SpheroidalS2

Notations

Traditional name

Radial spheroidal function of the second kind

Traditional notation

$S_{\nu,\mu}^{(2)}(\gamma, z)$

Mathematica StandardForm notation

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SpheroidalS2[ν, μ, γ, z]
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Primary definition

11.11.02.0001.01

$S_{\nu,\mu}^{(2)}(\gamma, z)$

$S_{\nu,\mu}^{(2)}(\gamma, z)$ is the radial spheroidal function of the second kind with variable z and parameters ν, μ, γ. It is defined as the normalizable solution $w(z) = S_{\nu,\mu}^{(2)}(\gamma, z)$ of the wave differential equation $(1-z^2)\, w''(z) - 2\, z\, w'(z) + \left(\lambda + \gamma^2 (1-z^2) - \mu^2 / (1-z^2)\right) w(z) = 0$ with parameter λ equal to spheroidal eigenvalue $\lambda = \lambda_{\nu,\mu}(\gamma)$. The parameter ν enumerates the spheroidal eigenvalues in such a manner that in the limit $(\gamma \to 0)$, the eigenvalues are $\lambda_{\nu,\mu}(0) = \nu\,(\mu+1)$ and $\lim_{\gamma\to 0} S_{\nu,\mu}^{(2)}(\gamma, z/\gamma) = y_\nu(z)$, where $y_\nu(z)$ is the spherical Bessel function of the second kind. The radial spheroidal functions are normalized according to the Meixner-Schäfke normalization scheme, meaning $\lim_{z\to\infty} S_{\nu,\mu}^{(2)}(\gamma, z/\gamma) / y_\nu(z) = 1$. $S_{\nu,\mu}^{(2)}(\gamma, z)$ is an analytical function in the variables ν, μ, γ and z.

Specific values

General characteristics

Domain and analyticity

$S_{\nu,\mu}^{(2)}(\gamma, z)$ is an analytical function of ν, μ, γ, z which is defined in $\mathbb{C}^4$.

11.11.04.0001.01

$(\nu * \mu * \gamma * z) \longrightarrow S_{\nu,\mu}^{(2)}(\gamma, z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Parity

$S_{\nu,\mu}^{(2)}(\gamma, z)$ is an even function with respect to μ.

11.11.04.0002.01

$$S_{\nu,-\mu}^{(2)}(\gamma, z) = S_{\nu,\mu}^{(2)}(\gamma, z)$$

Mirror symmetry

11.11.04.0003.01

$$S_{\overline{\nu},\overline{\mu}}^{(2)}(\overline{\gamma}, \bar{z}) == \overline{S_{\nu,\mu}^{(2)}(\gamma, z)}$$

Periodicity

No periodicity

Series representations

Generalized power series

Expansions at generic point $z == z_0$

11.11.06.0001.01

$$S_{\nu,\mu}^{(2)}(\gamma, z) \propto S_{\nu,\mu}^{(2)}(\gamma, z_0) + S_{\nu,\mu}^{(2)\prime}(\gamma, z_0)(z - z_0) + \frac{1}{2\left(1 - z_0^2\right)}\left(2 S_{\nu,\mu}^{(2)\prime}(\gamma, z_0) z_0 + S_{\nu,\mu}^{(2)}(\gamma, z_0)\left(\left(z_0^2 - 1\right)\gamma^2 - \lambda_{\nu,\mu}(\gamma) + \frac{\mu^2}{1 - z_0^2}\right)\right)(z - z_0)^2 - \frac{1}{6\left(z_0^2 - 1\right)^3}\left(S_{\nu,\mu}^{(2)\prime}(\gamma, z_0)\left(z_0^2 - 1\right)\left(\gamma^2 z_0^4 - 2\left(\gamma^2 + 3\right) z_0^2 + \gamma^2 - \mu^2 - \lambda_{\nu,\mu}(\gamma)\left(z_0^2 - 1\right) - 2\right) - 2 S_{\nu,\mu}^{(2)}(\gamma, z_0) z_0\left(-3\mu^2 + \gamma^2\left(z_0^2 - 1\right)^2 - 2\lambda_{\nu,\mu}(\gamma)\left(z_0^2 - 1\right)\right)\right)(z - z_0)^3 + \frac{1}{24\left(z_0^2 - 1\right)^4}\left(4 S_{\nu,\mu}^{(2)\prime}(\gamma, z_0) z_0\left(z_0^2 - 1\right)\left(\gamma^2 z_0^4 - 2\left(\gamma^2 + 3\right) z_0^2 + \gamma^2 - 3\mu^2 - 2\lambda_{\nu,\mu}(\gamma)\left(z_0^2 - 1\right) - 6\right) + S_{\nu,\mu}^{(2)}(\gamma, z_0)\left(\gamma^4\left(z_0^2 - 1\right)^4 + \lambda_{\nu,\mu}(\gamma)^2\left(z_0^2 - 1\right)^2 - 2\gamma^2\left(\mu^2 + 4 z_0^2 + 2\right)\left(z_0^2 - 1\right)^2 - 2\lambda_{\nu,\mu}(\gamma)\left(\gamma^2 z_0^4 - \left(2\gamma^2 + 9\right) z_0^2 + \gamma^2 - \mu^2 - 3\right)\left(z_0^2 - 1\right) + \mu^2\left(\mu^2 + 36 z_0^2 + 8\right)\right)\right)(z - z_0)^4 + \ldots /; (z \to z_0)$$

