(2\sqrt{\pi}\, \sqrt{\Gamma(10-\mu)}\, \sqrt{\Gamma(\mu+10)}\right)$$

07.37.03.0037.01

$$Y_{10}^\mu(\vartheta, \varphi) = \left(\sqrt{21}\, e^{i\mu\varphi} \left(654\,729\,075 \cos^{10}(\vartheta) - 654\,729\,075\, \mu \cos^9(\vartheta) + 310\,134\,825 \left(\mu^2 - 5\right) \cos^8(\vartheta) - 45\,945\,900 \left(2\mu^2 - 29\right) \mu \cos^7(\vartheta) + 9\,459\,450 \left(2\mu^4 - 56\mu^2 + 135\right) \cos^6(\vartheta) - 2\,837\,835 \left(\mu^4 - 45\mu^2 + 314\right) \mu \cos^5(\vartheta) + 315\,315 \left(\mu^6 - 65\mu^4 + 874\mu^2 - 1350\right) \cos^4(\vartheta) - 12\,870 \left(2\mu^6 - 175\mu^4 + 3773\mu^2 - 16\,830\right) \mu \cos^3(\vartheta) + 1485 \left(\mu^8 - 112\mu^6 + 3479\mu^4 - 29\,828\mu^2 + 33\,075\right) \cos^2(\vartheta) - 55 \left(\mu^8 - 138\mu^6 + 5754\mu^4 - 78\,877\mu^2 + 251\,865\right) \mu \cos(\vartheta) + \mu^{10} - 165\mu^8 + 8778\mu^6 - 172\,810\mu^4 + 1\,057\,221\mu^2 - 893\,025\right) \cos^2\!\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\!\left(\frac{\vartheta}{2}\right)^{-\mu/2}\right) \Big/ \left(2\sqrt{\pi}\, \sqrt{\Gamma(11-\mu)}\, \sqrt{\Gamma(\mu+11)}\right)$$

07.37.03.0038.01

$$Y_n^\mu(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}}\, \frac{\sqrt{\Gamma(n-\mu+1)}}{\sqrt{\Gamma(n+\mu+1)}}\, e^{i\varphi\mu} \cos^2\!\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\!\left(\frac{\vartheta}{2}\right)^{-\mu/2} \sum_{k=0}^{n} \frac{(-n)_k\, (n+1)_k}{\Gamma(k-\mu+1)\, k!} \sin^{2k}\!\left(\frac{\vartheta}{2}\right) /;\, n \in \mathbb{N}$$

07.37.03.0039.01

$$Y_{-n}^\mu(\vartheta, \varphi) = \sqrt{\frac{1-2n}{4\pi}}\, \frac{\sqrt{\Gamma(1-n-\mu)}\, e^{i\varphi\mu}}{\sqrt{\Gamma(1-n+\mu)}} \cos^2\!\left(\frac{\vartheta}{2}\right)^{\mu/2} \sin^2\!\left(\frac{\vartheta}{2}\right)^{-\mu/2} \sum_{k=0}^{n-1} \frac{(n)_k\, (1-n)_k}{\Gamma(k-\mu+1)\, k!} \sin^{2k}\!\left(\frac{\vartheta}{2}\right) /;\, n \in \mathbb{N}^+$$

07.37.03.0040.01

$$Y_n^m(\vartheta, \varphi) = 0 /;\, n \in \mathbb{N} \bigwedge m \in \mathbb{N}^+ \bigwedge n < m$$

07.37.03.0041.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^m\, 2^{n-1}\, e^{i\varphi m}}{\pi} \sqrt{\frac{(2n+1)\,(n-m)!}{(n+m)!}}\, \sin^2(\vartheta)^{m/2} \cos^{-m}(\vartheta) \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^k\, 2^{-2k}\, \Gamma\!\left(n-k+\frac{1}{2}\right) \cos^{n-2k}(\vartheta)}{k!\, \Gamma(n-m-2k+1)} /;\, n \in \mathbb{N} \land m \in \mathbb{N} \land m \le n$$

For fixed λ,φ

07.37.03.0042.01

$$Y_\lambda^0(0, \varphi) = \frac{\sqrt{1+2\lambda}}{2\sqrt{\pi}} /;\, -\lambda \notin \mathbb{N}^+$$

07.37.03.0043.01

$$Y_n^0(k\pi, \varphi) = (-1)^{n\left(2\left\lfloor \frac{k}{2} \right\rfloor - k\right)} \sqrt{\frac{2n+1}{4\pi}} /;\, k \in \mathbb{Z}$$

For fixed ϑ, φ

07.37.03.0044.01

$$Y_0^0(\vartheta, \varphi) = \frac{1}{2\sqrt{\pi}}$$

07.37.03.0045.01

$$Y_1^{-1}(\vartheta, \varphi) = \frac{1}{2} e^{-i\varphi} \sqrt{\frac{3}{2\pi}} \sqrt{\sin^2(\vartheta)}$$

07.37.03.0046.01

$$Y_1^0(\vartheta, \varphi) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos(\vartheta)$$

07.37.03.0047.01

$$Y_1^1(\vartheta, \varphi) = -\frac{1}{2} e^{i\varphi} \sqrt{\frac{3}{2\pi}} \sqrt{\sin^2(\vartheta)}$$

07.37.03.0048.01

$$Y_2^{-2}(\vartheta, \varphi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} e^{-2i\varphi} \sin^2(\vartheta)$$

07.37.03.0049.01

$$Y_2^{-1}(\vartheta, \varphi) = \frac{1}{2} e^{-i\varphi} \sqrt{\frac{15}{2\pi}} \cos(\vartheta) \sqrt{\sin^2(\vartheta)}$$

07.37.03.0050.01

$$Y_2^0(\vartheta, \varphi) = \frac{1}{8} \sqrt{\frac{5}{\pi}} (3\cos(2\vartheta) + 1)$$

07.37.03.0051.01

$$Y_2^1(\vartheta, \varphi) = -\frac{1}{2} e^{i\varphi} \sqrt{\frac{15}{2\pi}} \cos(\vartheta) \sqrt{\sin^2(\vartheta)}$$

07.37.03.0052.01

$$Y_2^2(\vartheta, \varphi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} e^{2i\varphi} \sin^2(\vartheta)$$

07.37.03.0053.01

$$Y_3^{-3}(\vartheta, \varphi) = \frac{1}{8} e^{-3i\varphi} \sqrt{\frac{35}{\pi}} \sin^2(\vartheta)^{3/2}$$

07.37.03.0054.01

$$Y_3^{-2}(\vartheta, \varphi) = \frac{1}{4} \sqrt{\frac{105}{2\pi}} e^{-2i\varphi} \cos(\vartheta) \sin^2(\vartheta)$$

07.37.03.0055.01

$$Y_3^{-1}(\vartheta, \varphi) == \frac{1}{16} e^{-i\varphi} \sqrt{\frac{21}{\pi}} (5\cos(2\vartheta) + 3) \sqrt{\sin^2(\vartheta)}$$

07.37.03.0056.01

$$Y_3^0(\vartheta, \varphi) == \frac{1}{16} \sqrt{\frac{7}{\pi}} (3\cos(\vartheta) + 5\cos(3\vartheta))$$

07.37.03.0057.01

$$Y_3^1(\vartheta, \varphi) == -\frac{1}{16} e^{i\varphi} \sqrt{\frac{21}{\pi}} (5\cos(2\vartheta) + 3) \sqrt{\sin^2(\vartheta)}$$

07.37.03.0058.01

$$Y_3^2(\vartheta, \varphi) == \frac{1}{4} \sqrt{\frac{105}{2\pi}} e^{2i\varphi} \cos(\vartheta) \sin^2(\vartheta)$$

07.37.03.0059.01

$$Y_3^3(\vartheta, \varphi) == -\frac{1}{8} e^{3i\varphi} \sqrt{\frac{35}{\pi}} \sin^2(\vartheta)^{3/2}$$

07.37.03.0060.01

$$Y_4^{-4}(\vartheta, \varphi) == \frac{3}{16} \sqrt{\frac{35}{2\pi}} e^{-4i\varphi} \sin^4(\vartheta)$$

07.37.03.0061.01

$$Y_4^{-3}(\vartheta, \varphi) == \frac{3}{8} \sqrt{\frac{35}{\pi}} e^{-3i\varphi} \cos(\vartheta) \sin^2(\vartheta)^{3/2}$$

07.37.03.0062.01

$$Y_4^{-2}(\vartheta, \varphi) == \frac{3}{8} \sqrt{\frac{5}{2\pi}} e^{-2i\varphi} \left(7\cos^2(\vartheta) - 1\right) \sin^2(\vartheta)$$

07.37.03.0063.01

$$Y_4^{-1}(\vartheta, \varphi) == \frac{3}{8} \sqrt{\frac{5}{\pi}} e^{-i\varphi} \cos(\vartheta) \left(7\cos^2(\vartheta) - 3\right) \sqrt{\sin^2(\vartheta)}$$

07.37.03.0064.01

$$Y_4^0(\vartheta, \varphi) == \frac{3}{16\sqrt{\pi}} \left(35\cos^4(\vartheta) - 30\cos^2(\vartheta) + 3\right)$$

07.37.03.0065.01

$$Y_4^1(\vartheta, \varphi) == -\frac{3}{8} \sqrt{\frac{5}{\pi}} e^{i\varphi} \cos(\vartheta) \left(7\cos^2(\vartheta) - 3\right) \sqrt{\sin^2(\vartheta)}$$

07.37.03.0066.01

$$Y_4^2(\vartheta, \varphi) == \frac{3}{8} \sqrt{\frac{5}{2\pi}} e^{2i\varphi} \left(7\cos^2(\vartheta) - 1\right) \sin^2(\vartheta)$$

07.37.03.0067.01

$$Y_4^3(\vartheta, \varphi) == -\frac{3}{8} \sqrt{\frac{35}{\pi}} e^{3 i \varphi} \cos(\vartheta) \sin^2(\vartheta)^{3/2}$$

07.37.03.0068.01

$$Y_4^4(\vartheta, \varphi) == \frac{3}{16} \sqrt{\frac{35}{2 \pi}} e^{4 i \varphi} \sin^4(\vartheta)$$

07.37.03.0069.01

$$Y_5^{-5}(\vartheta, \varphi) == \frac{3}{32} \sqrt{\frac{77}{\pi}} e^{-5 i \varphi} \sin^2(\vartheta)^{5/2}$$

07.37.03.0070.01

$$Y_5^{-4}(\vartheta, \varphi) == \frac{3}{16} \sqrt{\frac{385}{2 \pi}} e^{-4 i \varphi} \cos(\vartheta) \sin^4(\vartheta)$$

07.37.03.0071.01

$$Y_5^{-3}(\vartheta, \varphi) == \frac{1}{32} \sqrt{\frac{385}{\pi}} e^{-3 i \varphi} \left(9 \cos^2(\vartheta) - 1\right) \sin^2(\vartheta)^{3/2}$$

07.37.03.0072.01

$$Y_5^{-2}(\vartheta, \varphi) == \frac{1}{8} \sqrt{\frac{1155}{2 \pi}} e^{-2 i \varphi} \cos(\vartheta) \left(3 \cos^2(\vartheta) - 1\right) \sin^2(\vartheta)$$

07.37.03.0073.01

$$Y_5^0(\vartheta, \varphi) == \frac{1}{16} \sqrt{\frac{11}{\pi}} \left(63 \cos^5(\vartheta) - 70 \cos^3(\vartheta) + 15 \cos(\vartheta)\right)$$

07.37.03.0074.01

$$Y_5^1(\vartheta, \varphi) == -\frac{1}{16} \sqrt{\frac{165}{2 \pi}} e^{i \varphi} \left(21 \cos^4(\vartheta) - 14 \cos^2(\vartheta) + 1\right) \sqrt{\sin^2(\vartheta)}$$

07.37.03.0075.01

$$Y_5^2(\vartheta, \varphi) == \frac{1}{8} \sqrt{\frac{1155}{2 \pi}} e^{2 i \varphi} \cos(\vartheta) \left(3 \cos^2(\vartheta) - 1\right) \sin^2(\vartheta)$$

07.37.03.0076.01

$$Y_5^3(\vartheta, \varphi) == -\frac{1}{32} \sqrt{\frac{385}{\pi}} e^{3 i \varphi} \left(9 \cos^2(\vartheta) - 1\right) \sin^2(\vartheta)^{3/2}$$

07.37.03.0077.01

$$Y_5^4(\vartheta, \varphi) == \frac{3}{16} \sqrt{\frac{385}{2 \pi}} e^{4 i \varphi} \cos(\vartheta) \sin^4(\vartheta)$$

07.37.03.0078.01

$$Y_5^5(\vartheta, \varphi) = -\frac{3}{32}\sqrt{\frac{77}{\pi}}\, e^{5 i \varphi} \sin^2(\vartheta)^{5/2}$$

General characteristics

Domain and analyticity

$Y_\lambda^\mu(\vartheta, \varphi)$ is an analytical function of λ, μ, ϑ, φ which is defined over $\mathbb{C}^4$.

07.37.04.0001.01

$$(\lambda * \mu * \vartheta * \varphi) \longrightarrow Y_\lambda^\mu(\vartheta, \varphi) :: \mathbb{C}^4 \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

07.37.04.0002.01

$$Y_{-\lambda}^\mu(\vartheta, \varphi) = \frac{\sqrt{1 - 2\lambda}\sqrt{\Gamma(1 - \lambda - \mu)}\sqrt{\Gamma(\lambda + \mu)}}{\sqrt{2\lambda - 1}\sqrt{\Gamma(1 - \lambda + \mu)}\sqrt{\Gamma(\lambda - \mu)}} Y_{\lambda-1}^\mu(\vartheta, \varphi)$$

07.37.04.0003.01

$$Y_n^{-m}(\vartheta, \varphi) = (-1)^m e^{-2 i \varphi m} Y_n^m(\vartheta, \varphi) /; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.04.0004.01

$$Y_\lambda^\mu(-\vartheta, \varphi) = Y_\lambda^\mu(\vartheta, \varphi)$$

07.37.04.0005.01

$$Y_\lambda^\mu(\vartheta, -\varphi) = e^{-2 i \mu \varphi} Y_\lambda^\mu(\vartheta, \varphi)$$

07.37.04.0006.01

$$Y_n^m(\vartheta, -\varphi) = (-1)^m Y_n^{-m}(\vartheta, \varphi) /; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.04.0007.01

$$Y_n^m(-\vartheta, -\varphi) = (-1)^m Y_n^{-m}(\vartheta, \varphi) /; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

Mirror symmetry

07.37.04.0008.01

$$Y_n^{-m}(\vartheta, \varphi) = (-1)^m \overline{Y_n^m(\vartheta, \varphi)} /; n \in \mathbb{N} \bigwedge m \in \mathbb{Z} \bigwedge |m| \leq n \bigwedge \vartheta \in \mathbb{R} \bigwedge \varphi \in \mathbb{R}$$

Periodicity

$Y_\lambda^\mu(\vartheta, \varphi)$ is a periodic function with respect to ϑ and φ with periods 2π and $2\pi/\mu$ respectively.

07.37.04.0009.01

$$Y_\lambda^\mu(\vartheta + 2\pi k, \varphi) = Y_\lambda^\mu(\vartheta, \varphi) /; k \in \mathbb{Z}$$

07.37.04.0010.01

$$Y_\lambda^\mu\left(\vartheta, \varphi + \frac{2\pi k}{\mu}\right) = Y_\lambda^\mu(\vartheta, \varphi) /; k \in \mathbb{Z}$$

Phase shifts

07.37.04.0011.01

$$Y_\lambda^m(\vartheta, \varphi+\pi) = (-1)^m Y_\lambda^m(\vartheta, \varphi) /; m \in \mathbb{Z}$$

07.37.04.0012.01

$$Y_n^m(\pi-\vartheta, \varphi) = (-1)^{n+m} Y_n^m(\vartheta, \varphi) /; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.04.0013.01

$$Y_n^m(\pi-\vartheta, \varphi+\pi) = (-1)^n Y_n^m(\vartheta, \varphi) /; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

Poles and essential singularities

With respect to φ

For fixed λ,μ,ϑ, the function $Y_\lambda^\mu(\vartheta, \varphi)$ has only one singular point at $\varphi = \tilde{\infty}$. It is an essential singular point.

07.37.04.0014.01

$$\mathcal{S}ing_\varphi\left(Y_\lambda^\mu(\vartheta, \varphi)\right) = \{\{\tilde{\infty}, \infty\}\}$$

With respect to ϑ

For fixed λ, φ, μ /; $\frac{\mu}{2} \notin \mathbb{Z}$, the function $Y_\lambda^\mu(\vartheta, \varphi)$ does not have poles and essential singularities.

