Sqrt

Notations

Traditional name

Square root

Traditional notation

$\sqrt{z}$

Mathematica StandardForm notation

`Sqrt[z]`

Primary definition

01.01.02.0001.01

$$\sqrt{z} = z^{1/2}$$

01.01.02.0002.01

$$\sqrt{z} = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)_k \frac{(1-z)^k}{k!} \;/;\; |z-1| < 1$$

Specific values

Values at fixed points

01.01.03.0001.01

$$\sqrt{0} = 0$$

01.01.03.0002.01

$$\sqrt{-1} = i$$

01.01.03.0003.01

$$\sqrt{i} = \sqrt[4]{-1} = e^{\frac{i\pi}{4}}$$

01.01.03.0004.01

$$\sqrt{-i} = -(-1)^{3/4} = e^{-\frac{1}{4}(i\pi)}$$

Values of nested square roots

01.01.03.0011.01

$$\frac{b_n}{2}\sqrt{2+b_{n-1}\sqrt{2+b_{n-2}\sqrt{2+\ldots+b_2\sqrt{2+\sin\left(\frac{\pi b_1}{4}\right)}}}} = \cos\left(\pi\left(\frac{1}{2}-\sum_{k=1}^{n}2^{-k}\prod_{j=1}^{k}b_{k-j}\right)\right) /;$$

$$(b_k = -1 \lor b_k = 0 \lor b_k = 1) \land (2 \le k \le n \land k \in \mathbb{N}) \land b_1 \in \mathbb{R} \land -2 \le b_1 \le 2$$

L. D. Servi: Nested Square Roots of 2 American Mathematical Monthly 110, 326-329 (2003)

01.01.03.0012.01

$$\frac{b_n}{2}\sqrt{2-b_{n-1}\sqrt{2+b_{n-2}\sqrt{2+\ldots+b_2\sqrt{2+\sin\left(\frac{\pi b_1}{4}\right)}}}} = \sin\left(\pi\left(\frac{1}{2}-\sum_{k=1}^{n}2^{-k}\prod_{j=1}^{k}b_{k-j}\right)\right) /;$$

$$(b_n = 1 \lor b_n = -1) \land (2 \le k \le n-1 \land k \in \mathbb{N}) \land b_1 \in \mathbb{R} \land -2 \le b_1 \le 2$$

L. D. Servi: Nested Square Roots of 2 American Mathematical Monthly 110, 326-329 (2003)

Values at infinities

01.01.03.0005.01

$$\sqrt{\infty} = \infty$$

01.01.03.0006.01

$$\sqrt{-\infty} = i\,\infty$$

01.01.03.0007.01

$$\sqrt{i\,\infty} = \sqrt[4]{-1}\,\infty$$

01.01.03.0008.01

$$\sqrt{-i\,\infty} = -(-1)^{3/4}\,\infty$$

01.01.03.0009.01

$$\sqrt{\tilde{\infty}} = \tilde{\infty}$$

01.01.03.0010.01

$$\sqrt{(z\,\infty)} = \sqrt{\operatorname{sgn}(z)}\,\infty$$

General characteristics

Domain and analyticity

$\sqrt{z}$ is an analytical function of z which is defined over the whole complex z-plane.

01.01.04.0001.01

$$z \longrightarrow \sqrt{z} :: \mathbb{C} \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

01.01.04.0002.01

$$\sqrt{\bar{z}} == \overline{\sqrt{z}} \;/;\; z \notin (-\infty, 0)$$

Periodicity

No periodicity

Poles and essential singularities

The function $\sqrt{z}$ does not have poles and essential singularities.

01.01.04.0003.01

$$\mathcal{Sing}_z\left(\sqrt{z}\right) == \{\}$$

Branch points

The function $\sqrt{z}$ has two singular branch points: $z == 0$, $z == \tilde{\infty}$.

01.01.04.0004.01

$$\mathcal{BP}_z\left(\sqrt{z}\right) == \{0, \tilde{\infty}\}$$

01.01.04.0005.01

$$\mathcal{R}_z\left(\sqrt{z}, 0\right) == 2$$

01.01.04.0006.01

$$\mathcal{R}_z\left(\sqrt{z}, \tilde{\infty}\right) == 2$$

Branch cuts

The function $\sqrt{z}$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

01.01.04.0007.01

$$\mathcal{BC}_z\left(\sqrt{z}\right) == \{\{(-\infty, 0), -i\}\}$$

01.01.04.0008.01

$$\lim_{\epsilon \to +0} \sqrt{x + i\,\epsilon} == \sqrt{x} \;/;\; x < 0$$

01.01.04.0010.01

$$\lim_{\epsilon \to +0} \sqrt{x + i\,\epsilon} = \sqrt{|x|} \;/;\; x < 0$$

01.01.04.0011.01

$$\lim_{\epsilon \to +0} \sqrt{x + i\,\epsilon} == i \sqrt{-x} \;/;\; x < 0$$

01.01.04.0012.01

$$\lim_{\epsilon \to +0} \sqrt{x - i\,\epsilon} = -i \sqrt{|x|} \;/;\; x < 0$$

01.01.04.0009.01

$$\lim_{\epsilon \to +0} \sqrt{x - i\,\epsilon} == -i \sqrt{-x} \;/;\; x < 0$$

01.01.04.0013.01

$$\lim_{\epsilon \to +0} \sqrt{x - i\,\epsilon} = -\sqrt{x} \;/;\; x < 0$$

Analytic continuations

$\sqrt{z}$ is chosen to be the principal branch of the general square root function which has two sheets: $\pm\sqrt{z}$.

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

01.01.06.0013.01

$$\sqrt{z} \propto \left(\frac{1}{z_0}\right)^{\frac{1}{2}\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\frac{1}{2}\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} \left(1 + \frac{z-z_0}{2\,z_0} - \frac{(z-z_0)^2}{8\,z_0^2} + \ldots\right) /; (z \to z_0)$$

01.01.06.0014.01

$$\sqrt{z} \propto \left(\frac{1}{z_0}\right)^{\frac{1}{2}\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\frac{1}{2}\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} \left(1 + \frac{z-z_0}{2\,z_0} - \frac{(z-z_0)^2}{8\,z_0^2} + O\left((z-z_0)^3\right)\right)$$

01.01.06.0015.01

$$\sqrt{z} = \left(\frac{1}{z_0}\right)^{\frac{1}{2}\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\frac{1}{2}\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(-\frac{1}{2}\right)_k z_0^{-k} (z-z_0)^k$$

01.01.06.0016.01

$$\sqrt{z} == \left(\frac{1}{z_0}\right)^{\frac{1}{2}\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\frac{1}{2}\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} {}_1F_0\left(-\frac{1}{2};;-\frac{z-z_0}{z_0}\right)$$