11.11.06.0002.01

$$S_{\nu,\mu}^{(2)}(\gamma, z) \propto S_{\nu,\mu}^{(2)}(\gamma, z_0)(1 + O(z - z_0))$$

Expansions at $z == 0$

11.11.06.0003.01

$$S_{\nu,\mu}^{(2)}(\gamma, z) \propto S_{\nu,\mu}^{(2)}(\gamma, 0) + S_{\nu,\mu}^{(2)\prime}(\gamma, 0) z - \frac{1}{2}\left(\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)\right) S_{\nu,\mu}^{(2)}(\gamma, 0) z^2 + \frac{1}{6}\left(2 S_{\nu,\mu}^{(2)\prime}(\gamma, 0) - \left(\gamma^2 - \mu^2 + \lambda_{\nu,\mu}(\gamma)\right) S_{\nu,\mu}^{(2)\prime}(\gamma, 0)\right) z^3 + \ldots /; (z \to 0)$$

11.11.06.0004.01

$$S_{\nu,\mu}^{(2)}(\gamma, z) \propto S_{\nu,\mu}^{(2)}(\gamma, 0)(1 + O(z))$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

11.11.13.0001.01

$$\left(1-z^2\right) w''(z)-2\, z\, w'(z)+\left(\left(1-z^2\right) \gamma^2+\lambda_{\nu,\mu}(\gamma)-\frac{\mu^2}{1-z^2}\right) w(z)=0\; /;\; w(z)=\; c_1\, S_{\nu,\mu}^{(2)}(\gamma,\, z)+c_2\, S_{\nu,\mu}^{(1)}(\gamma,\, z)$$

11.11.13.0002.01

$$W_z\left(S_{\nu,\mu}^{(2)}(\gamma,\, z),\, S_{\nu,\mu}^{(1)}(\gamma,\, z)\right)=\frac{1}{1-z^2}\left(S_{\nu,\mu}^{(1)\prime}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)}(\gamma,\, 0)-S_{\nu,\mu}^{(1)}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)\prime}(\gamma,\, 0)\right)$$

11.11.13.0003.01

$$\left(1-g(z)^2\right) w''(z)+\left(\left(g(z)^2-1\right) \frac{g''(z)}{g'(z)}-2\, g(z)\, g'(z)\right) w'(z)+\left(\frac{\left(\gamma\, g(z)^2+\mu-\gamma\right)\left(-\gamma\, g(z)^2+\mu+\gamma\right) g'(z)^2}{g(z)^2-1}+\lambda_{\nu,\mu}(\gamma)\, g'(z)^2\right) w(z)=0\; /;$$
$$w(z)=c_1\, S_{\nu,\mu}^{(2)}(\gamma,\, g(z))\; +c_2\, S_{\nu,\mu}^{(1)}(\gamma,\, g(z))$$

11.11.13.0004.01

$$W_z\left(S_{\nu,\mu}^{(2)}(\gamma,\, g(z)),\, S_{\nu,\mu}^{(1)}(\gamma,\, g(z))\right)=\frac{g'(z)}{1-g[z]^2}\left(S_{\nu,\mu}^{(1)\prime}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)}(\gamma,\, 0)-S_{\nu,\mu}^{(1)}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)\prime}(\gamma,\, 0)\right)$$

11.11.13.0005.01

$$\left(1-g(z)^2\right) w''(z)+\left(\left(g(z)^2-1\right)\left(\frac{2\, h'(z)}{h(z)}+\frac{g''(z)}{g'(z)}\right)-2\, g(z)\, g'(z)\right) w'(z)+$$
$$\left(\frac{\left(\gamma\, g(z)^2+\mu-\gamma\right)\left(-\gamma\, g(z)^2+\mu+\gamma\right) g'(z)^2}{g(z)^2-1}+\lambda_{\nu,\mu}(\gamma)\, g'(z)^2+\frac{2\, g(z)\, h'(z)\, g'(z)}{h(z)}+\frac{\left(g(z)^2-1\right)\left(h(z)\, h''(z)-2\, h'(z)^2\right)}{h(z)^2}-\right.$$
$$\left.\frac{\left(g(z)^2-1\right) h'(z)\, g''(z)}{h(z)\, g'(z)}\right) w(z)=0\; /;\; w(z)=c_1\, h(z)\, S_{\nu,\mu}^{(2)}(\gamma,\, g(z))\; +c_2\, h(z)\, S_{\nu,\mu}^{(1)}(\gamma,\, g(z))$$