07.37.04.0015.01

$$\mathcal{S}ing_\vartheta\left(Y_\lambda^\mu(\vartheta, \varphi)\right) = \{\} /; \frac{\mu}{2} \notin \mathbb{Z}$$

For integer λ and integer $\frac{\mu}{2}$, the function $Y_\lambda^\mu(\vartheta, \varphi)$ is polynomial and has pole of order λ at $\cos(\vartheta) = \tilde{\infty}$ (for $\lambda \in \mathbb{N}^+$) or order $-\lambda-1$ at $\cos(\vartheta) = \tilde{\infty}$ (for $-\lambda \in \mathbb{N}^+$).

07.37.04.0016.01

$$\mathcal{S}ing_\vartheta\left(Y_\lambda^\mu(\vartheta, \varphi)\right) = \{\{\tilde{\infty}, \lambda\}\} /; \frac{\mu}{2} \in \mathbb{Z} \bigwedge \lambda \in \mathbb{N}^+$$

07.37.04.0017.01

$$\mathcal{S}ing_\vartheta\left(Y_\lambda^\mu(\vartheta, \varphi)\right) = \{\{\tilde{\infty}, -\lambda-1\}\} /; \frac{\mu}{2} \in \mathbb{Z} \bigwedge -\lambda \in \mathbb{N}^+$$

With respect to μ

For fixed λ,ϑ,φ, the function $Y_\lambda^\mu(\vartheta, \varphi)$ has only one singular point at $\mu = \tilde{\infty}$. It is an essential singular point. .

07.37.04.0018.01

$$\mathcal{S}ing_\mu\left(Y_\lambda^\mu(\vartheta, \varphi)\right) = \{\{\tilde{\infty}, \infty\}\}$$

With respect to λ

For fixed μ,ϑ,φ, the function $Y_\lambda^\mu(\vartheta, \varphi)$ has only one singular point at $\lambda = \tilde{\infty}$. It is an essential singular point. .

07.37.04.0019.01

$$\mathcal{S}ing_\lambda\left(Y_\lambda^\mu(\vartheta, \varphi)\right) = \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to φ

For fixed λ,μ,ϑ, the function $Y_\lambda^\mu(\vartheta, \varphi)$ does not have branch points.

07.37.04.0020.01

$$\mathcal{BP}_\varphi\left(Y_\lambda^\mu(\vartheta, \varphi)\right) = \{\}$$

With respect to ϑ

For fixed generic λ, φ, μ /; $\frac{\mu}{2} \notin \mathbb{Z}$, the function $Y_\lambda^\mu(\vartheta, \varphi)$ has the set of branch points where: $\cos(\vartheta) == -1$, $\cos(\vartheta) == 1$and $\cos(\vartheta) == \tilde{\infty}$. For fixed φ, noninteger λ and integer $\frac{\mu}{2}$, the function $Y_\lambda^\mu(\vartheta, \varphi)$ has the set of branch points where: $\cos(\vartheta) == -1$and $\cos(\vartheta) == \tilde{\infty}$. For fixed φ, integers λ and integers $\frac{\mu}{2}$, the function $Y_\lambda^\mu(\vartheta, \varphi)$ does not have branch points.

With respect to μ

For fixed $\lambda,\vartheta,\varphi$, the function $Y_\lambda^\mu(\vartheta, \varphi)$ has infinitely many branch points: $\mu == \lambda + n /; n \in \mathbb{N}^+$; $\mu == -\lambda - n /; n \in \mathbb{N}^+$ and $\mu == \tilde{\infty}$. All these are power-type branch points.

07.37.04.0021.01

$$\mathcal{BP}_\mu\left(Y_\lambda^\mu(\vartheta, \varphi)\right) == \left\{\left\{\lambda + n /; n \in \mathbb{N}^+\right\}, \left\{-\lambda - n /; n \in \mathbb{N}^+\right\}, \tilde{\infty}\right\}$$

07.37.04.0022.01

$$\mathcal{R}_\mu\left(Y_\lambda^\mu(\vartheta, \varphi), \lambda + n\right) == 2 /; n \in \mathbb{N}^+$$

07.37.04.0023.01

$$\mathcal{R}_\mu\left(Y_\lambda^\mu(\vartheta, \varphi), -\lambda - n\right) == 2 /; n \in \mathbb{N}^+$$

07.37.04.0024.01

$$\mathcal{R}_\mu\left(Y_\lambda^\mu(\vartheta, \varphi), \tilde{\infty}\right) == \log$$

With respect to λ

For fixed μ,ϑ,φ, the function $Y_\lambda^\mu(\vartheta, \varphi)$ has infinitely many branch points: $\lambda == \mu - n /; n \in \mathbb{N}^+$; $\lambda == -\mu - n /; n \in \mathbb{N}^+$; $\lambda == -\frac{1}{2}$ and $\mu == \tilde{\infty}$. All these are power-type branch points.

07.37.04.0025.01

$$\mathcal{BP}_\lambda\left(Y_\lambda^\mu(\vartheta, \varphi)\right) == \left\{\left\{\mu - n /; n \in \mathbb{N}^+\right\}, \left\{-\mu - n /; n \in \mathbb{N}^+\right\}, -\frac{1}{2}, \tilde{\infty}\right\}$$

07.37.04.0026.01

$$\mathcal{R}_\lambda\left(Y_\lambda^\mu(\vartheta, \varphi), \mu + n\right) == 2 /; n \in \mathbb{N}^+$$

07.37.04.0027.01

$$\mathcal{R}_\lambda\left(Y_\lambda^\mu(\vartheta, \varphi), -\mu - n\right) == 2 /; n \in \mathbb{N}^+$$

07.37.04.0028.01

$$\mathcal{R}_\lambda\left(Y_\lambda^\mu(\vartheta, \varphi), -\frac{1}{2}\right) == 2$$

07.37.04.0029.01

$$\mathcal{R}_\lambda\left(Y_\lambda^\mu(\vartheta, \varphi), \tilde{\infty}\right) == \log$$

Branch cuts

With respect to φ

For fixed λ,μ,ϑ, the function $Y_\lambda^\mu(\vartheta, \varphi)$ does not have branch cuts.

07.37.04.0030.01

$$\mathcal{BC}_\varphi\left(Y_\lambda^\mu(\vartheta, \varphi)\right) = \{\}$$

With respect to ϑ

For fixed generic λ, φ, μ /; $\frac{\mu}{2} \notin \mathbb{Z}$, the function $Y_\lambda^\mu(\vartheta, \varphi)$ is a single-valued function on the ϑ-plane cut along the intervals $-\infty < \cos(\vartheta) < -1$ and $1 < \cos(\vartheta) < \infty$.

For fixed φ, noninteger λ and integer $\frac{\mu}{2}$, the function $Y_\lambda^\mu(\vartheta, \varphi)$ is a single-valued function on the ϑ-plane cut along the interval $-\infty < \cos(\vartheta) < -1$.

For fixed φ, integers λ and integers $\frac{\mu}{2}$, the function $Y_\lambda^\mu(\vartheta, \varphi)$ is a polynomial and does not have branch cuts.

Series representations

Generalized power series

Expansions at $\sin\left(\frac{\vartheta}{2}\right) = 0$

07.37.06.0001.02

$$Y_\lambda^\mu(\vartheta, \varphi) \propto \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}\, e^{i\varphi\mu}}{\sqrt{\Gamma(\lambda+\mu+1)}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{\mu}{2}} \left(\frac{1}{\Gamma(1-\mu)} - \frac{(1-\mu)\mu + 2\lambda(\lambda+1)}{2\Gamma(2-\mu)} \sin^2\left(\frac{\vartheta}{2}\right) + \frac{\left(\mu^4 - 5\mu^3 - 4\left(\lambda^2+\lambda-2\right)\mu^2 + (8\lambda(\lambda+1)-4)\mu + 4(\lambda-1)\lambda(\lambda+1)(\lambda+2)\right)}{8\Gamma(3-\mu)} \sin^4\left(\frac{\vartheta}{2}\right) - \ldots\right) /; \left(\sin\left(\frac{\vartheta}{2}\right) \to 0\right)$$

07.37.06.0002.01

$$Y_\lambda^\mu(\vartheta, \varphi) = \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}\, e^{i\varphi\mu}}{\sqrt{\Gamma(\lambda+\mu+1)}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{\mu}{2}} \sum_{k=0}^\infty \sum_{j=0}^k \frac{(-\lambda)_j (\lambda+1)_j \left(-\frac{\mu}{2}\right)_{k-j}}{\Gamma(j-\mu+1)\, j!\, (k-j)!} \sin^{2k}\left(\frac{\vartheta}{2}\right) /; \left|\sin\left(\frac{\vartheta}{2}\right)\right| < 1$$

07.37.06.0003.01

$$Y_\lambda^\mu(\vartheta, \varphi) = \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}\, e^{i\varphi\mu}}{\sqrt{\Gamma(\lambda+\mu+1)}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{\mu}{2}} {}_1F_0\left(-\frac{\mu}{2};; \sin^2\left(\frac{\vartheta}{2}\right)\right) {}_2\tilde{F}_1\left(-\lambda, \lambda+1; 1-\mu; \sin^2\left(\frac{\vartheta}{2}\right)\right)$$

07.37.06.0004.01

$$Y_\lambda^\mu(\vartheta, \varphi) = \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}\, e^{i\varphi\mu}}{\Gamma(1-\mu)\sqrt{\Gamma(\lambda+\mu+1)}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{\mu}{2}} \left(1 + O\left(\sin^2\left(\frac{\vartheta}{2}\right)\right)\right) /; \left(\sin\left(\frac{\vartheta}{2}\right) \to 0\right) \bigwedge \mu \notin \mathbb{N}^+$$

07.37.06.0005.01

$$Y_n^\mu(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}} \frac{\sqrt{\Gamma(n-\mu+1)}\, e^{i\varphi\mu}}{\sqrt{\Gamma(n+\mu+1)}\, \sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sum_{k=0}^n \frac{(-n)_k (n+1)_k}{\Gamma(k-\mu+1)\, k!} \sin^{2k}\left(\frac{\vartheta}{2}\right) /; n \in \mathbb{N}$$

07.37.06.0006.01

$$Y_n^m(\vartheta,\varphi) = (-1)^m \sqrt{\frac{(2n+1)(m+n)!}{4\pi(n-m)!}}\, e^{i\varphi m} \tan\left(\frac{\vartheta}{2}\right)^m \left(\sin^2(\vartheta)\right)^{m/2} \sin(\vartheta)^{-m} \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(k+m)!\,k!} \sin\left(\frac{\vartheta}{2}\right)^{2k} /;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.06.0007.01

$$Y_n^m(\vartheta,\varphi) = \sqrt{\frac{(2n+1)(n-m)!}{\pi(m+n)!}}\, e^{i\varphi m}\, 2^{-m-1} \left(\sin^2(\vartheta)\right)^{m/2} \sum_{k=0}^{n-m} \frac{(-n)_{k+m} (n+1)_{k+m}}{(k+m)!\,k!} \sin\left(\frac{\vartheta}{2}\right)^{2k} /;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.06.0008.01

$$Y_n^m(\vartheta,\varphi) = (-1)^m \sqrt{\frac{(2n+1)(m+n)!}{\pi(n-m)!}}\, e^{i\varphi m}\, 2^{m-1} \left(\sin^2(\vartheta)\right)^{-\frac{m}{2}} \sum_{k=0}^{m+n} \frac{(-n)_{k-m} (n+1)_{k-m}}{(k-m)!\,k!} \sin\left(\frac{\vartheta}{2}\right)^{2k} /;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.06.0009.01

$$Y_{-n}^{\mu}(\vartheta,\varphi) = \sqrt{\frac{1-2n}{4\pi}}\, \frac{\sqrt{\Gamma(1-n-\mu)}\, e^{i\varphi\mu}}{\sqrt{\Gamma(1-n+\mu)}\, \sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} \sum_{k=0}^{n-1} \frac{(n)_k (1-n)_k}{\Gamma(k-\mu+1)\,k!} \sin^{2k}\left(\frac{\vartheta}{2}\right) /;\, n \in \mathbb{N}^+$$

Expansions at $\cos\left(\frac{\vartheta}{2}\right) = 0$

07.37.06.0010.01

$$Y_n^m(\vartheta,\varphi) = (-1)^{m+n} \sqrt{\frac{(2n+1)(n-m)!}{4\pi(m+n)!}}\, e^{i\varphi m} \csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \tan^m\left(\frac{\vartheta}{2}\right) \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(k-m)!\,k!} \cos\left(\frac{\vartheta}{2}\right)^{2k} /;$$

$$n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.06.0011.01

$$Y_n^m(\vartheta,\varphi) = (-1)^{n} \sqrt{\frac{(2n+1)(m+n)!}{4\pi(n-m)!}}\, e^{i\varphi m} \csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \cot^m\left(\frac{\vartheta}{2}\right) \sum_{k=0}^{n} \frac{(-n)_k (n+1)_k}{(k+m)!\,k!} \cos\left(\frac{\vartheta}{2}\right)^{2k} /;$$

$$n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.06.0012.01

$$Y_n^m(\vartheta,\varphi) = (-1)^{m+n} \sqrt{\frac{(2n+1)(n-m)!}{4\pi(m+n)!}}\, e^{i\varphi m} \sin^m\left(\frac{\vartheta}{2}\right) \cos^m\left(\frac{\vartheta}{2}\right) \csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=0}^{n-m} \frac{(-n)_{k+m} (n+1)_{k+m}}{(k+m)!\,k!} \cos\left(\frac{\vartheta}{2}\right)^{2k} /;$$

$$n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.06.0013.01

$$Y_n^m(\vartheta,\varphi) = (-1)^{n} \sqrt{\frac{(2n+1)(m+n)!}{4\pi(n-m)!}}\, e^{i\varphi m} \csc^m\left(\frac{\vartheta}{2}\right) \sec^m\left(\frac{\vartheta}{2}\right) \csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=0}^{m+n} \frac{(-n)_{k-m} (n+1)_{k-m}}{(k-m)!\,k!} \cos\left(\frac{\vartheta}{2}\right)^{2k} /;$$

$$n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

Expansions at $\tan\left(\frac{\vartheta}{2}\right) = 0$

07.37.06.0014.01

$$Y_n^m(\vartheta,\varphi) = (-1)^{\frac{m}{2}(\operatorname{sgn}(m)+1)} e^{i\varphi m} \sqrt{\frac{2n+1}{4\pi}} \sqrt{(m+n)!\,(n-m)!}\; n!\cos^{2n}\left(\frac{\vartheta}{2}\right)\csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=0}^{n-|m|} \frac{(-1)^k}{k!\,(k+|m|)!\,(n-k)!\,(n-|m|-k)!} \tan^{2k+|m|}\left(\frac{\vartheta}{2}\right) /;\ n\in\mathbb{N}\land m\in\mathbb{Z}\land |m|\le n$$

Expansions at $\cot\left(\frac{\vartheta}{2}\right) == 0$

07.37.06.0015.01

$$Y_n^m(\vartheta,\varphi) = (-1)^{\frac{m}{2}(\operatorname{sgn}(m)-1)+n} e^{i\,m\,\varphi} \sqrt{\frac{2n+1}{4\pi}} \sqrt{(n+m)!\,(n-m)!}\; n!\sin^{2n}\left(\frac{\vartheta}{2}\right)\csc^2 \left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=0}^{n-|m|} \frac{(-1)^k}{k!\,(k+|m|)!\,(n-k)!\,(n-|m|-k)!} \cot^{2k+|m|}\left(\frac{\vartheta}{2}\right) /;\ n\in\mathbb{N}\land m\in\mathbb{Z}\land |m|\le n$$

Expansions at $\sin(\vartheta) == 0$

07.37.06.0016.01

$$Y_n^m(\vartheta,\varphi)\,Y_n^m(\vartheta,-\varphi) = \frac{2n+1}{4\pi} \sum_{k=|m|}^{n} (-1)^{k+m} \frac{(k+n)!}{(-k+n)!} \frac{(2k-1)!!}{(2k)!!} \frac{\sin^{2k}(\vartheta)}{(k-m)!\,(k+m)!} /;\ n\in\mathbb{N}\land m\in\mathbb{Z}\land |m|\le n$$