01.01.06.0017.01

$$\sqrt{z} == \left(\frac{1}{z_0}\right)^{\frac{1}{2}\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\frac{1}{2}\left(\left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor + 1\right)} (1 + O(z-z_0))$$

Expansions on branch cuts

For the function itself

01.01.06.0018.01

$$\sqrt{z} \propto \sqrt{x}\; e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(1 + \frac{1}{2\,z_0}(z-x) - \frac{1}{8\,z_0^2}(z-x)^2 + \ldots\right) /; (z \to x) \land x \in \mathbb{R} \land x < 0$$

01.01.06.0019.01

$$\sqrt{z} \propto \sqrt{x}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \left(1 + \frac{1}{2 z_0} (z-x) - \frac{1}{8 z_0^2} (z-x)^2 + O\left((z-x)^3\right)\right) /;\ x \in \mathbb{R} \land x < 0$$

01.01.06.0020.01

$$\sqrt{z} \propto \sqrt{x}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(-\frac{1}{2}\right)_k x^{-k} (z-x)^k /;\ x \in \mathbb{R} \land x < 0$$

01.01.06.0021.01

$$\sqrt{z} = \sqrt{x}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} {}_1F_0\left(-\frac{1}{2};\ ;\ -\frac{z-x}{x}\right) /;\ x \in \mathbb{R} \land x < 0$$

01.01.06.0022.01

$$\sqrt{z} = \sqrt{x}\, e^{\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} (1 + O(z-x)) /;\ x \in \mathbb{R} \land x < 0$$

Expansions at $z = 1$

For the function itself

01.01.06.0005.02

$$\sqrt{z} \propto 1 + \frac{1}{2} (z-1) - \frac{1}{8} (z-1)^2 + \frac{1}{16} (z-1)^3 - \ldots /;\ (z \to 1)$$

01.01.06.0023.01

$$\sqrt{z} \propto 1 + \frac{1}{2} (z-1) - \frac{1}{8} (z-1)^2 + \frac{1}{16} (z-1)^3 - O\left((z-1)^4\right)$$

01.01.06.0006.01

$$\sqrt{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(-\frac{1}{2}\right)_k (z-1)^k /;\ |z-1| < 1$$

01.01.06.0007.01

$$\sqrt{z} = {}_1F_0\left(-\frac{1}{2};\ ;\ 1-z\right)$$

01.01.06.0008.02

$$\sqrt{z} \propto 1 + O(z-1)$$

01.01.06.0024.01

$$\sqrt{z} = F_\infty(z) /; \left(\left(F_n(z) = \sum_{k=0}^{n} \frac{(-1)^k (-1/2)_k (z-1)^k}{k!} = \sqrt{z} + (-1)^n (z-1)^{n+1} (-1/2)_{n+1}\, {}_2\tilde{F}_1(1, n+1/2; n+2; 1-z)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Expansions of $\sqrt{1+z}$ at $z = 0$

For the function itself

01.01.06.0001.01

$$\sqrt{1+z} \propto 1 + \frac{1}{2} z - \frac{1}{8} z^2 + \frac{1}{16} z^3 - \ldots /;\ (z \to 0)$$

01.01.06.0025.01

$$\sqrt{1+z} \propto 1+\frac{1}{2}z-\frac{1}{8}z^2+\frac{1}{16}z^3-O(z^4)$$

01.01.06.0002.01

$$\sqrt{1+z} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(-\frac{1}{2}\right)_k z^k \;/;\; |z|<1$$

01.01.06.0003.01

$$\sqrt{1+z} = {}_1F_0\left(-\frac{1}{2};\,;-z\right)$$

01.01.06.0004.02

$$\sqrt{1+z} \propto 1+O(z)$$

01.01.06.0026.01

$$\sqrt{1+z} = F_\infty(z) \;/;\; \left(\left(F_n(z) = \sum_{k=0}^{n}\frac{(-1)^k(-1/2)_k z^k}{k!} = \sqrt{1+z} + (-1)^n z^{n+1}(-1/2)_{n+1}\, {}_2\tilde{F}_1(1, n+1/2; n+2; -z)\right) \bigwedge n\in\mathbb{N}\right)$$

Summed form of the truncated series expansion.

Asymptotic series expansions

01.01.06.0009.01

$$\sqrt{z} \propto \sqrt{z} \;/;\; (|z|\to\infty)$$

Residue representations

Representations of $\sqrt{z}$

01.01.06.0027.01

$$\sqrt{z} = -\frac{1}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma\left(-s-\frac{1}{2}\right)(z-1)^{-s}\right)\Gamma(s)\right)(-j) \;/;\; |z-1|<1$$

01.01.06.0028.01

$$\sqrt{1} = \frac{1}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\operatorname{res}_s\left((\Gamma(s)(z-1)^{-s})\,\Gamma\left(-s-\frac{1}{2}\right)\right)\left(-\frac{1}{2}+j\right) \;/;\; |z-1|>1$$

Representations of $\sqrt{1+z}$

01.01.06.0010.01

$$\sqrt{z+1} = -\frac{1}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\operatorname{res}_s\left(\left(\Gamma\left(-s-\frac{1}{2}\right)z^{-s}\right)\Gamma(s)\right)(-j) \;/;\; |z|<1$$

01.01.06.0011.01

$$\sqrt{z+1} = \frac{1}{2\sqrt{\pi}}\sum_{j=0}^{\infty}\operatorname{res}_s\left((\Gamma(s)\,z^{-s})\,\Gamma\left(-s-\frac{1}{2}\right)\right)\left(-\frac{1}{2}+j\right) \;/;\; |z|>1$$

Other series representations

01.01.06.0012.01

$$\sqrt{a+\sqrt{z+1}} = \sqrt{a+1}\left(1-\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left(-\frac{z}{2a+2}\right)^k P_{k-1}^{\left(k-\frac{1}{2},\frac{1}{2}-k\right)}(a)\right)/;\ |z|<1$$

Integral representations

Contour integral representations

01.01.07.0001.01

$$\sqrt{z+1} = -\frac{1}{2\sqrt{\pi}}\frac{1}{2\pi i}\int_{\mathcal{L}}\Gamma(s)\,\Gamma\left(-s-\frac{1}{2}\right)z^{-s}\,ds$$

Limit representations

01.01.09.0001.01

$$\lim_{n\to\infty} y_n /;\ y_n = y_{n-1} - \frac{y_{n-1}^2 - z}{2\,y_n} \bigwedge y_0 = z \land z \notin \{-\infty, 0\}$$

P. W. Pedersen

Continued fraction representations

01.01.10.0005.01

$$\sqrt{1+z} = 1 + \cfrac{z}{4\left(\cfrac{1}{2} + \cfrac{z}{16\left(\cfrac{1}{2} + \cfrac{z}{16\left(\cfrac{1}{2} + \cfrac{z}{16\left(\cfrac{1}{2}+\cfrac{z}{16\ldots}\right)}\right)}\right)}\right)}$$