11.11.13.0006.01

$$W_z\left(h(z)\, S_{\nu,\mu}^{(2)}(\gamma,\, g(z)),\, h(z)\, S_{\nu,\mu}^{(1)}(\gamma,\, g(z))\right)=\frac{h(z)^2\, g'(z)}{1-g[z]^2}\left(S_{\nu,\mu}^{(1)\prime}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)}(\gamma,\, 0)-S_{\nu,\mu}^{(1)}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)\prime}(\gamma,\, 0)\right)$$

11.11.13.0007.01

$$\left(1-a^2\, z^{2\, r}\right) w''(z)-\frac{a^2\, (r-2\, s+1)\, z^{2\, r}+r+2\, s-1}{z}\, w'(z)+$$
$$\left(a^2\, r^2\, \lambda_{\nu,\mu}(\gamma)\, z^{2\, r-2}+\frac{-a^2\, r^2\left(\left(a^2\, z^{2\, r}-1\right)^2 \gamma^2-\mu^2\right) z^{2\, r}-s^2\left(a^2\, z^{2\, r}-1\right)^2+r\, s\left(a^4\, z^{4\, r}-1\right)}{z^2\left(a^2\, z^{2\, r}-1\right)}\right) w(z)=$$
$$0\; /;\; w(z)=c_1\, z^s\, S_{\nu,\mu}^{(2)}(\gamma,\, a\, z^r)\; +c_2\, z^s\, S_{\nu,\mu}^{(1)}(\gamma,\, a\, z^r)$$

11.11.13.0008.01

$$W_z\left(z^s\, S_{\nu,\mu}^{(2)}(\gamma,\, a\, z^r),\, z^s\, S_{\nu,\mu}^{(1)}(\gamma,\, a\, z^r)\right)=\frac{a\, r\, z^{r+2\, s-1}}{1-a^2\, z^{2\, r}}\left(S_{\nu,\mu}^{(1)\prime}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)}(\gamma,\, 0)-S_{\nu,\mu}^{(1)}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)\prime}(\gamma,\, 0)\right)$$

11.11.13.0009.01

$$\left(1-a^2 r^{2z}\right) w''(z)+\left(-a^2 \log(r)\, r^{2z}+2\, a^2 \log(s)\, r^{2z}-\log(r)-2 \log(s)\right) w'(z)+ \left(a^2 \log^2(r)\, \lambda_{\nu,\mu}(\gamma)\, r^{2z}+\frac{1}{a^2 r^{2z}-1}\left(-a^2\left(\left(a^2 r^{2z}-1\right)^2 \gamma^2-\mu^2\right)\log^2(r)\, r^{2z}-\left(a^2 r^{2z}-1\right)^2 \log^2(s)+\left(a^4 r^{4z}-1\right)\log(r)\log(s)\right)\right)$$
$$w(z)=0 \;/;\; w(z)=c_1\, s^z\, S_{\nu,\mu}^{(2)}(\gamma,\, a\, r^z)\;+c_2\, s^z\, S_{\nu,\mu}^{(1)}(\gamma,\, a\, r^z)$$

11.11.13.0010.01

$$W_z\left(s^z\, S_{\nu,\mu}^{(2)}(\gamma,\, a\, r^z),\, s^z\, S_{\nu,\mu}^{(1)}(\gamma,\, a\, r^z)\right) = \frac{a\, r^z\, s^{2z} \log(r)}{1-a^2 r^{2z}}\left(S_{\nu,\mu}^{(1)\prime}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)}(\gamma,\, 0)-S_{\nu,\mu}^{(1)}(\gamma,\, 0)\, S_{\nu,\mu}^{(2)\prime}(\gamma,\, 0)\right)$$

Differentiation

Low-order differentiation

With respect to z

11.11.20.0001.01

$$\text{SpheroidalS2}^{(0,0,0,1)}(\nu,\, \mu,\, \gamma,\, z) = S_{\nu,\mu}^{(2)\prime}(\gamma,\, z)$$

11.11.20.0002.01

$$\text{SpheroidalS2}^{(0,0,0,2)}(\nu,\, \mu,\, \gamma,\, z) = \frac{1}{1-z^2}\left(2\, z\, S_{\nu,\mu}^{(2)\prime}(\gamma,\, z)-\left(\left(1-z^2\right)\gamma^2+\lambda_{\nu,\mu}(\gamma)-\frac{\mu^2}{1-z^2}\right) S_{\nu,\mu}^{(2)}(\gamma,\, z)\right)$$

Integration

Operations

Limit operation

11.11.25.0001.01

$$\lim_{z\to\infty} \frac{S_{\nu,\mu}^{(2)}\left(\gamma,\, \frac{z}{\gamma}\right)}{y_\nu(z)} = 1$$

Representations through equivalent functions

Theorems

History

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