Expansions at $\cos(\vartheta) == 0$

07.37.06.0017.01

$$Y_n^m(\vartheta,\varphi) = e^{i\varphi m} \sqrt{\frac{(2n+1)\,(n-m)!}{(4\pi)\,(n+m)!}} \sin^m(\vartheta)\csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=\frac{(n-m)\bmod 2}{2}}^{\frac{n-m}{2}} (-1)^{\frac{n+m-2k}{2}} \frac{(n+m+2k-1)!!}{(n-m-2k)!!} \frac{\cos^{2k}(\vartheta)}{(2k)!} /;\ n\in\mathbb{N}\land m\in\mathbb{Z}\land |m|\le n$$

Expansions at $\tan(\vartheta) == 0$

07.37.06.0018.01

$$Y_n^m(\vartheta,\varphi) = e^{i\varphi m} \sqrt{\frac{(2n+1)\,(n+m)!\,(n-m)!}{4\pi}} \cos^n(\vartheta)\csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=\frac{|m|}{2}}^{\frac{n-(n-m)\bmod 2}{2}} \frac{(-1)^{\frac{2k+m}{2}}}{(2k+m)!!\,(2k-m)!!} \frac{\tan^{2k}(\vartheta)}{(n-2k)!} /;\ n\in\mathbb{N}\land m\in\mathbb{Z}\land |m|\le n$$

Expansions at $\cot(\vartheta) == 0$

07.37.06.0019.01

$$Y_n^m(\vartheta,\varphi) = e^{i\varphi m} \sqrt{\frac{(2n+1)\,(n+m)!\,(n-m)!}{4\pi}} \sin^n(\vartheta)\csc^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sum_{k=\frac{(n-m)\bmod 2}{2}}^{\frac{n-|m|}{2}} \frac{(-1)^{\frac{n+m-2k}{2}}}{(n+m-2k)!!\,(n-m-2k)!!} \frac{\cot^{2k}(\vartheta)}{(2k)!} /;\ n\in\mathbb{N}\land m\in\mathbb{Z}\land |m|\le n$$

Expansions at $\vartheta == 0$

07.37.06.0020.01

$$Y_\lambda^\mu(\vartheta, \varphi) \propto \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} 2^\mu e^{i\varphi\mu} \left(\vartheta^2\right)^{-\mu/2} \left(\frac{1}{\Gamma(1-\mu)} + \frac{1}{12}\left(-\frac{3\lambda(\lambda+1)}{\Gamma(2-\mu)} + \frac{1}{\Gamma(-\mu)}\right)\vartheta^2 + \frac{1}{1440}\left(\frac{30\lambda(\lambda+1)(\mu+1)}{\Gamma(2-\mu)} + \frac{45(\lambda-1)\lambda(\lambda+1)(\lambda+2)}{\Gamma(3-\mu)} + \frac{7-5\mu}{\Gamma(-\mu)}\right)\vartheta^4 + \frac{1}{362\,880}\left(-\frac{63\lambda(\lambda+1)(4+\mu(3+5\mu))}{\Gamma(2-\mu)} - \frac{945(\lambda-1)\lambda(\lambda+1)(\lambda+2)(\mu+2)}{\Gamma(3-\mu)} - \frac{945(\lambda-1)\lambda(\lambda+1)(\lambda+2)\left(\lambda^2+\lambda-6\right)}{\Gamma(4-\mu)} + \frac{124+7\mu(5\mu-21)}{\Gamma(-\mu)}\right)\vartheta^6 + O\left(\vartheta^8\right)\right) /; (\vartheta \to 0)$$

07.37.06.0021.01

$$Y_\lambda^\mu(\vartheta, \varphi) \propto \frac{1}{\Gamma(1-\mu)} \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} 2^\mu e^{i\varphi\mu} \left(\vartheta^2\right)^{-\mu/2} \left(1 + O\left(\vartheta^2\right)\right) /; (\vartheta \to 0) \bigwedge \mu \notin \mathbb{N}^+$$

Expansions at $\vartheta == \pi$

07.37.06.0022.01

$$Y_\lambda^\mu(\vartheta, \varphi) \propto \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} e^{i\mu\varphi} \left(\frac{2^{-\mu}\Gamma(-\mu)}{\Gamma(-\lambda-\mu)\Gamma(\lambda-\mu+1)} \left((\vartheta-\pi)^2\right)^{\mu/2} \left(1 + \frac{\mu(\mu+1) - 3\lambda(\lambda+1)}{12(\mu+1)}(\vartheta-\pi)^2 + \left(45\lambda^3(\lambda+2) - 30\lambda\left(\mu^2+\mu+1\right) - 15\lambda^2\left(2\mu^2+2\mu-1\right) + \mu\left(5\mu^3+22\mu^2+31\mu+14\right)\right) \big/ (1440(\mu+1)(\mu+2))(\vartheta-\pi)^4 + O\left((\vartheta-\pi)^6\right)\right) - \frac{2^\mu \sin(\lambda\pi)\Gamma(\mu)}{\pi} \left((\vartheta-\pi)^2\right)^{-\mu/2} \left(1 + \frac{3\lambda(\lambda+1) + \mu(1-\mu)}{12(\mu-1)}(\vartheta-\pi)^2 + \left(45\lambda^3(\lambda+2) - \lambda^2\left(30\mu^2-30\mu-15\right) - 30\lambda\left(\mu^2-\mu+1\right) + \mu\left(5\mu^3-22\mu^2+31\mu-14\right)\right) \big/ (1440(\mu-2)(\mu-1)) (\vartheta-\pi)^4 + O\left((\vartheta-\pi)^6\right)\right)\right) /; (\vartheta \to \pi) \bigwedge \mu \notin \mathbb{Z}$$

07.37.06.0023.01

$$Y_\lambda^\mu(\vartheta, \varphi) \propto \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} e^{i\mu\varphi} \left(\frac{2^{-\mu}\Gamma(-\mu)}{\Gamma(-\lambda-\mu)\Gamma(\lambda-\mu+1)} \left((\vartheta-\pi)^2\right)^{\mu/2} \left(1 + O\left((\vartheta-\pi)^2\right)\right) - \frac{2^\mu \sin(\lambda\pi)\Gamma(\mu)}{\pi} \left((\vartheta-\pi)^2\right)^{-\mu/2} \left(1 + O\left((\vartheta-\pi)^2\right)\right)\right) /; (\vartheta \to \pi) \bigwedge \mu \notin \mathbb{Z}$$

07.37.06.0024.01

$$Y_\lambda^0(\vartheta,\varphi) \propto \frac{\sin(\pi\lambda)}{2}\sqrt{\frac{2\lambda+1}{\pi^3}} \left(\log\left(\frac{(\vartheta-\pi)^2}{4}\right)\left(1-\frac{\lambda(\lambda+1)}{4}(\vartheta-\pi)^2+\frac{\lambda(3\lambda^3+6\lambda^2+\lambda-2)}{192}(\vartheta-\pi)^4+O\left((\vartheta-\pi)^6\right)\right)+\psi(-\lambda)+\psi(\lambda+1)+2\gamma- \left(\frac{\lambda(\lambda+1)}{4}(\psi(1-\lambda)+\psi(\lambda+2)+2\gamma-2)+\frac{1}{12}\right)(\vartheta-\pi)^2-\frac{1}{2880}(-60\lambda(\lambda+1)-60\lambda(\psi(1-\lambda)+\psi(\lambda+2)+2\gamma-2)(\lambda+1)-45(\lambda-1)\lambda(\lambda+2)(\psi(2-\lambda)+\psi(\lambda+3)+2\gamma-3)(\lambda+1)+2)(\vartheta-\pi)^4+O\left((\vartheta-\pi)^6\right)\right)/;(\vartheta\to\pi)$$

07.37.06.0025.01

$$Y_\lambda^0(\vartheta,\varphi) \propto \frac{\sin(\pi\lambda)}{2}\sqrt{\frac{2\lambda+1}{\pi^3}}\left(\log\left(\frac{(\vartheta-\pi)^2}{4}\right)\left(1+O\left((\vartheta-\pi)^2\right)\right)+\psi(-\lambda)+\psi(\lambda+1)+2\gamma+O\left((\vartheta-\pi)^2\right)\right)/;(\vartheta\to\pi)$$

Expansions at $\cos(\vartheta) == \infty$

07.37.06.0026.02

$$Y_\lambda^\mu(\vartheta,\varphi) \propto \frac{\sqrt{2\lambda+1}\sqrt{\Gamma(\lambda-\mu+1)}\,e^{i\varphi\mu}}{2\pi\sqrt{\Gamma(\lambda+\mu+1)}}\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\left(\frac{2^\lambda(\cos(\vartheta)-1)^\lambda}{\Gamma(\lambda-\mu+1)}\Gamma\left(\lambda+\frac{1}{2}\right)\left(1+\frac{\mu-\lambda}{1-\cos(\vartheta)}+\frac{(1-\lambda)(\mu-\lambda)(1+\mu-\lambda)}{(1-2\lambda)(1-\cos(\vartheta))^2}+\ldots\right)+\frac{2^{-1-\lambda}(z-1)^{-1-\lambda}}{\Gamma(-\lambda-\mu)}\Gamma\left(-\lambda-\frac{1}{2}\right)\left(1+\frac{1+\mu+\lambda}{1-\cos(\vartheta)}+\frac{(2+\lambda)(1+\mu+\lambda)(2+\mu+\lambda)}{(3+2\lambda)(\cos(\vartheta)-1)^2}+\ldots\right)\right)/;\left|\sin\left(\frac{\vartheta}{2}\right)\right|>1\bigwedge 2\lambda\notin\mathbb{Z}$$

07.37.06.0027.01

$$Y_\lambda^\mu(\vartheta,\varphi) == \frac{\sqrt{2\lambda+1}\sqrt{\Gamma(\lambda-\mu+1)}\,e^{i\varphi\mu}}{2\pi\sqrt{\Gamma(\lambda+\mu+1)}}\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\left(\frac{2^\lambda\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda-\mu+1)}(\cos(\vartheta)-1)^\lambda\sum_{k=0}^{\infty}\frac{(-\lambda)_k(\mu-\lambda)_k}{(-2\lambda)_k\,k!}\left(\frac{2}{1-\cos(\vartheta)}\right)^k+\frac{2^{-\lambda-1}\Gamma\left(-\lambda-\frac{1}{2}\right)}{\Gamma(-\lambda-\mu)}(\cos(\vartheta)-1)^{-1-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda+1)_k(\lambda+\mu+1)_k}{(2\lambda+2)_k\,k!}\left(\frac{2}{1-\cos(\vartheta)}\right)^k\right)/;\left|\sin\left(\frac{\vartheta}{2}\right)\right|>1\bigwedge 2\lambda\notin\mathbb{Z}$$

07.37.06.0028.01

$$Y_\lambda^\mu(\vartheta,\varphi) == \frac{\sqrt{2\lambda+1}\sqrt{\Gamma(\lambda-\mu+1)}\,e^{i\varphi\mu}}{2\pi\sqrt{\Gamma(\lambda+\mu+1)}}\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\left(\frac{2^\lambda(\cos(\vartheta)-1)^\lambda}{\Gamma(\lambda-\mu+1)}\Gamma\left(\lambda+\frac{1}{2}\right)\left({}_2F_1\left(\mu-\lambda,-\lambda;-2\lambda;\frac{2}{1-\cos(\vartheta)}\right)\right)+\frac{2^{-1-\lambda}(\cos(\vartheta)-1)^{-1-\lambda}}{\Gamma(-\mu-\lambda)}\Gamma\left(-\lambda-\frac{1}{2}\right)\left({}_2F_1\left(\lambda+1,\lambda+\mu+1;2\lambda+2;\frac{2}{1-\cos(\vartheta)}\right)\right)\right)/;\vartheta\notin\mathbb{R}\bigwedge 2\lambda\notin\mathbb{Z}$$

07.37.06.0029.01

$$Y_\lambda^\mu(\vartheta,\varphi) \propto \frac{\sqrt{2\lambda+1}\,\sqrt{\Gamma(\lambda-\mu+1)}\,e^{i\varphi\mu}}{2\pi\sqrt{\Gamma(\lambda+\mu+1)}}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}$$

$$\left(\frac{2^\lambda\cos^\lambda(\vartheta)}{\Gamma(\lambda-\mu+1)}\,\Gamma\left(\lambda+\frac{1}{2}\right)\left(1+O\left(\frac{1}{\cos(\vartheta)}\right)\right)+\frac{2^{-1-\lambda}\cos(\vartheta)^{-1-\lambda}}{\Gamma(-\lambda-\mu)}\,\Gamma\left(-\lambda-\frac{1}{2}\right)\left(1+O\left(\frac{1}{\cos(\vartheta)}\right)\right)\right)\text{/; }(|\cos(\vartheta)|\to\infty)\bigwedge 2\lambda\notin\mathbb{Z}$$

07.37.06.0030.01

$$Y_\lambda^\mu(\vartheta,\varphi) = \frac{e^{i\varphi\mu}\sin(\pi(\lambda-\mu))\sin(\pi\lambda)\sqrt{2\lambda+1}\,\sqrt{\Gamma(\lambda-\mu+1)}\,\sqrt{\Gamma(\lambda+\mu+1)}}{2^{\lambda+1}\pi^2\,\Gamma\left(\lambda+\frac{3}{2}\right)}\,(\cos(\vartheta)-1)^{-\lambda-1}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}$$

$$\sum_{k=0}^{\infty}\frac{(\lambda+1)_k\,(\lambda+\mu+1)_k}{k!\,(2\lambda+2)_k}\,(\psi(k+1)-\psi(-k-\lambda-\mu)-\psi(k+\lambda+1)+\psi(k+2\lambda+2))\left(\frac{2}{1-\cos(\vartheta)}\right)^k+$$

$$\frac{2^{\lambda+1}\sin(\pi(\mu-\lambda))\,\Gamma(\lambda+\mu+1)}{\pi\,\Gamma(-\lambda)\,\Gamma(2\lambda+2)}\,(\cos(\vartheta)-1)^{-\lambda-1}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\,\log\left(\frac{\cos(\vartheta)-1}{2}\right){}_2F_1\left(\lambda+1,\,\lambda+\mu+1;\,2\lambda+2;\,\frac{2}{1-\cos(\vartheta)}\right)+$$

$$\frac{2^{-\lambda}}{\Gamma(\lambda+1)}\,(\cos(\vartheta)-1)^\lambda\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\sum_{k=0}^{2\lambda}\frac{(2\lambda-k)!\,(-\lambda)_k}{k!\,\Gamma(-k+\lambda-\mu+1)}\left(\frac{2}{1-\cos(\vartheta)}\right)^k\text{/; }2\lambda+1\in\mathbb{N}\bigwedge\lambda-\mu\notin\mathbb{Z}$$

07.37.06.0031.01

$$Y_\lambda^\mu(\vartheta,\varphi) \propto \frac{e^{i\varphi\mu}\,2^\lambda\sqrt{2\lambda+1}\,\sqrt{\Gamma(\lambda-\mu+1)}\,\sqrt{\Gamma(\lambda+\mu+1)}\,\sin(\pi(\mu-\lambda))\cos^{-\lambda-1}(\vartheta)}{\pi^{3/2}\,\Gamma(-\lambda)\,\Gamma(2\lambda+2)}$$

$$\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\left(\log\left(\frac{\cos(\vartheta)}{2}\right)-\gamma-\psi(-\lambda-\mu)-\psi(\lambda+1)+\psi(2\lambda+2)\right)\left(1+O\left(\frac{1}{\cos(\vartheta)}\right)\right)+$$

$$\frac{2^\lambda\,\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\lambda-\mu+1)}\cos^\lambda(\vartheta)\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\left(1+O\left(\frac{1}{\cos(\vartheta)}\right)\right)\text{/; }2\lambda\in\mathbb{N}\bigwedge\lambda-\mu\notin\mathbb{Z}$$

07.37.06.0032.01

$$Y_\lambda^\mu(\vartheta,\varphi) = \frac{e^{i\varphi\mu}\,(-1)^{\mu-\lambda-1}\,2^\lambda\sqrt{2\lambda+1}\,\sqrt{\Gamma(\lambda-\mu+1)}\,\sqrt{\Gamma(\lambda+\mu+1)}}{\sqrt{\pi}\,\Gamma(2\lambda+2)\,\Gamma(-\lambda)}\,(\cos(\vartheta)-1)^{-\lambda-1}$$

$$\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\,{}_2F_1\left(\lambda+1,\,\lambda+\mu+1;\,2\lambda+2;\,\frac{2}{1-\cos(\vartheta)}\right)\text{/; }2\lambda+1\in\mathbb{N}\bigwedge\mu-\lambda\in\mathbb{N}^+\bigwedge\lambda\notin\mathbb{N}$$