Andreas Lauschke (2006)

01.01.10.0006.01

$$\sqrt{1+z} = 1 + 4\,\mathrm{K}_k\left(\frac{z}{16}, \frac{1}{2}\right)_1^{\infty}$$

Andreas Lauschke (2006)

01.01.10.0001.01

$$\sqrt{z+1} == 1 + \frac{z}{2\left(1 + \frac{z}{2\left(2 + \frac{3z}{2\left(3 + \frac{3z}{2\left(2 + \frac{5z}{2(5+\ldots)}\right)}\right)}\right)}\right)} /; z \notin (-\infty, -1)$$

01.01.10.0002.01

$$\sqrt{z+1} == 1 + \mathrm{K}_k\left(\left(\left\lfloor \frac{k}{2} \right\rfloor - \frac{(-1)^k}{2}\right) z, \frac{1}{2}\left(1-(-1)^k\right) k + (-1)^k + 1\right)_1^{\infty} /; z \notin (-\infty, -1)$$

01.01.10.0003.01

$$\sqrt{z} == z + \frac{z - z^2}{2z + \frac{z-z^2}{2z + \frac{z-z^2}{2z + \frac{z-z^2}{\ldots}}}} \bigwedge z \notin \{-\infty, 0\}$$

P. W. Pedersen

01.01.10.0004.01

$$\sqrt{z} == z + \mathrm{K}_k\left(z - z^2, 2z\right)_1^{\infty} /; z \notin (-\infty, 0)$$

P. W. Pedersen

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

01.01.13.0001.01

$$w'(z) - \frac{1}{2z} w(z) = 0 /; w(z) = c_1 \sqrt{z}$$

01.01.13.0002.01

$$w'(z) - \frac{1}{2z} w(z) == 0 /; w(z) == \sqrt{z} \bigwedge w[1] == 1$$

01.01.13.0003.01

$$4 w''(z) z^2 + w(z) == 0 /; w(z) == \sqrt{z}\, c_1 + \sqrt{z} \log(z)\, c_2$$

01.01.13.0004.01

$$W_z\left(\sqrt{z}, \sqrt{z} \log(z)\right) == 1$$

01.01.13.0005.01

$$w'(z) - \frac{g'(z)}{2 g(z)} w(z) = 0 /; w(z) = c_1 \sqrt{g(z)}$$

01.01.13.0006.01

$$w'(z) - \left(\frac{g'(z)}{2 g(z)} + \frac{h'(z)}{h(z)}\right) w(z) = 0 /; w(z) = c_1 h(z) \sqrt{g(z)}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

01.01.16.0001.01

$$\sqrt{-z} == -i \sqrt{z} /; \operatorname{Im}(z) > 0 \lor \operatorname{Im}(z) == 0 \land \operatorname{Re}(z) < 0$$

01.01.16.0002.01

$$\sqrt{-z} == i \sqrt{z} /; \operatorname{Im}(z) < 0 \lor \operatorname{Im}(z) == 0 \land \operatorname{Re}(z) > 0$$

01.01.16.0044.01

$$\sqrt{-z} == -i \sqrt{z} /; \arg(z) > 0$$

01.01.16.0045.01

$$\sqrt{-z} == i \sqrt{z} /; \arg(z) \leq 0$$

01.01.16.0003.01

$$\sqrt{-z} == \exp\left(\frac{1}{2} i \pi \left(1 + 2 \left\lfloor -\frac{\arg(z)}{2\pi} \right\rfloor\right)\right) \sqrt{z}$$

01.01.16.0004.01

$$\sqrt{-z} == \exp\left(\frac{\pi \sqrt{-z^2}}{2z}\right) \sqrt{z}$$

01.01.16.0046.01

$$\sqrt{i z} == \sqrt[4]{-1} \sqrt{z} /; \arg(z) \leq \frac{\pi}{2}$$

01.01.16.0047.01

$$\sqrt{i z} == -\sqrt[4]{-1} \sqrt{z} /; \arg(z) > \frac{\pi}{2}$$

01.01.16.0048.01

$$\sqrt{i z} == (-1)^{\left\lfloor \frac{1}{4} - \frac{\arg(z)}{2\pi} \right\rfloor + \frac{1}{4}} \sqrt{z}$$

01.01.16.0049.01

$$\sqrt{-i\,z} = -(-1)^{3/4}\sqrt{z} \;/;\; \arg(z) > -\frac{\pi}{2}$$

01.01.16.0050.01

$$\sqrt{-i\,z} = (-1)^{3/4}\sqrt{z} \;/;\; \arg(z) \leq -\frac{\pi}{2}$$

01.01.16.0051.01

$$\sqrt{-i\,z} = -(-1)^{\left\lfloor \frac{3}{4}-\frac{\arg(z)}{2\pi}\right\rfloor+\frac{3}{4}}\sqrt{z}$$

01.01.16.0005.01

$$\sqrt{\frac{1}{z}} = \frac{1}{\sqrt{z}} \;/;\; z \notin (-\infty, 0)$$

01.01.16.0052.01

$$\sqrt{\frac{1}{z}} = \frac{1}{\sqrt{z}} \;/;\; \arg(z) \neq \pi$$

01.01.16.0053.01

$$\sqrt{\frac{1}{z}} = -\frac{1}{\sqrt{z}} \;/;\; \arg(z) = \pi$$

01.01.16.0054.01

$$\sqrt{\frac{1}{z}} = \frac{(-1)^{\left\lfloor \frac{\arg(z)}{2\pi}+\frac{1}{2}\right\rfloor}}{\sqrt{z}}$$

01.01.16.0055.01

$$\sqrt{-\frac{1}{z}} = \frac{i}{\sqrt{z}} \;/;\; \mathrm{Im}(z) \geq 0$$

01.01.16.0056.01

$$\sqrt{-\frac{1}{z}} = -\frac{i}{\sqrt{z}} \;/;\; \mathrm{Im}(z) < 0$$

01.01.16.0057.01

$$\sqrt{-\frac{1}{z}} = \frac{i\,(-1)^{\left\lfloor \frac{\arg(z)}{2\pi}\right\rfloor}}{\sqrt{z}}$$

01.01.16.0058.01

$$\sqrt{\frac{i}{z}} = \frac{\sqrt[4]{-1}}{\sqrt{z}} \;/;\; \arg(z) \geq -\frac{\pi}{2}$$

01.01.16.0059.01

$$\sqrt{\frac{i}{z}} = -\frac{\sqrt[4]{-1}}{\sqrt{z}} \;/;\; \arg(z) < -\frac{\pi}{2}$$

01.01.16.0060.01

$$\sqrt{\frac{i}{z}} = \frac{(-1)^{\left\lfloor \frac{\arg(z)}{2\pi}+\frac{1}{4}\right\rfloor+\frac{1}{4}}}{\sqrt{z}}$$