07.37.06.0033.01

$$Y_\lambda^\mu(\vartheta,\varphi) \propto \frac{e^{i\varphi\mu}(-1)^{\mu-\lambda-1}2^\lambda\sqrt{2\lambda+1}\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu+1)}}{\sqrt{\pi}\,\Gamma(2\lambda+2)\,\Gamma(-\lambda)}\cos^{-\lambda-1}(\vartheta)\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\left(1+O\left(\frac{1}{\cos(\vartheta)}\right)\right)/;$$

$$(|\cos(\vartheta)|\to\infty)\bigwedge 2\lambda+1\in\mathbb{N}\bigwedge\mu-\lambda\in\mathbb{N}^+\bigwedge\lambda\notin\mathbb{N}$$

07.37.06.0034.01

$$Y_\lambda^\mu(\vartheta,\varphi) = \frac{e^{i\varphi\mu}2^{\lambda-1}\Gamma\left(\lambda+\frac{1}{2}\right)\sqrt{2\lambda+1}}{\pi\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu+1)}}(\cos(\vartheta)-1)^\lambda\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\sum_{k=0}^{\lambda-\mu}\frac{(\mu-\lambda)_k(-\lambda)_k}{k!\,(-2\lambda)_k}\left(\frac{2}{1-\cos(\vartheta)}\right)^k -$$

$$\frac{(-1)^{\lambda-\mu}2^{\lambda+1}\Gamma(\lambda+\mu+1)}{\Gamma(-\lambda)\,\Gamma(2\lambda+2)}(\cos(\vartheta)-1)^{-\lambda-1}\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\left({}_2F_1\left(\lambda+1,\lambda+\mu+1;2\lambda+2;\frac{2}{1-\cos(\vartheta)}\right)\right)/;$$

$$2\lambda+1\in\mathbb{N}\bigwedge\lambda-\mu\in\mathbb{Z}\bigwedge-\lambda\le\mu\le\lambda+1$$

07.37.06.0035.01

$$Y_\lambda^\mu(\vartheta,\varphi) \propto \frac{e^{i\varphi\mu}2^{\lambda-1}\Gamma\left(\lambda+\frac{1}{2}\right)\sqrt{2\lambda+1}}{\pi\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu+1)}}\cos^\lambda(\vartheta)\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\left(1+O\left(\frac{1}{\cos(\vartheta)}\right)\right)/;$$

$$(|\cos(\vartheta)|\to\infty)\bigwedge 2\lambda+1\in\mathbb{N}\bigwedge\lambda-\mu\in\mathbb{Z}\bigwedge-\lambda\le\mu\le\lambda+1$$

07.37.06.0036.01

$$Y_\lambda^\mu(\vartheta,\varphi) = \sqrt{\frac{2\lambda+1}{\pi}}\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}e^{i\varphi\mu}\frac{(-1)^{2\lambda+1}2^\lambda}{\Gamma(-\mu-\lambda)\,\Gamma(-\lambda)}(\cos(\vartheta)-1)^{-\lambda-1}\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}$$

$$\sum_{k=0}^{-\mu-\lambda-1}\frac{(\lambda+1)_k(\lambda+\mu+1)_k}{k!\,(2\lambda+k+1)!}\left(\log\left(\frac{\cos(\vartheta)-1}{2}\right)+\psi(k+1)-\psi(-k-\lambda-\mu)-\psi(k+\lambda+1)+\psi(k+2\lambda+2)\right)\left(\frac{2}{1-\cos(\vartheta)}\right)^k +$$

$$\frac{2^\lambda\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\lambda-\mu+1)}(\cos(\vartheta)-1)^\lambda\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\sum_{k=0}^{2\lambda}\frac{(\mu-\lambda)_k(-\lambda)_k}{k!\,(-2\lambda)_k}\left(\frac{2}{1-\cos(\vartheta)}\right)^k+\frac{(-1)^{2\lambda}2^{1-\mu}\sin(\pi\lambda)\,\Gamma(1-\mu)}{\pi\,\Gamma(1-\lambda-\mu)\,\Gamma(\lambda-\mu+2)}$$

$$(\cos(\vartheta)-1)^{\mu-1}\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{}_3F_2\left(1,1,1-\mu;-\lambda-\mu+1,\lambda-\mu+2;\frac{2}{1-\cos(\vartheta)}\right)/;2\lambda+1\in\mathbb{N}\bigwedge\lambda-\mu\in\mathbb{Z}\bigwedge\lambda+\mu<0$$

07.37.06.0037.01

$$Y_\lambda^\mu(\vartheta,\varphi) \propto \sqrt{\frac{2\lambda+1}{\pi}}\, \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\, e^{i\varphi\mu} \frac{(-1)^{2\lambda+1}\, 2^\lambda}{\Gamma(-\mu-\lambda)\,\Gamma(-\lambda)\,\Gamma(2\lambda+2)} \left(\log\left(\frac{\cos(\vartheta)}{2}\right) - \psi(-\lambda-\mu) - \psi(\lambda+1) + \psi(2\lambda+2) - \gamma\right)$$

$$\cos^{-\lambda-1}(\vartheta)\, \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \left(1 + O\left(\frac{1}{\cos(\vartheta)}\right)\right) + \frac{2^\lambda\, \Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}\, \Gamma(\lambda-\mu+1)} \cos^\lambda(\vartheta)\, \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \left(1 + O\left(\frac{1}{\cos(\vartheta)}\right)\right) +$$

$$\frac{(-1)^{2\lambda}\, 2^{1-\mu}\, \sin(\pi\lambda)\,\Gamma(1-\mu)}{\pi\,\Gamma(-\lambda-\mu+1)\,\Gamma(\lambda-\mu+2)} \cos^{\mu-1}(\vartheta)\, \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \left(1 + O\left(\frac{1}{\cos(\vartheta)}\right)\right) /;$$

$$(|\cos(\vartheta)| \to \infty) \bigwedge 2\lambda + 1 \in \mathbb{N} \bigwedge \lambda - \mu \in \mathbb{Z} \bigwedge \lambda + \mu < 0$$

Integral representations

On the real axis

Of the direct function

07.37.07.0001.01

$$Y_n^m(\vartheta,\varphi) = \frac{1}{4\pi^{3/2}\, i^{3m}\, n!} \sqrt{(2n+1)\,(n-m)!\,(n+m)!} \int_0^{2\pi} (\cos(\vartheta) + i\cos(t-\varphi)\sin(\vartheta))^n\, e^{i m t}\, dt \,/;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land -n \le m \le n$$

07.37.07.0002.01

$$Y_n^m(\vartheta,\varphi) = (-1)^m\, e^{i m\varphi} \sqrt{\frac{2n+1}{4\pi}\, \frac{(n+m)!}{(n-m)!}}\, \frac{\csc^m(\vartheta)}{(m-1)!} \int_{\cos(\vartheta)}^1 P_n(t)\,(t-\cos(\vartheta))^{m-1}\, dt \,/;\, n \in \mathbb{N} \land m \in \mathbb{N} \land m \le n$$

07.37.07.0003.01

$$Y_n^m(\vartheta,\varphi) = \frac{i^m}{2\pi^{3/2}}\, e^{i m\varphi} \sqrt{(2n+1)\,(n+m)!\,(n-m)!}\, \frac{1}{n!} \int_0^\pi (\cos(\vartheta) + i\sin(\vartheta)\cos(t))^n \cos(m t)\, dt \,/;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \le n$$

07.37.07.0004.01

$$Y_n^m(\vartheta,\varphi) = \frac{i^m}{4\pi^{3/2}} \sqrt{(2n+1)\,(n+m)!\,(n-m)!}\, \frac{1}{n!} \int_0^{2\pi} (\cos(\vartheta) + i\sin(\vartheta)\cos(t-\varphi))^n\, e^{i m t}\, dt \,/;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \le n$$

07.37.07.0005.01

$$Y_n^m(\vartheta,\varphi) = \frac{(-1)^m}{2\pi^{3/2}}\, e^{i m\varphi} \sqrt{(2n+1)\,\frac{(n+m)!}{(n-m)!}}\, \frac{\sin(\vartheta)^m}{(2m-1)!!} \int_0^\pi (\cos(\vartheta) + i\sin(\vartheta)\cos(t))^{n-m} \sin(t)^{2m}\, dt \,/;\, n \in \mathbb{N} \land m \in \mathbb{N} \land m \le n$$

07.37.07.0006.01

$$Y_n^m(\vartheta,\varphi) = (-1)^m\, e^{i m\varphi} \sqrt{\frac{2n+1}{4\pi}\, \frac{1}{(n+m)!\,(n-m)!}} \int_0^\infty e^{-t\cos(\vartheta)}\, J_m(t\sin(\vartheta))\, t^n\, dt \,/;\, n \in \mathbb{N} \bigwedge m \in \mathbb{Z} \bigwedge |m| \le n \bigwedge \cos(\vartheta) > 0$$

Involving the direct function

07.37.07.0007.01

$$\left|Y_n^m(\vartheta,\varphi)\right|^2 = \frac{2n+1}{4\pi} \int_0^\infty J_m\left(\frac{t}{2}\sin(\vartheta)\right)^2 J_{2n+1}(t)\, dt \,/;\, n \in \mathbb{N} \bigwedge m \in \mathbb{Z} \bigwedge |m| \le n \bigwedge \vartheta \in \mathbb{R} \bigwedge \varphi \in \mathbb{R} \bigwedge 0 < \sin(\vartheta) < 1$$

Multiple integral representations

07.37.07.0008.01

$$Y_n^m(\vartheta, \varphi) = e^{i m \varphi} \sqrt{\frac{2n+1}{4\pi} \frac{(n+m)!}{(n-m)!}} \csc(\vartheta)^m \underbrace{\int_1^{\cos(\vartheta)} \ldots \int_1^{t_3} \int_1^{t_2}}_{m} P_n(t_1)\, dt_1\, dt_2 \ldots dt_m \;/;\; n \in \mathbb{N} \land m \in \mathbb{N} \land m \leq n$$

Integral representations of negative integer order

07.37.07.0009.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^{n+m} r^{n+1}}{(n-m)!} \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)^m \frac{\partial^{n-m} \frac{1}{r}}{\partial z^{n-m}} \;/;$$

$$r = \sqrt{x^2+y^2+z^2} \bigwedge x = r\cos(\varphi)\sin(\vartheta) \bigwedge y = r\sin(\varphi)\sin(\vartheta) \bigwedge z = r\cos(\vartheta) \bigwedge n \in \mathbb{N} \bigwedge m \in \mathbb{N} \bigwedge m \leq n$$

07.37.07.0010.01

$$Y_n^m(\vartheta, \varphi) = \frac{1}{2^n n!} \sqrt{\frac{(2n+1)(n+m)!}{4\pi(n-m)!}}\, e^{i m \varphi} \sin^2(\vartheta)^{-\frac{m}{2}} \frac{\partial^{n-m}\left(z^2-1\right)^n}{\partial z^{n-m}} \;/;\; z = \cos(\vartheta) \land n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.07.0011.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^m}{2^n n!} \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}\, e^{i m \varphi} \sin^2(\vartheta)^{\frac{m}{2}} \frac{\partial^{n+m}\left(z^2-1\right)^n}{\partial z^{n+m}} \;/;\; z = \cos(\vartheta) \land n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.07.0012.01

$$Y_n^m(\vartheta, \varphi) = \frac{e^{i m \varphi}}{2^{n+1}\sqrt{\pi}} \sqrt{\frac{2n+1}{(n-m)!\,(n+m)!}} \cos^m\left(\frac{\vartheta}{2}\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \cot^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^{-m}\left(\frac{\vartheta}{2}\right) \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \frac{\partial^n\left((z+1)^{n-m}(z-1)^{n+m}\right)}{\partial z^n} \;/;$$

$$z = \cos(\vartheta) \bigwedge n \in \mathbb{N} \bigwedge m \in \mathbb{Z} \bigwedge |m| \leq n$$

07.37.07.0013.01

$$Y_n^m(\vartheta, \varphi) = \frac{(-1)^m e^{i m \varphi}}{2^{n+1}\sqrt{\pi}} \sqrt{\frac{2n+1}{(n-m)!\,(n+m)!}} \cos^{-m}\left(\frac{\vartheta}{2}\right) \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \cot^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^m\left(\frac{\vartheta}{2}\right) \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \frac{\partial^n\left((z+1)^{n+m}(z-1)^{n-m}\right)}{\partial z^n} \;/;$$

$$z = \cos(\vartheta) \bigwedge n \in \mathbb{N} \bigwedge m \in \mathbb{Z} \bigwedge |m| \leq n$$

07.37.07.0014.01

$$Y_\lambda^m(\vartheta, \varphi) = \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-m+1)}}{\sqrt{\Gamma(\lambda+m+1)}} 2^{-\lambda} e^{i m \varphi} \sin^m(\vartheta) \frac{\partial^m P_\lambda(z)}{\partial z^m} \;/;\; z = \cos(\vartheta) \land m \in \mathbb{N}$$

07.37.07.0015.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{(2n+1)(n+m)!}{4\pi(n-m)!}} \frac{e^{i m \varphi}}{2^n n!} \frac{1}{\sin^2(\vartheta)^{m/2}} \frac{\partial^{n-m}\left(z^2-1\right)^n}{\partial z^{n-m}} \;/;\; z = \cos(\vartheta) \land n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.07.0016.01

$$Y_n^m(\vartheta, \varphi) = (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}} \frac{e^{i m \varphi}}{2^n n!} \sin^2(\vartheta)^{m/2} \frac{\partial^{n+m}\left(z^2-1\right)^n}{\partial z^{n+m}} \;/;\; z = \cos(\vartheta) \land n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.07.0017.01

$$Y_n^m(\vartheta, \varphi) == \sqrt{\frac{2n+1}{4\pi (n+m)!\,(n-m)!}}\,\frac{e^{i m \varphi}}{2^n} \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \cot^m\left(\frac{\vartheta}{2}\right) \cot^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \frac{\partial^n \left((z+1)^{n-m} (z-1)^{n+m}\right)}{\partial z^n} \,/;\, z == \cos(\vartheta) \land n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.07.0018.01

$$Y_n^m(\vartheta, \varphi) == (-1)^m \sqrt{\frac{2n+1}{4\pi (n+m)!\,(n-m)!}}\,\frac{e^{i m \varphi}}{2^n} \cos^2\left(\frac{\vartheta}{2}\right)^{m/2} \cot^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{m}{2}} \tan^m\left(\frac{\vartheta}{2}\right) \frac{\partial^n \left((z+1)^{n+m} (z-1)^{n-m}\right)}{\partial z^n} \,/;\, z == \cos(\vartheta) \land n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.07.0019.01

$$Y_n^m(\vartheta, \varphi) == (-1)^m \sqrt{\frac{(2n+1)\,(n-m)!}{4\pi (n+m)!}}\, e^{i m \varphi} \sin^m(\vartheta) \frac{\partial^m P_n(z)}{\partial z^m} \,/;\, z == \cos(\vartheta) \land n \in \mathbb{N} \land m \in \mathbb{N} \land m \leq n$$

Generating functions

07.37.11.0001.01

$$\frac{1}{\left(w^2 - 2\cos(\vartheta)\, w + 1\right)^{m+1/2}} == \frac{(-1)^m}{(2m-1)!!\,\sin(\vartheta)^m} \sum_{n=m}^{\infty} w^{n-m} \sqrt{\frac{4\pi}{2n+1}\,\frac{(n+m)!}{(n-m)!}}\; Y_n^m(\vartheta, 0) \,/;\, m \in \mathbb{N} \land \vartheta \in \mathbb{R} \land |w| < 1$$

07.37.11.0002.01

$$\frac{1}{\left(w^2 - 2\cos(\vartheta)\, w + 1\right)^{m+1/2}} == \frac{(-1)^m}{(2m-1)!!\,\sin(\vartheta)^m} \sum_{n=m}^{\infty} \frac{1}{w^{n+m+1}} \sqrt{\frac{4\pi}{2n+1}\,\frac{(n+m)!}{(n-m)!}}\; Y_n^m(\vartheta, 0) \,/;\, m \in \mathbb{N} \land \vartheta \in \mathbb{R} \land |w| > 1$$

07.37.11.0003.01

$$\frac{\left(\left(1 + \sqrt{1 - 2w\cos(\vartheta) + w^2}\right)^2 - w^2\right)^{-m}}{\sqrt{1 - 2w\cos(\vartheta) + w^2}} == \frac{(-1)^m}{2^m \sin^m(\vartheta)} \sum_{n=m}^{\infty} w^{n-m}\, n! \sqrt{\frac{4\pi}{(2n+1)\,(n+m)!\,(n-m)!}}\; Y_n^m(\vartheta, 0) \,/;\, m \in \mathbb{N} \land \vartheta \in \mathbb{R} \land |w| < 1$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