01.01.16.0061.01

$$\sqrt{-\frac{i}{z}} = -\frac{(-1)^{3/4}}{\sqrt{z}} \;/;\; \arg(z) < \frac{\pi}{2}$$

01.01.16.0062.01

$$\sqrt{-\frac{i}{z}} = \frac{(-1)^{3/4}}{\sqrt{z}} \;/;\; \arg(z) \geq \frac{\pi}{2}$$

01.01.16.0063.01

$$\sqrt{-\frac{i}{z}} = -\frac{(-1)^{\left\lfloor \frac{\arg(z)}{2\pi} + \frac{3}{4} \right\rfloor + \frac{3}{4}}}{\sqrt{z}}$$

Half-angle formulas

01.01.16.0006.01

$$\sqrt{\frac{z}{2}} = \frac{\sqrt{z}}{\sqrt{2}}$$

Multiple arguments

For products

01.01.16.0007.01

$$\sqrt{a\, z} = \sqrt{a}\, \sqrt{z} \;/;\; a > 0$$

01.01.16.0008.01

$$\sqrt{z_1\, z_2} = \sqrt{z_1}\, \sqrt{z_2} \;/;\; z_1 + z_2 \geq 0$$

01.01.16.0009.01

$$\sqrt{z - z^2} = \sqrt{z}\, \sqrt{1 - z}$$

01.01.16.0010.01

$$\sqrt{-z^2 - z} = \sqrt{-z}\, \sqrt{z + 1}$$

01.01.16.0064.01

$$\sqrt{z_1\, z_2} = \sqrt{z_1}\, \sqrt{z_2} \;/;\; \arg(z_1) \leq 0 \bigwedge -\arg(z_1) - \pi < \arg(z_2) \bigvee \arg(z_1) \geq 0 \bigwedge \arg(z_2) \leq \pi - \arg(z_1)$$

01.01.16.0065.01

$$\sqrt{z_1\, z_2} = -\sqrt{z_1}\, \sqrt{z_2} \;/;\; \arg(z_1) \geq 0 \bigwedge \arg(z_2) > \pi - \arg(z_1) \bigvee \arg(z_1) \leq 0 \bigwedge \arg(z_2) \leq -\arg(z_1) - \pi$$

01.01.16.0011.01

$$\sqrt{z_1\, z_2} = \sqrt{z_1}\, \sqrt{z_2}\, \exp\left(\pi\, i \left\lfloor \frac{\pi - \arg(z_1) - \arg(z_2)}{2\pi} \right\rfloor\right)$$

01.01.16.0066.01

$$\sqrt{\prod_{k=1}^{n} z_k} = e^{\pi\, i \left\lfloor \frac{\pi - \sum_{k=1}^{n} \arg(z_k)}{2\pi} \right\rfloor} \prod_{k=1}^{n} \sqrt{z_k} \;/;\; n \in \mathbb{N}^+$$

For quotients

01.01.16.0012.01

$$\sqrt{\frac{z}{z+1}} == \frac{\sqrt{z}}{\sqrt{z+1}}$$

01.01.16.0013.01

$$\sqrt{\frac{z}{z-1}} == \frac{\sqrt{-z}}{\sqrt{1-z}}$$

01.01.16.0014.01

$$\sqrt{\frac{z_1}{z_2}} == \frac{1}{\sqrt{\frac{z_2}{z_2-z_1}}} \sqrt{\frac{z_1}{z_2-z_1}} \text{ /; } z_2 - z_1 \in \mathbb{R} \bigwedge z_2 \neq z_1$$

01.01.16.0015.01

$$\sqrt{\frac{z_1}{z_2}} == \frac{\sqrt{z_1}}{\sqrt{z_2}} \text{ /; } z_2 - z_1 \geq 0$$

01.01.16.0016.01

$$\sqrt{\frac{z_1}{z_2}} == \frac{\sqrt{-z_1}}{\sqrt{-z_2}} \text{ /; } z_2 - z_1 < 0$$

01.01.16.0067.01

$$\sqrt{\frac{z_1}{z_2}} == \frac{\sqrt{z_1}}{\sqrt{z_2}} \text{ /; } \arg(z_1) \leq 0 \bigwedge \arg(z_2) < \arg(z_1) + \pi \bigvee \arg(z_1) > 0 \bigwedge \arg(z_2) \geq \arg(z_1) - \pi$$

01.01.16.0068.01

$$\sqrt{\frac{z_1}{z_2}} == -\frac{\sqrt{z_1}}{\sqrt{z_2}} \text{ /; } \arg(z_1) \geq 0 \bigwedge \arg(z_2) < \arg(z_1) - \pi \bigvee \arg(z_1) \leq 0 \bigwedge \arg(z_2) \geq \arg(z_1) + \pi$$

01.01.16.0069.01

$$\sqrt{\frac{z_1}{z_2}} == \frac{\sqrt{z_1}}{\sqrt{z_2}} e^{i \pi \left\lfloor \frac{\pi - \arg(z_1) + \arg(z_2)}{2\pi} \right\rfloor}$$

01.01.16.0017.01

$$\sqrt{\frac{a+z}{b+z}} == \frac{1}{\sqrt{\frac{b+z}{b-a}}} \sqrt{\frac{a+z}{b-a}} \text{ /; } a - b \in \mathbb{R} \bigwedge a \neq b$$

01.01.16.0018.01

$$\sqrt{\frac{z}{1-z}} == \sqrt{z} \sqrt{\frac{1}{1-z}}$$

01.01.16.0019.01

$$\sqrt{-\frac{z}{1+z}} == \sqrt{-z} \sqrt{\frac{1}{1+z}}$$

01.01.16.0020.01

$$\sqrt{\frac{z_1}{z_2}} == \sqrt{\frac{z_1}{z_1+z_2}} \sqrt{\frac{z_1+z_2}{z_2}} \text{ /; } z_1 + z_2 \in \mathbb{R} \bigwedge z_1 + z_2 \neq 0$$

01.01.16.0021.01

$$\sqrt{\frac{z_1}{z_2}} = \sqrt{z_1}\,\sqrt{\frac{1}{z_2}}\ /;\ z_1+z_2 \geq 0$$

01.01.16.0022.01

$$\sqrt{\frac{z_1}{z_2}} = \sqrt{-z_1}\,\sqrt{-\frac{1}{z_2}}\ /;\ z_1+z_2 < 0$$

01.01.16.0023.01

$$\sqrt{\frac{a+z}{b-z}} = \sqrt{\frac{a+z}{a+b}}\,\sqrt{\frac{a+b}{b-z}}\ /;\ a+b \in \mathbb{R} \bigwedge a+b \neq 0$$

Power of arguments

01.01.16.0070.01

$$\sqrt{z^2} = z\ /;\ -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

01.01.16.0071.01

$$\sqrt{z^2} = -z\ /;\ -\pi < \arg(z) \leq -\frac{\pi}{2} \bigvee \frac{\pi}{2} < \arg(z) \leq \pi$$

01.01.16.0024.01

$$\sqrt{z^2} = \sqrt{-i\,z}\,\sqrt{i\,z}$$

01.01.16.0025.01

$$\sqrt{-z^2} = \sqrt{z}\,\sqrt{-z}$$

01.01.16.0072.01

$$\sqrt{z^a} = z^{a/2}\ /;\ a \in \mathbb{R} \bigwedge -\pi < a\arg(z) \leq \pi$$

01.01.16.0073.01

$$\sqrt{z^a} = (-1)^k\,z^{a/2}\ /;\ a \in \mathbb{R} \bigwedge -2\,\pi\,k-\pi < a\arg(z) \leq \pi-2\,\pi\,k \bigwedge k \in \mathbb{Z}$$

01.01.16.0026.01

$$\sqrt{z^a} = z^{a/2}\ /;\ -\pi < \mathrm{Im}(a\log(z)) \leq \pi \bigvee -1 < a \leq 1$$

01.01.16.0074.01

$$\sqrt{z^a} = (-1)^k\,z^{a/2}\ /;\ -2\,\pi\,k-\pi < \mathrm{Im}(a\log(z)) \leq \pi-2\,\pi\,k \bigwedge k \in \mathbb{Z}$$

01.01.16.0027.01

$$\sqrt{z^a} = z^{a/2}\exp\left(\pi\,i\left\lfloor\frac{\pi-\mathrm{Im}(a\log(z))}{2\,\pi}\right\rfloor\right)$$

Exponent of arguments

01.01.16.0075.01

$$\sqrt{e^z} = e^{z/2}\ /;\ -\pi < \mathrm{Im}(z) \leq \pi$$

01.01.16.0076.01

$$\sqrt{e^z} = (-1)^k e^{z/2} /; -2\pi k - \pi < \operatorname{Im}(z) \leq \pi - 2\pi k \bigwedge k \in \mathbb{Z}$$

01.01.16.0077.01

$$\sqrt{e^z} = e^{z/2} e^{\pi i \left\lfloor \frac{\pi - \operatorname{Im}(z)}{2\pi} \right\rfloor}$$

01.01.16.0078.01

$$\sqrt{e^{iz}} = e^{\frac{iz}{2}} e^{\pi i \left\lfloor \frac{\pi - \operatorname{Re}(z)}{2\pi} \right\rfloor}$$

Some functions of arguments

01.01.16.0079.01

$$\sqrt{c z^m} = \sqrt{c} \, z^{m/2} e^{\pi i \left\lfloor \frac{\pi - \arg(c) - \operatorname{Im}(m \log(z))}{2\pi} \right\rfloor}$$

01.01.16.0080.01

$$\sqrt{c e^z} = \sqrt{c} \, e^{z/2} e^{\pi i \left\lfloor \frac{\pi - \arg(c) - \operatorname{Im}(z)}{2\pi} \right\rfloor}$$

01.01.16.0081.01

$$\sqrt{x^{a_1} y^{a_2}} = e^{i\pi \left\lfloor \frac{\pi - \operatorname{Im}(a_1 \log(x)) - \operatorname{Im}(a_2 \log(y))}{2\pi} \right\rfloor} x^{\frac{a_1}{2}} y^{\frac{a_2}{2}}$$

01.01.16.0082.01

$$\sqrt{x^{a_1} y^{a_2} z^{a_3}} = e^{i\pi \left\lfloor \frac{\pi - \operatorname{Im}(a_1 \log(x)) - \operatorname{Im}(a_2 \log(y)) - \operatorname{Im}(a_3 \log(z))}{2\pi} \right\rfloor} x^{\frac{a_1}{2}} y^{\frac{a_2}{2}} z^{\frac{a_3}{2}}$$

01.01.16.0083.01

$$\sqrt{\prod_{k=1}^{n} z_k^{a_k}} = e^{i\pi \left\lfloor \frac{\pi - \sum_{k=1}^{n} \operatorname{Im}(a_k \log(z_k))}{2\pi} \right\rfloor} \prod_{k=1}^{n} z_k^{\frac{a_k}{2}}$$

Products, sums, and powers of the direct function

Products of the direct function

01.01.16.0028.01

$$\sqrt{z} \sqrt{z} = z$$

01.01.16.0029.01

$$\sqrt{x_1} \sqrt{x_2} = \sqrt{x_1 x_2} /; x_1 > 0 \bigwedge x_2 > 0 \bigvee x_1 x_2 < 0$$

01.01.16.0030.01

$$\sqrt{x_1} \sqrt{x_2} = -\sqrt{x_1 x_2} /; x_1 < 0 \bigwedge x_2 < 0$$

01.01.16.0084.01

$$\sqrt{z_1} \sqrt{z_2} = \sqrt{z_1 z_2} /; \arg(z_1) \leq 0 \bigwedge -\arg(z_1) - \pi < \arg(z_2) \bigvee \arg(z_1) \geq 0 \bigwedge \arg(z_2) \leq \pi - \arg(z_1)$$

01.01.16.0085.01

$$\sqrt{z_1} \sqrt{z_2} = -\sqrt{z_1 z_2} /; \arg(z_1) \geq 0 \bigwedge \arg(z_2) > \pi - \arg(z_1) \bigvee \arg(z_1) \leq 0 \bigwedge \arg(z_2) \leq -\arg(z_1) - \pi$$

01.01.16.0031.01

$$\sqrt{z_1} \sqrt{z_2} = \sqrt{z_1 z_2} \exp\left(-\pi i \left\lfloor \frac{\pi - \arg(z_1) - \arg(z_2)}{2\pi} \right\rfloor\right)$$

01.01.16.0086.01

$$\prod_{k=1}^{n}\sqrt{z_k} == \sqrt{\prod_{k=1}^{n} z_k}\; e^{-\pi i \left\lfloor \frac{\pi - \sum_{k=1}^{n}\arg(z_k)}{2\pi} \right\rfloor} /;\; n \in \mathbb{N}^{+}$$

01.01.16.0087.01

$$\frac{\sqrt{z_1}}{\sqrt{z_2}} == \sqrt{\frac{z_1}{z_2}} /;\; z_2 - z_1 \geq 0$$

01.01.16.0088.01

$$\frac{\sqrt{z_1}}{\sqrt{z_2}} == \sqrt{\frac{z_1}{z_2}} /;\; \arg(z_1) \leq 0 \bigwedge \arg(z_2) < \arg(z_1) + \pi \bigvee \arg(z_1) > 0 \bigwedge \arg(z_2) \geq \arg(z_1) - \pi$$