With respect to ϑ

07.37.13.0001.01

$$\frac{1}{\sin(\vartheta)} \frac{\partial}{\partial \vartheta}\left(\sin(\vartheta) \frac{\partial Y_\lambda^\mu(\vartheta, \varphi)}{\partial \vartheta}\right) + \left(\lambda(\lambda+1) - \frac{\mu^2}{\sin^2(\vartheta)}\right) Y_\lambda^\mu(\vartheta, \varphi) == 0$$

With respect to φ

07.37.13.0002.01

$$i \frac{\partial Y_\lambda^\mu(\vartheta, \varphi)}{\partial \varphi} + \mu\, Y_\lambda^\mu(\vartheta, \varphi) == 0$$

Partial differential equations

07.37.13.0003.01

$$e^{i\varphi} \left(\frac{\partial Y_\lambda^\mu(\vartheta, \varphi)}{\partial \vartheta} + i \cot(\vartheta) \frac{\partial Y_\lambda^\mu(\vartheta, \varphi)}{\partial \varphi} \right) == \frac{\sqrt{\Gamma(\lambda - \mu + 1)}\, \sqrt{\Gamma(\lambda + \mu + 2)}}{\sqrt{\Gamma(\lambda + \mu + 1)}\, \sqrt{\Gamma(\lambda - \mu)}} Y_\lambda^{\mu+1}(\vartheta, \varphi)$$

07.37.13.0004.01

$$\frac{1}{\sin(\vartheta)} \frac{\partial}{\partial \vartheta} \left(\sin(\vartheta) \frac{\partial Y_\lambda^\mu(\vartheta, \varphi)}{\partial \vartheta} \right) + \frac{1}{\sin^2(\vartheta)} \frac{\partial^2 Y_\lambda^\mu(\vartheta, \varphi)}{\partial \varphi^2} + \lambda\, (\lambda + 1)\, Y_\lambda^\mu(\vartheta, \varphi) == 0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

07.37.16.0001.01

$$Y_{-\lambda}^\mu(\vartheta, \varphi) == \frac{\sqrt{1 - 2\lambda}\, \sqrt{\Gamma(1 - \lambda - \mu)}\, \sqrt{\Gamma(\lambda + \mu)}}{\sqrt{2\lambda - 1}\, \sqrt{\Gamma(1 - \lambda + \mu)}\, \sqrt{\Gamma(\lambda - \mu)}} Y_{\lambda-1}^\mu(\vartheta, \varphi)$$

07.37.16.0002.01

$$Y_n^{-m}(\vartheta, \varphi) == \overline{(-1)^m\, Y_n^m(\vartheta, \varphi)} \,/;\, m \in \mathbb{Z} \bigwedge n \in \mathbb{Z} \bigwedge \vartheta \in \mathbb{R} \bigwedge \varphi \in \mathbb{R}$$

07.37.16.0003.01

$$Y_n^{-m}(\vartheta, \varphi) == (-1)^m\, e^{-2 i \varphi m}\, Y_n^m(\vartheta, \varphi) \,/;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.16.0004.01

$$Y_\lambda^\mu(-\vartheta, \varphi) == Y_\lambda^\mu(\vartheta, \varphi)$$

07.37.16.0005.01

$$Y_\lambda^\mu(\vartheta, -\varphi) == e^{-2 i \mu \varphi}\, Y_\lambda^\mu(\vartheta, \varphi)$$

07.37.16.0006.01

$$Y_n^m(\vartheta, -\varphi) == (-1)^m\, Y_n^{-m}(\vartheta, \varphi) \,/;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

07.37.16.0007.01

$$Y_n^m(-\vartheta, -\varphi) == (-1)^m\, Y_n^{-m}(\vartheta, \varphi) \,/;\, n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \leq n$$

Products, sums, and powers of the direct function

Products involving the direct function

Clebsch-Gordan series for product of two spherical harmonics

07.37.16.0008.01

$$Y_{n_1}^{m_1}(\vartheta,\varphi)\,Y_{n_2}^{m_2}(\vartheta,\varphi) = \sqrt{\frac{(2n_1+1)(2n_2+1)}{4\pi}} \sum_{k=\max(|n_1-n_2|,|m_1+m_2|)}^{n_1+n_2} \frac{1}{\sqrt{2k+1}} Y_k^{m_1+m_2}(\vartheta,\varphi)\,\langle n_1\, n_2\, 0\, 0 \mid n_1\, n_2\, k\, 0\rangle \langle n_1\, n_2\, m_1\, m_2 \mid n_1\, n_2\, k\, m_1+m_2\rangle \;/;\; n_1 \in \mathbb{N} \land n_2 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land m_2 \in \mathbb{Z} \land |m_1| \le n_1 \land |m_2| \le n_2$$

Clebsch-Gordan double series for product of three spherical harmonics

07.37.16.0009.01

$$Y_{n_1}^{m_1}(\vartheta,\varphi)\,Y_{n_2}^{m_2}(\vartheta,\varphi)\,Y_{n_3}^{m_3}(\vartheta,\varphi) = \frac{\sqrt{(2n_1+1)(2n_2+1)(2n_3+1)}}{4\pi} \sum_{k_1=\max(|n_1-n_2|,|m_1+m_2|)}^{n_1+n_2} \sum_{k_2=\max(|k_1-n_3|,|m_1+m_2+m_3|)}^{k_1+n_3} \frac{1}{\sqrt{2k_2+1}} Y_{k_2}^{m_1+m_2+m_3}(\vartheta,\varphi) \langle n_1\, n_2\, 0\, 0 \mid n_1\, n_2\, k_1\, 0\rangle \langle k_1\, n_3\, 0\, 0 \mid k_1\, n_3\, k_2\, 0\rangle \langle n_1\, n_2\, m_1\, m_2 \mid n_1\, n_2\, k_1\, m_1+m_2\rangle \langle k_1\, n_3\, m_1+m_2\, m_3 \mid k_1\, n_3\, k_2\, m_1+m_2+m_3\rangle \;/;\; n_j \in \mathbb{N} \land m_j \in \mathbb{Z} \land |m_j| \le n_j \land j \in \{1,2,3\}$$

Clebsch-Gordan multiple series for product of several spherical harmonics

07.37.16.0010.01

$$\prod_{j=1}^{p} Y_{n_j}^{m_j}(\vartheta,\varphi) = \frac{1}{(4\pi)^{\frac{p-1}{2}}} \prod_{j=1}^{p} \sqrt{2n_j+1} \sum_{k_1=\max(|n_1-n_2|,|M_2|)}^{n_1+n_2} \sum_{k_2=\max(|k_1-n_3|,|M_3|)}^{k_1+n_3} \cdots \sum_{k_{p-1}=\max(|k_{p-2}-n_p|,|M_p|)}^{k_{p-2}+n_p} \frac{1}{\sqrt{2k_{p-1}+1}} Y_{k_{p-1}}^{M_p}(\vartheta,\varphi) \prod_{j=1}^{p-1} \langle k_{j-1}\, n_{j+1}\, 0\, 0 \mid k_{j-1}\, n_{j+1}\, k_j\, 0\rangle \langle k_{j-1}\, n_{j+1}\, M_j\, m_{j+1} \mid k_{j-1}\, n_{j+1}\, k_j\, M_{j+1}\rangle \;/;$$

$$p \in \mathbb{Z} \bigwedge p > 1 \bigwedge n_k \in \mathbb{N} \bigwedge m_k \in \mathbb{Z} \bigwedge |m_k| \le n_k \bigwedge k_0 = n_1 \bigwedge M_0 = 0 \bigwedge M_j = \sum_{k=1}^{j} m_k$$

Identities

Recurrence identities

Consecutive neighbors

07.37.17.0001.01

$$Y_\lambda^\mu(\vartheta,\varphi) = \frac{(2\lambda+3)\sqrt{2\lambda+1}\,\sqrt{\Gamma(\lambda-\mu+1)}\,\sqrt{\Gamma(\lambda+\mu+2)}}{(\lambda+\mu+1)\sqrt{2\lambda+3}\,\sqrt{\Gamma(\lambda+\mu+1)}\,\sqrt{\Gamma(\lambda-\mu+2)}} \cos(\vartheta)\, Y_{\lambda+1}^\mu(\vartheta,\varphi) + \frac{(\mu-\lambda-2)\sqrt{2\lambda+1}\,\sqrt{\Gamma(\lambda-\mu+1)}\,\sqrt{\Gamma(\lambda+\mu+3)}}{(\lambda+\mu+1)\sqrt{2\lambda+5}\,\sqrt{\Gamma(\lambda+\mu+1)}\,\sqrt{\Gamma(\lambda-\mu+3)}} Y_{\lambda+2}^\mu(\vartheta,\varphi)$$

07.37.17.0002.01

$$Y_\lambda^\mu(\vartheta,\varphi)=\frac{(2\lambda-1)\sqrt{2\lambda+1}\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu)}}{(\lambda-\mu)\sqrt{2\lambda-1}\sqrt{\Gamma(\lambda+\mu+1)}\sqrt{\Gamma(\lambda-\mu)}}\cos(\vartheta)\,Y_{\lambda-1}^\mu(\vartheta,\varphi)-\frac{(\lambda+\mu-1)\sqrt{2\lambda+1}\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu-1)}}{(\lambda-\mu)\sqrt{2\lambda-3}\sqrt{\Gamma(\lambda+\mu+1)}\sqrt{\Gamma(\lambda-\mu-1)}}Y_{\lambda-2}^\mu(\vartheta,\varphi)$$

07.37.17.0003.01

$$Y_\lambda^\mu(\vartheta,\varphi)=\frac{2(\mu+1)\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu+2)}}{(\mu(\mu+1)-\lambda(\lambda+1))\sqrt{\Gamma(\lambda+\mu+1)}\sqrt{\Gamma(\lambda-\mu)}}e^{-i\varphi}\cot(\vartheta)\,Y_\lambda^{\mu+1}(\vartheta,\varphi)+\frac{\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu+3)}}{(\mu(\mu+1)-\lambda(\lambda+1))\sqrt{\Gamma(\lambda+\mu+1)}\sqrt{\Gamma(\lambda-\mu-1)}}e^{-2i\varphi}Y_\lambda^{\mu+2}(\vartheta,\varphi)$$

07.37.17.0004.01

$$Y_\lambda^\mu(\vartheta,\varphi)=\frac{2(1-\mu)\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu)}}{\sqrt{\Gamma(\lambda+\mu+1)}\sqrt{\Gamma(\lambda-\mu+2)}}e^{i\varphi}\cot(\vartheta)\,Y_\lambda^{\mu-1}(\vartheta,\varphi)+\frac{((\mu-1)(\mu-2)-\lambda(\lambda+1))\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu-1)}}{\sqrt{\Gamma(\lambda+\mu+1)}\sqrt{\Gamma(\lambda-\mu+3)}}e^{2i\varphi}Y_\lambda^{\mu-2}(\vartheta,\varphi)$$

Functional identities

Relations between contiguous functions

07.37.17.0005.01

$$Y_\lambda^\mu(\vartheta,\varphi)=\sec(\vartheta)\left(\frac{(\lambda+\mu)\sqrt{\Gamma(\lambda+\mu)}\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{2\lambda+1}\sqrt{2\lambda-1}\sqrt{\Gamma(\lambda-\mu)}\sqrt{\Gamma(\lambda+\mu+1)}}Y_{\lambda-1}^\mu(\vartheta,\varphi)+\frac{(\lambda-\mu+1)\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu+2)}}{\sqrt{2\lambda+1}\sqrt{2\lambda+3}\sqrt{\Gamma(\lambda+\mu+1)}\sqrt{\Gamma(\lambda-\mu+2)}}Y_{\lambda+1}^\mu(\vartheta,\varphi)\right)$$

07.37.17.0006.01

$$Y_\lambda^\mu(\vartheta,\varphi)=-\frac{\tan(\vartheta)}{2\mu}\left(\frac{(\lambda(\lambda+1)-\mu(\mu-1))\sqrt{\Gamma(\lambda+\mu)}\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda-\mu+2)}\sqrt{\Gamma(\lambda+\mu+1)}}e^{i\varphi}Y_\lambda^{\mu-1}(\vartheta,\varphi)+\frac{\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu+2)}}{\sqrt{\Gamma(\lambda+\mu+1)}\sqrt{\Gamma(\lambda-\mu)}}e^{-i\varphi}Y_\lambda^{\mu+1}(\vartheta,\varphi)\right)$$

Additional relations between contiguous functions

Below relations are correct only under some restrictions on the parameters

07.37.17.0007.01

$$Y_\lambda^\mu(\vartheta, \varphi) = \csc(\vartheta)\, e^{i\varphi} \left(\sqrt{\frac{(\lambda-\mu+1)(\lambda-\mu+2)}{(2\lambda+1)(2\lambda+3)}}\; Y_{\lambda+1}^{\mu-1}(\vartheta, \varphi) - \sqrt{\frac{(\lambda+\mu-1)(\lambda+\mu)}{(2\lambda-1)(2\lambda+1)}}\; Y_{\lambda-1}^{\mu-1}(\vartheta, \varphi) \right)$$

07.37.17.0008.01

$$Y_\lambda^\mu(\vartheta, \varphi) = \csc(\vartheta)\, e^{-i\varphi} \left(\sqrt{\frac{(\lambda-\mu-1)(\lambda-\mu)}{(2\lambda-1)(2\lambda+1)}}\; Y_{\lambda-1}^{\mu+1}(\vartheta, \varphi) - \sqrt{\frac{(\lambda+\mu+1)(\lambda+\mu+2)}{(2\lambda+1)(2\lambda+3)}}\; Y_{\lambda+1}^{\mu+1}(\vartheta, \varphi) \right)$$

07.37.17.0009.01

$$Y_\lambda^\mu(\vartheta, \varphi) = e^{i\varphi} \csc(\vartheta) \left(\sqrt{\frac{\lambda-\mu+1}{\lambda+\mu}}\; \cos(\vartheta)\, Y_\lambda^{\mu-1}(\vartheta, \varphi) - \sqrt{\frac{(2\lambda+1)(\lambda+\mu-1)}{(2\lambda-1)(\lambda+\mu)}}\; Y_{\lambda-1}^{\mu-1}(\vartheta, \varphi) \right)$$

07.37.17.0010.01

$$Y_\lambda^\mu(\vartheta, \varphi) = e^{i\varphi} \csc(\vartheta) \left(\sqrt{\frac{(2\lambda+1)(\lambda-\mu+2)}{(2\lambda+3)(\lambda-\mu+1)}}\; Y_{\lambda+1}^{\mu-1}(\vartheta, \varphi) - \cos(\vartheta) \sqrt{\frac{\lambda+\mu}{\lambda-\mu+1}}\; Y_\lambda^{\mu-1}(\vartheta, \varphi) \right)$$

07.37.17.0011.01

$$Y_\lambda^\mu(\vartheta, \varphi) = e^{-i\varphi} \csc(\vartheta) \left(\sqrt{\frac{(2\lambda+1)(\lambda-\mu-1)}{(2\lambda-1)(\lambda-\mu)}}\; Y_{\lambda-1}^{\mu+1}(\vartheta, \varphi) - \cos(\vartheta) \sqrt{\frac{\lambda+\mu+1}{\lambda-\mu}}\; Y_\lambda^{\mu+1}(\vartheta, \varphi) \right)$$

07.37.17.0012.01

$$Y_\lambda^\mu(\vartheta, \varphi) = e^{-i\varphi} \csc(\vartheta) \left(\cos(\vartheta) \sqrt{\frac{\lambda-\mu}{\lambda+\mu+1}}\; Y_\lambda^{\mu+1}(\vartheta, \varphi) - \sqrt{\frac{(2\lambda+1)(\lambda+\mu+2)}{(2\lambda+3)(\lambda+\mu+1)}}\; Y_{\lambda+1}^{\mu+1}(\vartheta, \varphi) \right)$$

Relations of special kind

07.37.17.0013.01

$$Y_\lambda^\mu(\vartheta, 0) = e^{-i\varphi\mu}\, Y_\lambda^\mu(\vartheta, \varphi)$$

07.37.17.0014.01

$$Y_\lambda^\mu(\vartheta, 0) = Y_\lambda^\mu\left(\vartheta, \frac{2\pi p}{\mu}\right) /;\ p \in \mathbb{Z}$$