01.01.16.0089.01

$$\frac{\sqrt{z_1}}{\sqrt{z_2}} == -\sqrt{\frac{z_1}{z_2}} /;\; \arg(z_1) \geq 0 \bigwedge \arg(z_2) < \arg(z_1) - \pi \bigvee \arg(z_1) \leq 0 \bigwedge \arg(z_2) \geq \arg(z_1) + \pi$$

01.01.16.0090.01

$$\frac{\sqrt{z_1}}{\sqrt{z_2}} == \sqrt{\frac{z_1}{z_2}}\; e^{-i\pi \left\lfloor \frac{\pi - \arg(z_1) + \arg(z_2)}{2\pi} \right\rfloor}$$

Powers of the direct function

01.01.16.0032.01

$$\sqrt{z}^{\,2} == z$$

01.01.16.0033.01

$$\sqrt{z}^{\,a} == z^{a/2}$$

Power of the product of the direct function

01.01.16.0091.01

$$\left(\sqrt{x}\,\sqrt{y}\right)^a = x^{a/2}\, y^{a/2}$$

01.01.16.0092.01

$$\left(\sqrt{x}\,\sqrt{y}\,\sqrt{z}\right)^a = e^{2 i a \pi \left\lfloor \frac{2\pi - \arg(x) - \arg(y) - \arg(z)}{4\pi} \right\rfloor} x^{a/2}\, y^{a/2}\, z^{a/2}$$

01.01.16.0093.01

$$\left(\prod_{k=1}^{n}\sqrt{z_k}\right)^a = e^{2 i a \pi \left\lfloor \frac{2\pi - \sum_{k=1}^{n}\arg(z_k)}{4\pi} \right\rfloor} \prod_{k=1}^{n} z_k^{a/2}$$

Nested direct and inverse functions

01.01.16.0034.01

$$\sqrt{-1-z}\,\sqrt{\frac{1}{1+z}}\,\sqrt{\frac{1}{z}}\,\sqrt{-z} == -1$$

01.01.16.0035.01

$$\sqrt{\frac{z-1}{z}}\,\sqrt{\frac{z}{1-z}} == \sqrt{-\frac{1}{z}}\,\sqrt{z}$$

01.01.16.0036.01

$$\frac{\sqrt{\sqrt{z^2+1}-1}}{\sqrt{1-\sqrt{z^2+1}}} = \frac{\sqrt{z^2}}{\sqrt{-z^2}}$$

01.01.16.0037.01

$$\frac{\sqrt{\sqrt{1-z^2}-1}}{\sqrt{1-\sqrt{1-z^2}}} = \frac{\sqrt{-z^2}}{\sqrt{z^2}}$$

01.01.16.0038.01

$$\frac{\sqrt{\sqrt{z^2+1}-z}}{\sqrt{z-\sqrt{z^2+1}}} = -\sqrt{z}\sqrt{-\frac{1}{z}}$$

01.01.16.0039.01

$$\frac{\sqrt{z+\sqrt{z^2+1}}}{\sqrt{-z-\sqrt{z^2+1}}} = -\sqrt{-z}\sqrt{\frac{1}{z}}$$

01.01.16.0040.01

$$\frac{\sqrt{\sqrt{1-z^2}-i z}}{\sqrt{i z-\sqrt{1-z^2}}} = -\sqrt{i z}\sqrt{\frac{i}{z}}$$

01.01.16.0041.01

$$\frac{\sqrt{i z+\sqrt{1-z^2}}}{\sqrt{-i z-\sqrt{1-z^2}}} = -\sqrt{-i z}\sqrt{-\frac{i}{z}}$$

01.01.16.0042.01

$$\sqrt{\frac{z^2}{1-\sqrt{1-z^2}}}\sqrt{\frac{1-\sqrt{1-z^2}}{z^2}} = 1$$

01.01.16.0043.01

$$\sqrt{\frac{z}{\sqrt{z^2+1}-z}}\sqrt{\frac{\sqrt{z^2+1}-z}{z}} = \sqrt{\frac{1}{z^2+1}}\sqrt{z^2+1}+\sqrt{\frac{1}{z}}\sqrt{z}-1$$

01.01.16.0094.01

$$\sqrt{\sqrt{z}-1}\,\sqrt{\sqrt{z}+1}=\sqrt{z-1}$$

Brychkov Yu.A. (2006)

01.01.16.0095.01

$$\sqrt{\sqrt{z+1}-1}\,\sqrt{\sqrt{z+1}+1}=\sqrt{z}$$

Brychkov Yu.A. (2006)

01.01.16.0096.01

$$\frac{\sqrt{2}\,\sqrt{1-\sqrt{1-z}}\,\sqrt{\sqrt{1-z}+i\sqrt{z}}}{i\left(1-\sqrt{1-z}\right)+\sqrt{z}}=1$$

01.01.16.0097.01

$$\frac{\sqrt{1-z}\left(1+z-i\sqrt{1-z^2}\right)}{\sqrt{z+1}\left(1-z+i\sqrt{1-z^2}\right)}=-i$$

01.01.16.0098.01

$$\frac{\sqrt{1-z}\left(1+z+i\sqrt{1-z^2}\right)}{\sqrt{z+1}\left(1-z-i\sqrt{1-z^2}\right)}=i$$

01.01.16.0099.01

$$\frac{1}{\sqrt{1-\sqrt{z}}}+\frac{1}{\sqrt{1+\sqrt{z}}}=\frac{\sqrt{2}\,\sqrt{\sqrt{1-z}+1}}{\sqrt{1-z}}$$

01.01.16.0100.01

$$\frac{1}{\sqrt{1-\sqrt{z}}}-\frac{1}{\sqrt{1+\sqrt{z}}}=\frac{\sqrt{2}\,\sqrt{1-\sqrt{1-z}}}{\sqrt{1-z}}$$

01.01.16.0101.01

$$\sqrt{1+\sqrt{z}}+\sqrt{1-\sqrt{z}}=\sqrt{2}\,\sqrt{1+\sqrt{1-z}}$$

01.01.16.0102.01

$$\sqrt{1+\sqrt{z}}-\sqrt{1-\sqrt{z}}=\sqrt{2}\,\sqrt{1-\sqrt{1-z}}$$

Identities

Functional identities

01.01.17.0001.01

$$\sqrt{z}=\sqrt{z-1}\,\sqrt{\frac{z}{z-1}}\;/;\;z\notin(0,1)$$

01.01.17.0002.01

$$w(x_1\, x_2) == w(x_1)\, w(x_2) \,/;\, w(x) == \sqrt{x} \bigwedge x_1 > 0 \bigwedge x_2 > 0 \bigvee x_1\, x_2 < 0$$

Complex characteristics

Real part

01.01.19.0001.01

$$\operatorname{Re}\left(\sqrt{x + i\, y}\right) == \sqrt[4]{x^2 + y^2}\, \cos\left(\frac{1}{2} \tan^{-1}(x,\, y)\right)$$