Complex characteristics

Conjugate value

07.37.19.0001.01

$$\overline{Y_n^m(\vartheta, \varphi)} = (-1)^m\, Y_n^{-m}(\vartheta, \varphi) /;\ n \in \mathbb{Z} \land m \in \mathbb{Z} \land \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

Differentiation

Low-order differentiation

With respect to λ

07.37.20.0001.01

$$\frac{\partial Y_\lambda^\mu(\vartheta,\varphi)}{\partial\lambda} = \frac{2+(2\lambda+1)(2\pi\cot(\lambda\pi)+\psi(\lambda-\mu+1)-\psi(\lambda+\mu+1))}{2(2\lambda+1)} Y_\lambda^\mu(\vartheta,\varphi) - \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} e^{i\varphi\mu} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \sum_{k=0}^{\infty} \frac{(-\lambda)_k(\lambda+1)_k(\psi(k-\lambda)-\psi(k+\lambda+1))}{k!\,\Gamma(k-\mu+1)} \sin^{2k}\left(\frac{\vartheta}{2}\right) /; \lambda \notin \mathbb{Z}$$

07.37.20.0002.01

$$\frac{\partial Y_\lambda^\mu(\vartheta,\varphi)}{\partial\lambda} = \frac{2+(2\lambda+1)(\psi(\lambda-\mu+1)-\psi(\lambda+\mu+1))}{2(2\lambda+1)} Y_\lambda^\mu(\vartheta,\varphi) + \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} e^{i\varphi\mu} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k-\mu+1)} \sin^{2k}\left(\frac{\vartheta}{2}\right) \sum_{j=1}^{k} \lambda^j S_k^{(j)} \sum_{r=1}^{k} \frac{(-1)^r(\lambda+1)^{r-1}(\lambda j+j+\lambda r)}{\lambda} S_k^{(r)}$$

07.37.20.0003.01

$$\frac{\partial Y_\lambda^\mu(\vartheta,\varphi)}{\partial\lambda} = \frac{2+(2\lambda+1)(\psi(\lambda-\mu+1)-\psi(\lambda+\mu+1))}{2(2\lambda+1)} Y_\lambda^\mu(\vartheta,\varphi) - \frac{2\lambda+1}{2\Gamma(2-\mu)} \sqrt{\frac{2\lambda+1}{\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} e^{i\mu\varphi} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2-1}} F_{2\times 0\times 2}^{2\times 1\times 3}\left(\begin{matrix} 1-\lambda,\ \lambda+2;\ 1;\ 1,\ -\lambda,\ \lambda+1; \\ 2,\ 2-m;;\ \lambda+2,\ 1-\lambda; \end{matrix}\ \sin^2\left(\frac{\vartheta}{2}\right), \sin^2\left(\frac{\vartheta}{2}\right)\right)$$

07.37.20.0004.01

$$\frac{\partial^2 Y_\lambda^\mu(\vartheta,\varphi)}{\partial\lambda^2} = e^{i\varphi\mu} \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \left(\sum_{k=0}^{\infty} \frac{(-\lambda)_k(\lambda+1)_k}{k!\,\Gamma(k-\mu+1)} \left(\psi(k-\lambda)^2 - 2(\pi\cot(\lambda\pi)+\psi(k+\lambda+1))\psi(k-\lambda) + \psi(k+\lambda+1)^2 + 2\pi\cot(\lambda\pi)\psi(k+\lambda+1) + \psi^{(1)}(k-\lambda) + \psi^{(1)}(k+\lambda+1)\right) \sin^{2k}\left(\frac{\vartheta}{2}\right) - \frac{2+(2\lambda+1)(\psi(\lambda-\mu+1)-\psi(\lambda+\mu+1))}{2\lambda+1} \sum_{k=0}^{\infty} \frac{(-\lambda)_k(\lambda+1)_k(\psi(k-\lambda)-\psi(k+\lambda+1))}{k!\,\Gamma(k-\mu+1)} \sin^{2k}\left(\frac{\vartheta}{2}\right)\right) + \left(\frac{1}{4(2\lambda+1)^2}\left((2\lambda+1)^2\psi(\lambda-\mu+1)^2 - 2(2\lambda+1)((2\lambda+1)\psi(\lambda+\mu+1)-2)\psi(\lambda-\mu+1) + (2\lambda+1)^2\psi(\lambda+\mu+1)^2 + 2\left((2\lambda+1)^2\left(\psi^{(1)}(\lambda-\mu+1)-\psi^{(1)}(\lambda+\mu+1)\right)-2\right) - 4(2\lambda+1)\psi(\lambda+\mu+1)\right) + \frac{\pi\cot(\lambda\pi)((2\lambda+1)(\psi(\lambda-\mu+1)-\psi(\lambda+\mu+1))+2)}{2\lambda+1} - \pi^2\right) Y_\lambda^\mu(\vartheta,\varphi)$$

07.37.20.0005.01

$$\frac{\partial^2 Y_\lambda^\mu(\vartheta,\varphi)}{\partial\lambda^2} = e^{i\varphi\mu}\sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}$$

$$\left(\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(k-\mu+1)}\sin^{2k}\left(\frac{\vartheta}{2}\right)\sum_{i=1}^{k}\lambda^{i-2}S_k^{(i)}\sum_{r=1}^{k}(-1)^r(\lambda+1)^{r-2}\left((r-1)\,r\,\lambda^2+i^2(\lambda+1)^2+i(\lambda+1)(\lambda(2r-1)-1)\right)S_k^{(r)}+\right.$$

$$\frac{2+(2\lambda+1)(\psi(\lambda-\mu+1)-\psi(\lambda+\mu+1))}{2\lambda+1}$$

$$\left.\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(k-\mu+1)}\sin^{2k}\left(\frac{\vartheta}{2}\right)\sum_{j=1}^{k}\lambda^j S_k^{(j)}\sum_{r=1}^{k}\frac{(-1)^r(\lambda+1)^{r-1}(\lambda j+j+\lambda r)}{\lambda}S_k^{(r)}\right)+$$

$$\frac{1}{4(2\lambda+1)^2}\left(\left((2\lambda+1)^2\psi(\lambda-\mu+1)^2-2(2\lambda+1)((2\lambda+1)\psi(\lambda+\mu+1)-2)\psi(\lambda-\mu+1)+(2\lambda+1)^2\psi(\lambda+\mu+1)^2+\right.\right.$$

$$\left.\left.2\left((2\lambda+1)^2\left(\psi^{(1)}(\lambda-\mu+1)-\psi^{(1)}(\lambda+\mu+1)\right)-2\right)-4(2\lambda+1)\psi(\lambda+\mu+1)\right)Y_\lambda^\mu(\vartheta,\varphi)\right)$$

With respect to μ

07.37.20.0006.01

$$\frac{\partial Y_\lambda^\mu(\vartheta,\varphi)}{\partial m} = \frac{1}{2}\left(2i\varphi+\log\left(\cos^2\left(\frac{\vartheta}{2}\right)\right)-\log\left(\sin^2\left(\frac{\vartheta}{2}\right)\right)-\psi(\lambda-\mu+1)-\psi(\lambda+\mu+1)\right)Y_\lambda^\mu(\vartheta,\varphi)+$$

$$\sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\,e^{i\varphi\mu}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\sum_{k=0}^{\infty}\frac{(-\lambda)_k(\lambda+1)_k\,\psi(k-\mu+1)}{k!\,\Gamma(k-\mu+1)}\sin^{2k}\left(\frac{\vartheta}{2}\right)$$

07.37.20.0007.01

$$\frac{\partial Y_\lambda^\mu(\vartheta,\varphi)}{\partial\mu} =$$

$$\frac{\lambda(\lambda+1)}{(\mu-1)\,\Gamma(2-\mu)}\sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\,e^{i\varphi\mu}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2-2}}\,F_{2\times0\times1}^{2\times1\times2}\left(\begin{matrix}1-\lambda,\ \lambda+2;\ 1;\ 1,\ 1-\mu;\\ 2,\ 2-\mu;;\ 2-\mu;\end{matrix}\ \sin^2\left(\frac{\vartheta}{2}\right),\ \sin^2\left(\frac{\vartheta}{2}\right)\right)-$$

$$\frac{1}{2}\left(\log\left(\sin^2\left(\frac{\vartheta}{2}\right)\right)-\log\left(\cos^2\left(\frac{\vartheta}{2}\right)\right)-2i\varphi-2\psi(1-\mu)+\psi(\lambda-\mu+1)+\psi(\lambda+\mu+1)\right)Y_\lambda^\mu(\vartheta,\varphi)$$

07.37.20.0008.01

$$\frac{\partial^2 Y_\lambda^\mu(\vartheta,\varphi)}{\partial\mu^2} = i\, e^{i\mu\varphi}\,(2\varphi + i\,\psi(\lambda-\mu+1) + i\,\psi(\lambda+\mu+1))\sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\sum_{k=0}^{\infty}\frac{(-\lambda)_k\,(\lambda+1)_k\,\psi(k-\mu+1)}{k!\,\Gamma(k-\mu+1)}\sin^{2k}\left(\frac{\vartheta}{2}\right) + e^{i\varphi\mu}\sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\sum_{k=0}^{\infty}\frac{(-\lambda)_k\,(\lambda+1)_k}{k!\,\Gamma(k-\mu+1)}\left(\psi(k-\mu+1)^2 + \left(\log\left(\cos^2\left(\frac{\vartheta}{2}\right)\right) - \log\left(\sin^2\left(\frac{\vartheta}{2}\right)\right)\right)\psi(k-\mu+1) - \psi^{(1)}(k-\mu+1)\right)\sin^{2k}\left(\frac{\vartheta}{2}\right) + \frac{1}{4}\left(-4\varphi^2 + 4\,i\left(\log\left(\cos^2\left(\frac{\vartheta}{2}\right)\right) - \log\left(\sin^2\left(\frac{\vartheta}{2}\right)\right)\right)\left(\log\left(\cos^2\left(\frac{\vartheta}{2}\right)\right) - \log\left(\sin^2\left(\frac{\vartheta}{2}\right)\right)\right)^2\varphi + \psi(\lambda-\mu+1)^2 + \psi(\lambda+\mu+1)^2 - 2\left(2\,i\,\varphi + \log\left(\cos^2\left(\frac{\vartheta}{2}\right)\right) - \log\left(\sin^2\left(\frac{\vartheta}{2}\right)\right)\right)\psi(\lambda+\mu+1) + 2\,\psi(\lambda-\mu+1)\left(-2\,i\,\varphi - \log\left(\cos^2\left(\frac{\vartheta}{2}\right)\right) + \log\left(\sin^2\left(\frac{\vartheta}{2}\right)\right) + \psi(\lambda+\mu+1)\right) + 2\,\psi^{(1)}(\lambda-\mu+1) - 2\,\psi^{(1)}(\lambda+\mu+1)\right)Y_\lambda^\mu(\vartheta,\varphi)$$

With respect to ϑ

07.37.20.0009.01

$$\frac{\partial Y_\lambda^\mu(\vartheta,\varphi)}{\partial\vartheta} = \mu\cot(\vartheta)\,Y_\lambda^\mu(\vartheta,\varphi) + \frac{\sqrt{\Gamma(\lambda-\mu+1)}\,\sqrt{\Gamma(\lambda+\mu+2)}}{\sqrt{\Gamma(\lambda+\mu+1)}\,\sqrt{\Gamma(\lambda-\mu)}}\,e^{-i\varphi}\,Y_\lambda^{\mu+1}(\vartheta,\varphi)$$

07.37.20.0010.01

$$\frac{\partial Y_\lambda^\mu(\vartheta,\varphi)}{\partial\vartheta} = \sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\,e^{i\varphi\mu}\csc(\vartheta)\left(\lambda\cos(\vartheta)\,P_\lambda^\mu(\cos(\vartheta)) - (\lambda+\mu)\,P_{\lambda-1}^\mu(\cos(\vartheta))\right)$$

07.37.20.0011.01

$$\frac{\partial^2 Y_\lambda^\mu(\vartheta,\varphi)}{\partial\vartheta^2} = \mu\left(\mu\cot^2(\vartheta) - \csc^2(\vartheta)\right)Y_\lambda^\mu(\vartheta,\varphi) + \frac{\sqrt{\Gamma(\lambda-\mu+1)}\,\sqrt{\Gamma(\lambda+\mu+2)}}{\sqrt{\Gamma(\lambda+\mu+1)}\,\sqrt{\Gamma(\lambda-\mu)}}\,(2\mu+1)\,e^{-i\varphi}\cot(\vartheta)\,Y_\lambda^{\mu+1}(\vartheta,\varphi) + \frac{\sqrt{\Gamma(\lambda-\mu+1)}\,\sqrt{\Gamma(\lambda+\mu+3)}}{\sqrt{\Gamma(\lambda+\mu+1)}\,\sqrt{\Gamma(\lambda-\mu-1)}}\,e^{-2i\varphi}\,Y_\lambda^{\mu+2}(\vartheta,\varphi)$$

07.37.20.0012.01

$$\frac{\partial^2 Y_\lambda^\mu(\vartheta,\varphi)}{\partial\vartheta^2} = \frac{1}{2}\sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\,e^{i\varphi\mu}\csc^2(\vartheta)\left(2(\lambda+\mu)\cos(\vartheta)\,P_{\lambda-1}^\mu(\cos(\vartheta)) + \left(2\mu^2 - 2\lambda - 2\lambda^2\sin^2(\vartheta)\right)P_\lambda^\mu(\cos(\vartheta))\right)$$

With respect to φ

07.37.20.0013.01

$$\frac{\partial Y_\lambda^\mu(\vartheta,\varphi)}{\partial\varphi} = i\,\mu\,Y_\lambda^\mu(\vartheta,\varphi)$$

07.37.20.0014.01

$$\frac{\partial^2 Y_\lambda^\mu(\vartheta, \varphi)}{\partial \varphi^2} = -\mu^2 Y_\lambda^\mu(\vartheta, \varphi)$$

Symbolic differentiation

With respect to φ

07.37.20.0015.02

$$\frac{\partial^n Y_\lambda^\mu(\vartheta, \varphi)}{\partial \varphi^n} = (i \mu)^n Y_\lambda^\mu(\vartheta, \varphi) /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to φ

07.37.20.0016.01

$$\frac{\partial^\alpha Y_\lambda^\mu(\vartheta, \varphi)}{\partial \varphi^\alpha} = \varphi^{-\alpha} (i \varphi \mu)^\alpha Q(-\alpha, 0, i \varphi \mu) Y_\lambda^\mu(\vartheta, \varphi)$$

Integration

Indefinite integration

Involving only one direct function with respect to φ

07.37.21.0001.01

$$\int Y_\lambda^\mu(\vartheta, \varphi) \, d\varphi = -\frac{i}{\mu} Y_\lambda^\mu(\vartheta, \varphi)$$

Involving one direct function and elementary functions with respect to φ

Involving power function

07.37.21.0002.01

$$\int \varphi^{\alpha-1} Y_\lambda^\mu(\vartheta, \varphi) \, d\varphi = -\varphi^\alpha (-i \mu \varphi)^{-\alpha} e^{-i \varphi \mu} \Gamma(\alpha, -i \mu \varphi) Y_\lambda^\mu(\vartheta, \varphi)$$

Involving functions of the direct function and elementary functions with respect to ϑ

Involving elementary functions of the direct function and elementary functions

Involving products of the direct function and trigonometric functions

07.37.21.0003.01

$$\int\left((\lambda-\kappa)(\kappa+\lambda+1)-\frac{\mu^2-\nu^2}{\sin^2(\vartheta)}\right)Y_\lambda^\mu(\vartheta,\varphi)\,Y_\kappa^\nu(\vartheta,\varphi)\sin(\vartheta)\,d\vartheta=(\nu-\mu)\cos(\vartheta)\,Y_\lambda^\mu(\vartheta,\varphi)\,Y_\kappa^\nu(\vartheta,\varphi)+ e^{-i\varphi}\sin(\vartheta)\left(\frac{\sqrt{\Gamma(\kappa-\nu+1)}\sqrt{\Gamma(\kappa+\nu+2)}}{\sqrt{\Gamma(\kappa+\nu+1)}\sqrt{\Gamma(\kappa-\nu)}}Y_\lambda^\mu(\vartheta,\varphi)\,Y_\kappa^{\nu+1}(\vartheta,\varphi)-\frac{\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu+2)}}{\sqrt{\Gamma(\lambda+\mu+1)}\sqrt{\Gamma(\lambda-\mu)}}Y_\lambda^{\mu+1}(\vartheta,\varphi)\,Y_\kappa^\nu(\vartheta,\varphi)\right)$$

Definite integration

Involving the direct function

07.37.21.0004.01

$$\int_0^\pi\sin(\vartheta)\,Y_n^m(\vartheta,\varphi)^2\,d\vartheta=\frac{e^{2im\varphi}}{2\pi}\;/;\;m\in\mathbb{N}\land n\in\mathbb{N}\land m\le n$$

07.37.21.0005.01

$$\int_0^\pi\sin(\vartheta)\,Y_k^m(\vartheta,\varphi)\,Y_n^m(\vartheta,\varphi)\,d\vartheta=\frac{e^{2im\varphi}}{2\pi}\delta_{n,k}\;/;\;k\in\mathbb{N}\bigwedge m\in\mathbb{N}\bigwedge n\in\mathbb{N}$$

07.37.21.0006.01

$$\int_0^\pi\csc(\vartheta)\,Y_n^m(\vartheta,\varphi)\,\overline{Y_n^{m_1}(\vartheta,\varphi)}\,d\vartheta=\frac{2n+1}{4\pi m}\delta_{m,m_1}\;/;\;n\in\mathbb{N}^+\bigwedge m\in\mathbb{N}\bigwedge m_1\in\mathbb{N}\bigwedge m\le n\bigwedge m_1\le n$$

07.37.21.0007.01

$$\int_0^{\frac{\pi}{2}}\sin^{m+1}(\vartheta)\cos^p(\vartheta)\,Y_n^m(\vartheta,\varphi)\,d\vartheta=(-1)^m\sqrt{\frac{2n+1}{4\pi}\frac{(n+m)!}{(n-m)!}}\;\frac{p!\,e^{im\varphi}}{(p+n+m+1)!!\,(p-n+m)!!}\;/;\;p\in\mathbb{N}\bigwedge m\ge 0$$

07.37.21.0008.01

$$\int_a^b\left((n_1-n_2)(n_1+n_2+1)-\frac{m_1^2-m_2^2}{1-z^2}\right)Y_{n_1}^{m_1}\left(\cos^{-1}(z),\varphi\right)Y_{n_2}^{m_2}\left(\cos^{-1}(z),\varphi\right)dz=\left(b\,Y_{n_1}^{m_1}\left(\cos^{-1}(b),\varphi\right)Y_{n_2}^{m_2}\left(\cos^{-1}(b),\varphi\right)(n_1-n_2)+Y_{n_1-1}^{m_2}\left(\cos^{-1}(b),\varphi\right)Y_{n_1}^{m_1}\left(\cos^{-1}(b),\varphi\right)(n_2+m_2)\right)-\left(a\,Y_{n_1}^{m_1}\left(\cos^{-1}(a),\varphi\right)Y_{n_2}^{m_2}\left(\cos^{-1}(a),\varphi\right)(n_1-n_2)+Y_{n_1-1}^{m_2}\left(\cos^{-1}(a),\varphi\right)Y_{n_1}^{m_1}\left(\cos^{-1}(a),\varphi\right)(n_2+m_2)\right)\;/;\;n_1\in\mathbb{N}\bigwedge n_2\in\mathbb{N}\bigwedge m_1\in\mathbb{Z}\bigwedge m_2\in\mathbb{Z}\bigwedge a\in\mathbb{R}\bigwedge b\in\mathbb{R}\bigwedge|m_1|\le n_1\bigwedge|m_2|\le n_2$$

Multiple integration

07.37.21.0009.01

$$\int_0^\pi\int_0^{2\pi}\sin(\vartheta)\,Y_n^m(\vartheta,\varphi)\,d\varphi\,d\vartheta=\sqrt{4\pi}\;\delta_{n,0}\,\delta_{m,0}\;/;\;n\in\mathbb{N}\land m\in\mathbb{Z}\land|m|\le n$$

07.37.21.0010.01

$$\int_0^\pi\int_0^{2\pi}\sin(\vartheta)\,Y_{n_1}^{m_1}(\vartheta,\varphi)\,Y_{n_2}^{m_2}(\vartheta,\varphi)\,Y_{n_3}^{m_3}(\vartheta,-\varphi)\,d\varphi\,d\vartheta=\sqrt{\frac{(2n_1+1)(2n_2+1)}{4\pi(2n_3+1)}}\;\langle n_1\,n_2\,0\,0\mid n_1\,n_2\,n_3\,0\rangle\,\langle n_1\,n_2\,m_1\,m_2\mid n_1\,n_2\,n_3\,m_3\rangle$$

07.37.21.0011.01

$$\int_0^\pi \int_0^{2\pi} \sin(\vartheta)\, Y_{n_1}^{m_1}(\vartheta, \varphi)\, Y_{n_2}^{m_2}(\vartheta, \varphi)\, \overline{Y_{n_3}^{m_3}(\vartheta, \varphi)}\, d\varphi\, d\vartheta = \sqrt{\frac{(2n_1+1)(2n_2+1)}{4\pi(2n_3+1)}} \langle n_1\, n_2\, 0\, 0 \mid n_1\, n_2\, n_3\, 0\rangle \langle n_1\, n_2\, m_1\, m_2 \mid n_1\, n_2\, n_3\, m_3\rangle \,/;$$
$$n_1 \in \mathbb{N} \land n_2 \in \mathbb{N} \land n_3 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land m_2 \in \mathbb{Z} \land m_3 \in \mathbb{Z} \land |m_1| \le n_1 \land |m_2| \le n_2 \land |m_3| \le n_3$$

07.37.21.0012.01

$$\int_0^\pi \int_0^{2\pi} \sin(\vartheta)\, Y_{n_1}^{m_1}(\vartheta, \varphi)\, Y_{n_2}^{m_2}(\vartheta, \varphi)\, Y_{n_3}^{m_3}(\vartheta, \varphi)\, d\varphi\, d\vartheta = \sqrt{\frac{(2n_1+1)(2n_2+1)(2n_3+1)}{4\pi}} \begin{pmatrix} n_1 & n_2 & n_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \end{pmatrix} /;$$
$$n_1 \in \mathbb{N} \land n_2 \in \mathbb{N} \land n_3 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land m_2 \in \mathbb{Z} \land m_3 \in \mathbb{Z} \land |m_1| \le n_1 \land |m_2| \le n_2 \land |m_3| \le n_3$$

Orthonormality relations:

07.37.21.0013.01

$$\int_0^\pi \int_0^{2\pi} \sin(\vartheta)\, \overline{Y_{n_1}^{m_1}(\vartheta, \varphi)}\, Y_{n_2}^{m_2}(\vartheta, \varphi)\, d\varphi\, d\vartheta = \delta_{n_1,n_2}\, \delta_{m_1,m_2} \,/;\, n_1 \in \mathbb{Z} \land n_2 \in \mathbb{Z} \land m_1 \in \mathbb{Z} \land m_2 \in \mathbb{Z}$$

07.37.21.0014.01

$$\int_0^\pi \int_0^{2\pi} \sin(\vartheta)\, Y_{n_1}^{m_1}(\vartheta, \varphi)\, Y_{n_2}^{m_2}(\vartheta, -\varphi)\, d\varphi\, d\vartheta = (-1)^{m_2}\, \delta_{m_1,m_2}\, \delta_{n_1,n_2} \,/;\, n_1 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land |m_1| \le n_1$$

07.37.21.0015.01

$$\int_0^\pi \int_0^{2\pi} \sin(\vartheta)\, Y_{n_1}^{m_1}(\vartheta, \varphi)\, Y_{n_2}^{m_2}(\vartheta, \varphi)\, d\varphi\, d\vartheta = (-1)^{m_2}\, \delta_{m_1,-m_2}\, \delta_{n_1,n_2} \,/;\, n_1 \in \mathbb{N} \land m_1 \in \mathbb{Z} \land |m_1| \le n_1$$

Summation

Finite summation

Involving the direct function

07.37.23.0001.01

$$\sum_{m=-n}^{n} \overline{Y_n^m(\vartheta_1, \varphi_1)}\, Y_n^m(\vartheta_2, \varphi_2) = \frac{2n+1}{4\pi} P_n(\cos(\vartheta_1)\cos(\vartheta_2) + \cos(\varphi_1 - \varphi_2)\sin(\vartheta_1)\sin(\vartheta_2)) \,/;\, n \in \mathbb{N} \land \vartheta_k \in \mathbb{R} \land \varphi_k \in \mathbb{R} \land k \in \{1, 2\}$$

07.37.23.0002.01

$$\sum_{m=-n}^{n} \left|Y_n^m(\vartheta, \varphi)\right|^2 = \frac{2n+1}{4\pi} \,/;\, n \in \mathbb{N} \land \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

07.37.23.0003.01

$$\sum_{m=-n}^{n} m \left|Y_n^m(\vartheta, \varphi)\right|^2 = 0 \,/;\, n \in \mathbb{N} \land \varphi \in \mathbb{R}$$

07.37.23.0004.01

$$\sum_{m=-n}^{n} m^2 \left|Y_n^m(\vartheta, \varphi)\right|^2 = \frac{n(n+1)(2n+1)}{8\pi} \sin^2(\vartheta) \,/;\, n \in \mathbb{N} \land \vartheta \in \mathbb{R} \land \varphi \in \mathbb{R}$$

07.37.23.0005.01

$$\sum_{m=-n}^{n}\sqrt{\left(n^2-m^2\right)\left((n+1)^2-m^2\right)}\;Y_{n-1}^{m}(\vartheta,\varphi)\,\overline{Y_{n+1}^{m}(\vartheta,\varphi)}=\frac{n(n+1)}{8\pi}\sqrt{(2n-1)(2n+3)}\,\left(3\cos^2(\vartheta)-1\right)/;\,n\in\mathbb{N}\land\vartheta\in\mathbb{R}\land\varphi\in\mathbb{R}$$

07.37.23.0006.01

$$\sum_{m=-n}^{n}\frac{i^m}{\sqrt{(n-m)!\,(n+m)!}}\,Y_n^m(\vartheta,\varphi)=\sqrt{\frac{2n+1}{4\pi}}\;\frac{(\cos(\vartheta)+i\sin(\vartheta)\cos(\varphi))^n}{n!}\;/;\,n\in\mathbb{N}\land\vartheta\in\mathbb{R}\land\varphi\in\mathbb{R}$$

07.37.23.0007.01

$$\sum_{l=m}^{m+p}\frac{(-1)^l\,w^{p-l+m}}{(p-l+m)!\,\sqrt{2l+1}\,\sqrt{(l-m)!\,(l+m)!}}\,Y_l^m(\vartheta,\varphi)=$$

$$(-1)^p\,\frac{(2m-1)!!}{2\sqrt{\pi}\,(2m+p)!}\left(\sin(\vartheta)\,e^{i\varphi}\right)^m\left(1-2w\cos(\vartheta)+w^2\right)^{p/2}C_p^{m+1/2}\!\left(\frac{\cos(\vartheta)-w}{\sqrt{1-2w\cos(\vartheta)+w^2}}\right)/;\,m\in\mathbb{N}\land p\in\mathbb{N}$$

Involving Clebsch-Gordan functions

The inverse Clebsch-Gordan series:

07.37.23.0008.01

$$\sum_{k=\max(M-n_2,-n_1)}^{\min(M+n_2,n_1)}\langle n_1\,n_2\,k\,M-k\mid n_1\,n_2\,L\,M\rangle\,Y_{n_1}^{k}(\vartheta,\varphi)\,Y_{n_2}^{M-k}(\vartheta,\varphi)=\sqrt{\frac{(2n_1+1)(2n_2+1)}{4\pi(2L+1)}}\;\langle n_1\,n_2\,0\,0\mid n_1\,n_2\,L\,0\rangle\,Y_L^M(\vartheta,\varphi)\,/;$$

$$n_1\in\mathbb{N}\land n_2\in\mathbb{N}\land L\in\mathbb{Z}\land M\in\mathbb{Z}\land|n_1-n_2|\le L\le n_1+n_2\land -L\le M\le L$$

Infinite summation

07.37.23.0009.01

$$\sum_{n=m}^{\infty}\frac{Y_n^m(\vartheta,\varphi)\,w^{n-m}}{n!\,\sqrt{(2n+1)(n-m)!\,(m+n)!}}=\frac{\left(-\sin(\vartheta)\,e^{i\varphi}\right)^m}{2^{m+1}\sqrt{\pi}\;m!^2}\,{}_0F_1\!\left(;m+1;w\cos^2\!\left(\frac{\vartheta}{2}\right)\right){}_0F_1\!\left(;m+1;-w\sin^2\!\left(\frac{\vartheta}{2}\right)\right)/;$$

$$m\in\mathbb{N}\land n\in\mathbb{N}\land\vartheta\in\mathbb{R}\land\varphi\in\mathbb{R}\land|w|<1$$

07.37.23.0010.01

$$\sum_{n=m}^{\infty}\frac{(n-m+p)!}{\sqrt{(2n+1)(n+m)!\,(n-m)!}}\,Y_n^m(\vartheta,\varphi)\,w^{n-m}=$$

$$\frac{p!}{2^{m+1}\sqrt{\pi}\;m!}\,\frac{\left(-\sin(\vartheta)\,e^{i\varphi}\right)^m}{(1-w\cos(\vartheta))^{p+1}}\,{}_2F_1\!\left(\frac{p+1}{2},\,\frac{p}{2}+1;\,m+1;\,-\left(\frac{w\sin(\vartheta)}{1-w\cos(\vartheta)}\right)^2\right)/;\,m\in\mathbb{N}\land p\in\mathbb{N}\land\vartheta\in\mathbb{R}\land\varphi\in\mathbb{R}\land|w|<1$$

07.37.23.0011.01

$$\sum_{n=m}^{\infty}\frac{(n+m-p)!\,(n-m+p-1)!}{n!\,\sqrt{(2n+1)(n-m)!\,(n+m)!}}\,Y_n^m(\vartheta,\varphi)\,w^{n-m}=$$

$$\frac{(2m-p)!\,(p-1)!}{2^{m+1}\sqrt{\pi}\;m!^2}\left(-\sin(\vartheta)\,e^{i\varphi}\right)^m{}_2F_1\!\left(p,\,2m-p+1;\,m+1;\,\frac{1-w-\sqrt{1-2w\cos(\vartheta)+w^2}}{2}\right)$$

$${}_2F_1\!\left(p,\,2m-p+1;\,m+1;\,\frac{1+w-\sqrt{1-2w\cos(\vartheta)+w^2}}{2}\right)/;\,m\in\mathbb{N}\land p\in\mathbb{N}\land p\le 2m\land\vartheta\in\mathbb{R}\land\varphi\in\mathbb{R}\land|w|<1$$

07.37.23.0012.01

$$\sum_{n=m}^{\infty} \sqrt{\frac{(n-m)!}{(2n+1)(n+m)!}}\, L_{n-m}^{2m}(z)\, Y_n^m(\vartheta, \varphi)\, w^{n-m} = \frac{1}{2^{m+1}\sqrt{\pi}\, m!} \frac{\left(-\sin(\vartheta)\, e^{i\varphi}\right)^m}{\left(1 - 2w\cos(\vartheta) + w^2\right)^{m+1/2}} \exp\left(-\frac{z\,w\,(\cos(\vartheta) - w)}{1 - 2w\cos(\vartheta) + w^2}\right) {}_0F_1\left(; m+1; -\frac{z^2 w^2 \sin^2(\vartheta)}{4\left(1 - 2w\cos(\vartheta) + w^2\right)^2}\right) /; m \in \mathbb{N} \bigwedge \vartheta \in \mathbb{R} \bigwedge \varphi \in \mathbb{R} \bigwedge |w| < 1$$

07.37.23.0013.01

$$\sum_{n=m}^{\infty} i^{n-m} \sqrt{\frac{(2n+1)(n+m)!}{(n-m)!}}\, J_{n+1/2}(w)\, Y_n^m(\vartheta, \varphi) = \frac{\sqrt{w}\,\left(-w\sin(\vartheta)\, e^{i\varphi}\right)^m}{\sqrt{2}\,\pi} e^{i w \cos(\vartheta)} /; m \in \mathbb{N} \bigwedge \vartheta \in \mathbb{R} \bigwedge \varphi \in \mathbb{R}$$

07.37.23.0014.01

$$\sum_{\substack{n=m,\\ \Delta n=2}}^{\infty} i^{n-m} \sqrt{\frac{(2n+1)(n+m)!}{(n-m)!}}\, J_{n+1/2}(w)\, Y_n^m(\vartheta, \varphi) = \frac{\sqrt{w}}{\sqrt{2}\,\pi} \left(-w\sin(\vartheta)\, e^{i\varphi}\right)^m \cos(w\cos(\vartheta)) /; m \in \mathbb{N} \bigwedge \vartheta \in \mathbb{R} \bigwedge \varphi \in \mathbb{R}$$