Imaginary part

01.01.19.0002.01

$$\operatorname{Im}\left(\sqrt{x + i\, y}\right) == \sqrt[4]{x^2 + y^2}\, \sin\left(\frac{1}{2} \tan^{-1}(x,\, y)\right)$$

Absolute value

01.01.19.0003.01

$$\left|\sqrt{x + i\, y}\right| == \sqrt[4]{x^2 + y^2}$$

Argument

01.01.19.0004.01

$$\arg\left(\sqrt{x + i\, y}\right) == \frac{1}{2} \tan^{-1}(x,\, y)$$

Conjugate value

01.01.19.0005.01

$$\overline{\sqrt{x + i\, y}} == \sqrt[4]{x^2 + y^2}\, \cos\left(\frac{1}{2} \tan^{-1}(x,\, y)\right) - i \sqrt[4]{x^2 + y^2}\, \sin\left(\frac{1}{2} \tan^{-1}(x,\, y)\right)$$

Signum value

01.01.19.0006.01

$$\operatorname{sgn}(x^a) == x^{i \operatorname{Im}(a)} \,/;\, x > 0$$

01.01.19.0007.01

$$\operatorname{sgn}(z^a) == \sqrt{z}\, e^{-\frac{1}{2} \operatorname{Re}(\log(z))}$$

01.01.19.0008.01

$$\operatorname{sgn}\left(\sqrt{z}\right) == e^{\frac{i}{2} \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))}$$

01.01.19.0009.01

$$\operatorname{sgn}\left(\sqrt{z}\right) == e^{\frac{i}{2} \arg(z)}$$

Differentiation

Low-order differentiation

01.01.20.0001.01

$$\frac{\partial \sqrt{z}}{\partial z} = \frac{1}{2\sqrt{z}}$$

01.01.20.0002.01

$$\frac{\partial^2 \sqrt{z}}{\partial z^2} = -\frac{1}{4 z^{3/2}}$$

Symbolic differentiation

01.01.20.0003.02

$$\frac{\partial^n \sqrt{z}}{\partial z^n} = \left(\frac{3}{2} - n\right)_n z^{\frac{1}{2}-n} \;/;\; n \in \mathbb{N}$$

01.01.20.0006.01

$$\frac{\partial^n \sqrt{b + c z}}{\partial z^n} = c^n \left(\frac{3}{2} - n\right)_n (b + c z)^{\frac{1}{2}-n} \;/;\; n \in \mathbb{N}$$

01.01.20.0007.01

$$\frac{\partial^n \sqrt{c z^2 + b}}{\partial z^n} = \sum_{k=0}^{n} \frac{(2k - n + 1)_{2(n-k)} \left(\frac{3}{2} - k\right)_k c^k \left(c z^2 + b\right)^{\frac{1}{2}-k}}{(n-k)!\, (2z)^{n-2k}} \;/;\; n \in \mathbb{N}$$

01.01.20.0008.01

$$\frac{\partial^n \sqrt{c z^2 + b}}{\partial z^n} = 2^n (c z)^n \left(c z^2 + b\right)^{\frac{1}{2}-n} \left(\frac{3}{2} - n\right)_n {}_2F_1\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{3}{2} - n; \frac{b}{c z^2} + 1\right) \;/;\; n \in \mathbb{N}$$

01.01.20.0009.01

$$\frac{\partial^n \sqrt{b + c\sqrt{z}}}{\partial z^n} = \sum_{k=0}^{n} \frac{(-1)^{n-k} (k)_{2(n-k)} \left(\frac{3}{2} - k\right)_k c^k \left(b + c\sqrt{z}\right)^{\frac{1}{2}-k}}{(n-k)!\, \left(2\sqrt{z}\right)^{2n-k}} \;/;\; n \in \mathbb{N}$$

01.01.20.0010.01

$$\frac{\partial^n \sqrt{c z^\nu + b}}{\partial z^n} = \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{(-1)^j (-n - \nu j + k \nu + 1)_n \left(\frac{3}{2} - k\right)_k c^k (c z^\nu + b)^{\frac{1}{2}-k}}{j!\, (k-j)!\, z^{n-\nu k}} \;/;\; n \in \mathbb{N}$$

01.01.20.0011.01

$$\frac{\partial^n g(z)^a}{\partial z^n} = a \binom{n-a}{n} \sum_{k=0}^{n} \frac{(-1)^k}{a-k} \binom{n}{k} g(z)^{a-k} \frac{\partial^n g(z)^k}{\partial z^n} \;/;\; n \in \mathbb{N}$$

01.01.20.0004.02

$$\frac{\partial^n f\left(\sqrt{z}\right)}{\partial z^n} = \sum_{k=0}^{n} \frac{(-1)^k (n-k)_{2k}}{k!\, \left(2\sqrt{z}\right)^{k+n}} f^{(n-k)}\left(\sqrt{z}\right) \;/;\; n \in \mathbb{N}$$

Fractional integro-differentiation

01.01.20.0005.01

$$\frac{\partial^{\alpha}\sqrt{z}}{\partial z^{\alpha}} = \frac{\sqrt{\pi}}{2\,\Gamma\left(\frac{3}{2}-\alpha\right)}\,z^{\frac{1}{2}-\alpha}$$

Integration

Indefinite integration

For the direct function itself

01.01.21.0001.01

$$\int \sqrt{z}\,dz = \frac{2\,z^{3/2}}{3}$$

Definite integration

For the direct function itself

01.01.21.0002.01

$$\int_0^1 \sqrt{t}\,dt = \frac{2}{3}$$

Involving the direct function

01.01.21.0003.01

$$\int_0^1 \sqrt{t}\,(t+1)^b\,dt = \frac{2}{3}\,{}_2F_1\left(\frac{3}{2},\,-b;\,\frac{5}{2};\,-1\right)$$

Involving related functions

01.01.21.0004.01

$$\int_0^1 \sqrt{t}\,\log(t)\,dt = -\frac{4}{9}$$

Integral transforms

Laplace transforms

01.01.22.0001.01

$$\mathcal{L}_t\left[\sqrt{t}\right](z) = \frac{\sqrt{\pi}}{2\,z^{3/2}}\;/;\;\mathrm{Re}(z) > 0$$

01.01.22.0002.01

$$\mathcal{L}_t^{-1}\left[\sqrt{t}\right](p) = -\frac{1}{2\,p^{3/2}\,\sqrt{\pi}}$$