07.37.23.0015.01

$$\sum_{\substack{n=m+1,\\ \Delta n=2}}^{\infty} i^{n-m-1} \sqrt{\frac{(2n+1)(n+m)!}{(n-m)!}}\, J_{n+1/2}(w)\, Y_n^m(\vartheta, \varphi) = \frac{\sqrt{w}}{\sqrt{2}\,\pi} \left(-w\sin(\vartheta)\, e^{i\varphi}\right)^m \sin(w\cos(\vartheta)) /; m \in \mathbb{N} \bigwedge \vartheta \in \mathbb{R} \bigwedge \varphi \in \mathbb{R}$$

07.37.23.0016.01

$$\sum_{n=m}^{\infty} i^{n-m}\, J_{n+1/2}(w)\, Y_n^m(\vartheta_1, \varphi_1)\, \overline{Y_n^m(\vartheta_2, \varphi_2)} = \frac{\sqrt{2w}}{4\pi^{3/2}} J_m(w\sin(\vartheta_1)\sin(\vartheta_2))\, e^{i w \cos(\vartheta_1)\cos(\vartheta_2)}\, e^{i m (\varphi_1 - \varphi_2)} /; m \in \mathbb{N} \bigwedge \vartheta_k \in \mathbb{R} \bigwedge \varphi_k \in \mathbb{R} \bigwedge k \in \{1, 2\}$$

Multiple infinite summation

07.37.23.0017.01

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \overline{Y_n^m(\vartheta_1, \varphi_1)}\, Y_n^m(\vartheta_2, \varphi_2) = \delta(\varphi_1 - \varphi_2)\, \delta(\cos(\vartheta_1) - \cos(\vartheta_2)) /; \vartheta_k \in \mathbb{R} \bigwedge \varphi_k \in \mathbb{R} \bigwedge k \in \{1, 2\}$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

07.37.26.0001.01

$$Y_\lambda^\mu(\vartheta, \varphi) = \sqrt{\frac{2\lambda+1}{4\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} e^{i\varphi\mu} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} {}_2\tilde{F}_1\left(-\lambda, \lambda+1; 1-\mu; \sin^2\left(\frac{\vartheta}{2}\right)\right)$$

Involving ${}_2F_1$

07.37.26.0002.01

$$Y_\lambda^\mu(\vartheta, \varphi) == \frac{1}{2\,\Gamma(1-\mu)} \sqrt{\frac{2\,\lambda+1}{\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} e^{i\,\varphi\,\mu} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} {}_2F_1\left(-\lambda, \lambda+1; 1-\mu; \sin^2\left(\frac{\vartheta}{2}\right)\right) /; \mu \notin \mathbb{N}^+$$

07.37.26.0003.01

$$Y_\lambda^\mu(\vartheta, \varphi) == \sqrt{\frac{2\,\lambda+1}{4\,\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}\; e^{i\,\varphi\,\mu}}{\sqrt{\Gamma(\lambda+\mu+1)}} \left(\frac{\Gamma(-\mu)}{\Gamma(-\lambda-\mu)\,\Gamma(\lambda-\mu+1)\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2} {}_2F_1\left(-\lambda, \lambda+1; \mu+1; \cos^2\left(\frac{\vartheta}{2}\right)\right) - \frac{\sin(\pi\,\lambda)\,\Gamma(\mu)}{\pi \sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \cos^2\left(\frac{\vartheta}{2}\right)^{-\frac{\mu}{2}} {}_2F_1\left(\lambda-\mu+1, -\lambda-\mu; 1-\mu; \cos^2\left(\frac{\vartheta}{2}\right)\right)\right) /; \mu \notin \mathbb{Z}$$

07.37.26.0004.01

$$Y_\lambda^\mu(\vartheta, \varphi) == \sqrt{\frac{2\,\lambda+1}{4\,\pi}} \frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}} e^{i\,\varphi\,\mu} \frac{1}{\sqrt{\pi}} \frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}} \left(\frac{2^\lambda\,(\cos(\vartheta)-1)^\lambda}{\Gamma(\lambda-\mu+1)} \Gamma\left(\lambda+\frac{1}{2}\right) \left({}_2F_1\left(\mu-\lambda, -\lambda; -2\,\lambda; \frac{2}{1-\cos(\vartheta)}\right)\right) + \frac{2^{-1-\lambda}\,(\cos(\vartheta)-1)^{-1-\lambda}}{\Gamma(-m-\lambda)} \Gamma\left(-\lambda-\frac{1}{2}\right) \left({}_2F_1\left(\lambda+1, \lambda+\mu+1; 2\,\lambda+2; \frac{2}{1-\cos(\vartheta)}\right)\right)\right) /; \vartheta \notin \mathbb{R} \bigwedge 2\,\lambda \notin \mathbb{Z}$$

07.37.26.0005.01

$$Y_n^m(\vartheta, \varphi) == \frac{(-1)^{\frac{m}{2}(\mathrm{sgn}(m)+1)}\, e^{i\,m\,\varphi}}{|m|!\,2^{|m|+1}} \sqrt{\frac{(2\,n+1)\,(n+|m|)!}{\pi\,(n-|m|)!}} \sin^2(\vartheta)^{\frac{|m|}{2}} {}_2F_1\left(|m|-n, n+|m|+1; |m|+1; \sin^2\left(\frac{\vartheta}{2}\right)\right) /; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \le n$$

07.37.26.0006.01

$$Y_n^m(\vartheta, \varphi) == \frac{(-1)^{n-m}\,(-1)^{\frac{m}{2}(\mathrm{sgn}(m)+1)}\,(2\,n)!\, e^{i\,m\,\varphi}}{2^{|m|+1}\,n!} \sqrt{\frac{2\,n+1}{\pi\,(n+m)!\,(n-m)!}} \sin^{2\,n-2\,|m|}\left(\frac{\vartheta}{2}\right) \sin^2(\vartheta)^{\frac{|m|}{2}} {}_2F_1\left(-n, |m|-n; -2\,n; \csc^2\left(\frac{\vartheta}{2}\right)\right) /; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{Z} \bigwedge |m| \le n$$

07.37.26.0007.01

$$Y_n^m(\vartheta, \varphi) == \frac{(-1)^{n-m}\,(-1)^{\frac{m}{2}(\mathrm{sgn}(m)+1)}\, e^{i\,m\,\varphi}}{|m|!\,2^{|m|+1}} \sqrt{\frac{(2\,n+1)\,(n+|m|)!}{\pi\,(n-|m|)!}} \sin^2(\vartheta)^{\frac{|m|}{2}} {}_2F_1\left(|m|-n, n+|m|+1; |m|+1; \cos^2\left(\frac{\vartheta}{2}\right)\right) /; n \in \mathbb{N} \land m \in \mathbb{Z} \land |m| \le n$$

07.37.26.0008.01

$$Y_n^m(\vartheta, \varphi) == \frac{(-1)^{\frac{m}{2}(\mathrm{sgn}(m)+1)}\,(2\,n)!\, e^{i\,m\,\varphi}}{2^{|m|+1}\,n!} \sqrt{\frac{2\,n+1}{\pi\,(n+m)!\,(n-m)!}} \cos^{2\,n-2\,|m|}\left(\frac{\vartheta}{2}\right) \sin^2(\vartheta)^{\frac{|m|}{2}} {}_2F_1\left(-n, |m|-n; -2\,n; \sec^2\left(\frac{\vartheta}{2}\right)\right) /; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{Z} \bigwedge |m| \le n$$

07.37.26.0009.01

$$Y_n^m(\vartheta,\varphi)=\frac{(-1)^{\frac{m}{2}(\operatorname{sgn}(m)+1)}\,e^{i m\varphi}}{2^{|m|+1}\,|m|!}\sqrt{\frac{(2n+1)(n+|m|)!}{\pi(n-|m|)!}}\cos^{2n-2|m|}\left(\frac{\vartheta}{2}\right)\sin^2(\vartheta)^{\frac{|m|}{2}}\,{}_2F_1\left(|m|-n,-n;|m|+1;-\tan^2\left(\frac{\vartheta}{2}\right)\right)/;$$

$n\in\mathbb{N}\land m\in\mathbb{Z}\land|m|\leq n$

07.37.26.0010.01

$$Y_n^m(\vartheta,\varphi)=\frac{(-1)^{n-m}\,(-1)^{\frac{1}{2}m(\operatorname{sgn}(m)+1)}\,e^{i m\varphi}}{2^{|m|+1}\,|m|!}\sqrt{\frac{(2n+1)(n+|m|)!}{\pi(n-|m|)!}}\sin^{2n-2|m|}\left(\frac{\vartheta}{2}\right)\sin^2(\vartheta)^{\frac{|m|}{2}}\,{}_2F_1\left(|m|-n,-n;|m|+1;-\cot^2\left(\frac{\vartheta}{2}\right)\right)/;$$

$n\in\mathbb{N}\land m\in\mathbb{Z}\land|m|\leq n$

Through Meijer *G*

Classical cases

07.37.26.0011.01

$$Y_\lambda^\mu(\vartheta,\varphi)=-\frac{\sin(\pi\lambda)}{\pi}\sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\,e^{i\varphi\mu}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\,G_{2,2}^{1,2}\left(-\sin^2\left(\frac{\vartheta}{2}\right)\middle|\begin{matrix}\lambda+1,-\lambda\\0,\mu\end{matrix}\right)/;\lambda\notin\mathbb{Z}$$

07.37.26.0012.01

$$Y_n^\mu(\vartheta,\varphi)=-\frac{1}{\pi}\sqrt{\frac{2n+1}{4\pi}}\,\frac{\sqrt{\Gamma(n-\mu+1)}}{\sqrt{\Gamma(n+\mu+1)}}\,e^{i\varphi\mu}\,\frac{\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}{\sin^2\left(\frac{\vartheta}{2}\right)^{\mu/2}}\lim_{\nu\to n}\sin(\pi\nu)\,G_{2,2}^{1,2}\left(-\sin^2\left(\frac{\vartheta}{2}\right)\middle|\begin{matrix}\nu+1,-\nu\\0,\mu\end{matrix}\right)/;n\in\mathbb{Z}$$

Through other functions

Involving Legendre functions

07.37.26.0013.01

$$Y_\lambda^\mu(\vartheta,\varphi)=\sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\,e^{i\varphi\mu}\,P_\lambda^\mu(\cos(\vartheta))$$

07.37.26.0014.01

$$Y_\lambda^\mu(\vartheta,\varphi)=\sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\sqrt{\Gamma(\lambda-\mu+1)}}{\sqrt{\Gamma(\lambda+\mu+1)}}\,e^{i\varphi\mu}\,(-\cos(\vartheta)-1)^{-\frac{\mu}{2}}\,(\cos(\vartheta)+1)^{\mu/2}\,\mathfrak{P}_\lambda^\mu(\cos(\vartheta))$$

Involving some hypergeometric-type functions

07.37.26.0015.01

$$Y_\lambda^\mu(\vartheta,\varphi)=\sqrt{\frac{2\lambda+1}{4\pi}}\,\frac{\Gamma(\lambda+1)}{\sqrt{\Gamma(\lambda-\mu+1)}\sqrt{\Gamma(\lambda+\mu+1)}}\,e^{i\mu\varphi}\cos^2\left(\frac{\vartheta}{2}\right)^{\mu/2}\cot^2\left(\frac{\vartheta}{2}\right)^{\mu/2}\sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{\mu}{2}}\tan^\mu\left(\frac{\vartheta}{2}\right)P_\lambda^{(-\mu,\mu)}(\cos(\vartheta))$$

07.37.26.0016.01

$$Y_\lambda^\mu(\vartheta,\varphi)=\frac{2^{-2\mu-1}\,e^{i\mu\varphi}\sqrt{2\lambda+1}\,\Gamma\left(\frac{1}{2}-\mu\right)\sqrt{\Gamma(\lambda+\mu+1)}}{\pi\sqrt{\Gamma(\lambda-\mu+1)}}\csc^2\left(\frac{\vartheta}{2}\right)^{\mu/2}\sin^2\left(\frac{\vartheta}{2}\right)^{-\frac{\mu}{2}}\tan^\mu\left(\frac{\vartheta}{2}\right)C_{\lambda+\mu}^{\frac{1}{2}-\mu}(\cos(\vartheta))$$

Representations through equivalent functions

With related functions

Involving Wigner-D functions

07.37.27.0001.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}}\, D_{0,-m}^n(0, \vartheta, \varphi) \;/;\; n \in \mathbb{Z} \land m \in \mathbb{Z}$$

07.37.27.0002.01

$$Y_n^m(\vartheta, \varphi) = (-1)^m \sqrt{\frac{2n+1}{4\pi}}\, \overline{D_{0,m}^n(0, \vartheta, \varphi)} \;/;\; n \in \mathbb{Z} \land m \in \mathbb{Z}$$

07.37.27.0003.01

$$Y_n^m(\vartheta, \varphi) = (-1)^m \sqrt{\frac{2n+1}{4\pi}}\, D_{-m,0}^n(\varphi, \vartheta, 0) \;/;\; n \in \mathbb{Z} \land m \in \mathbb{Z}$$

07.37.27.0004.01

$$Y_n^m(\vartheta, \varphi) = \sqrt{\frac{2n+1}{4\pi}}\, \overline{D_{m,0}^n(\varphi, \vartheta, 0)} \;/;\; n \in \mathbb{Z} \land m \in \mathbb{Z}$$

Zeros

When $Y_n^m(\vartheta, \varphi)$ is not identically zero, it possesses a finite number of zeros in the interval $0 < \vartheta < \pi$, all of which are nondegenerate.

For integers m and n with $n \geq |m|$, the function $Y_n^m(\vartheta, \varphi)$ has $n - |m|$ zeros in the interval $0 < \vartheta < \pi$. If $m \neq 0$, there are also two more zeros at $\vartheta = 0, \pi$. All of these zeros are symmetric about $\vartheta = \pi/2$.

Theorems

Eigenfunction to the angular part of the Laplace operator in spherical coordinates

The function $Y_n^m(\vartheta, \varphi)$ is an eigenfunction to the angular part $L^2$of the Laplace operator in spherical coordinates $-L^2 = \frac{1}{\sin(\vartheta)^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\sin(\vartheta)} \frac{\partial}{\partial \vartheta} \left(\sin(\vartheta) \frac{\partial}{\partial \vartheta}\right)$ with eigenvalue $n(n+1)$.

Eigenfunction of the *z* component of the quantum mechanical angular momentum operator

The function $Y_n^m(\vartheta, \varphi)$ is an eigenfunction to the z-component of the quantum mechanical angular momentum operator $\hat{L}_z = -i\, \partial / \partial \varphi$ with eigenvalue m.

Eigenfunctions of the curl operator in spherical coordinates

The function $\boldsymbol{u}(r, \vartheta, \varphi) == \lambda\, \boldsymbol{r} \times \nabla \psi(r, \vartheta, \varphi) + \nabla \times (\boldsymbol{r} \times \nabla \psi(r, \vartheta, \varphi))$ with $\psi(r, \vartheta, \varphi) == g(r)\, Y_n^m(\vartheta, \varphi)$ and $g(r) == \frac{\sqrt{2}\, i}{\lambda \sqrt{n(n+1)}} \left(c_1 \frac{J_{n+1/2}(\lambda r)}{\sqrt{\lambda r}} + \frac{Y_{n+1/2}(\lambda r)}{\sqrt{\lambda r}} \right)$ are eigenfunctions of the curl operator in spherical coordinates $\nabla \times \boldsymbol{u}(r, \vartheta, \varphi) == \lambda\, \boldsymbol{u}(r, \vartheta, \varphi)$.

Multiple expansion theorem

Any function $f(\vartheta, \varphi)$ that is square integrable over $0 \le \vartheta \le \pi,\ 0 \le \varphi \le 2\pi$ can be expanded in a series of spherical harmonics $Y_n^m(\vartheta, \varphi)$, with series coefficients $a_{n,m}$ called the multipole moments:

$$f(\vartheta, \varphi) == \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m}\, Y_n^m(\vartheta, \varphi) \;/;\; a_{n,m} == \int_0^{2\pi} \int_0^{\pi} \sin(\vartheta)\, \overline{Y_n^m(\vartheta, \varphi)}\, f(\vartheta, \varphi)\, d\vartheta\, d\varphi.$$

History

A.M. Legendre (1785); P.S. Laplace (1785) gave the name “spherical harmonic”; K.F. Gauss (1828).

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