Operations

Limit operation

01.01.25.0001.01

$$\lim_{z\to\infty}\sqrt{z}\,b^{-z}=0\;/;\;b>1$$

01.01.25.0002.01

$$\lim_{z\to 0}\sqrt{z}\,\log(z)=0$$

01.01.25.0003.01

$$\lim_{z\to\infty}\frac{\log(z)}{\sqrt{z}}=0$$

Representations through more general functions

Through hypergeometric functions

Involving $_p\tilde{F}_q$

01.01.26.0001.01

$$\sqrt{z}={}_1\tilde{F}_0\left(-\frac{1}{2};;1-z\right)$$

Involving $_pF_q$

01.01.26.0002.01

$$\sqrt{z}={}_1F_0\left(-\frac{1}{2};;1-z\right)$$

01.01.26.0003.01

$$\sqrt{z}={}_0F_0\left(;;\frac{1}{2}\log(z)\right)$$

Involving $_2F_1$

01.01.26.0004.01

$$\sqrt{z}={}_2F_1\left(-\frac{1}{2},b;b;1-z\right)$$

Through Meijer G

Classical cases for the direct function itself

01.01.26.0005.01

$$\sqrt{1+z}=-\frac{1}{2\sqrt{\pi}}\,G_{1,1}^{1,1}\left(z\,\middle|\,\begin{matrix}\frac{3}{2}\\0\end{matrix}\right)$$

See power function.

Classical cases involving unit step θ

01.01.26.0006.01

$$\theta(1-|z|)\sqrt{1-z}=\frac{1}{2}\sqrt{\pi}\,G_{1,1}^{1,0}\left(z\,\middle|\,\begin{matrix}\frac{3}{2}\\0\end{matrix}\right)$$

01.01.26.0007.01

$$\theta(|z|-1)\sqrt{z-1} = \frac{1}{2}\sqrt{\pi}\, G_{1,1}^{0,1}\left(z \middle| \begin{matrix} \frac{3}{2} \\ 0 \end{matrix}\right)$$

See power function.

Through other functions

01.01.26.0008.01

$$\sqrt{z} = \left(x;\, x^2 - z\right)_2^{-1}$$

Representations through equivalent functions

With inverse function

01.01.27.0001.01

$$\sqrt{z}^{\,2} = z$$

The left side of above formula corresponds to composition $f^{(-1)}(f(z))$ /; $f(z) = \sqrt{z}$, which generically equal to z.

01.01.27.0008.01

$$\sqrt{z^2} = e^{i\pi\left\lfloor \frac{\pi - 2\arg(z)}{2\pi} \right\rfloor} z$$

The left side of above formula corresponds to composition $f\left(f^{(-1)}(z)\right)$ /; $f(z) = \sqrt{z}$, which generically does not equal to z.

01.01.27.0002.01

$$\sqrt{z^2} = z \;/;\; -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}$$

The left side of above formula corresponds to composition $f\left(f^{(-1)}(z)\right)$ /; $f(z) = \sqrt{z}$, which equal to z under restriction $-\frac{\pi}{2} < \operatorname{Arg}(z) \leq \frac{\pi}{2}$.

01.01.27.0003.01

$$\sqrt{z^2} = -z \;/;\; \frac{\pi}{2} < \arg(z) \leq \pi \bigvee -\pi < \arg(z) \leq -\frac{\pi}{2}$$

The left side of above formula corresponds to composition $f\left(f^{(-1)}(z)\right)$ /; $f(z) = \sqrt{z}$, which equal to $-z$ under restriction $\frac{\pi}{2} < \operatorname{Arg}(z) \leq \pi \bigvee -\pi < \operatorname{Arg}(z) \leq -\frac{\pi}{2}$.

01.01.27.0004.01

$$\sqrt{z^2} = \sqrt{-i z}\,\sqrt{i z}$$

01.01.27.0005.01

$$\sqrt{-z^2} = \sqrt{z}\,\sqrt{-z}$$

With related functions

01.01.27.0006.01

$$\sqrt{z} = z^a /; a = \frac{1}{2}$$

01.01.27.0007.01

$$\sqrt{z} = (x; x^2 - z)_2^{-1}$$

Inequalities

01.01.29.0001.01

$$\sqrt{x} < x /; x > 1$$

Zeros

01.01.30.0001.01

$$\sqrt{z} = 0 /; z = 0$$

Theorems

Wigner's semi-circle law

The probability density $p(x)$ for the eigenvalues of a unitary $n \times n$ Gaussian ensemble matrix is $p(x) = \theta(2n - x^2) \frac{1}{\pi} \sqrt{2n - x^2}$.

The maximal power efficiency output of an endo-reversible Carnot heat engine operating in finite times with a linear Newtonian heat transfer

The maximal power efficiency η output of an endo-reversible Carnot heat engine operating in finite times with a linear Newtonian heat transfer is given by $\eta = 1 - \sqrt{T_c / T_h}$, where T_c and T_h are the cold and hot temperature of the heat reservoirs.

Differential-Algebraic Constants

To ensure the correctness of many formulas given in this collection over the whole complex plane, it is often necessary to work with expressions of the form $\sqrt{z^2} = \sqrt{i z} \sqrt{-i z}$, $\sqrt{z^2} / z$, $z \sqrt{1/z^2}$, $\sqrt{-z^2} = \sqrt{z} \sqrt{-z}$, etc.. While in a textbook-mathematics setting these expressions are often simplified to z, ± 1, 1, $\pm i$ z, etc, this cannot be done inside *Mathematica*. From a complex function point of view the Riemann surface of such functions are made from disconnected sheets. Inside *Mathematica* all branch cuts of all functions (that have branch cuts) follow uniquely from the branch cut of the power function (the logarithm function respectively). As a result the branch cuts related to functions such as $\sqrt{z^2}$, $\sqrt{z^2} / z$, $z \sqrt{1/z^2}$, $\sqrt{-z^2}$ exist, although they do not start and end at branch points, but mostly extend from $-i\infty$ to $i\infty$ or from $-\infty$ to ∞.
In details we have:

$$\mathcal{BC}_z\left(\sqrt{z^2}\right) = \mathcal{BC}_z\left(\sqrt{i z} \sqrt{-i z}\right) = \{\{(-i\infty, 0), 1\}, \{(0, i\infty), -1\}\}$$

$$\lim_{\epsilon \to +0} \sqrt{(x+\epsilon)^2} = -\sqrt{x^2} \;/;\; i\,x > 0$$

$$\lim_{\epsilon \to +0} \sqrt{(x-\epsilon)^2} = -\sqrt{x^2} \;/;\; i\,x < 0$$

$$\mathcal{BC}_z\left(\sqrt{-z^2}\right) = \mathcal{BC}_z\left(\sqrt{z}\,\sqrt{-z}\right) = \{\{(-\infty, 0), -i\}, \{(0, \infty), i\}\}$$

$$\lim_{\epsilon \to +0} \sqrt{-(x - i\,\epsilon)^2} = -\sqrt{-x^2} \;/;\; x < 0$$

$$\lim_{\epsilon \to +0} \sqrt{-(x + i\,\epsilon)^2} = -\sqrt{-x^2} \;/;\; x > 0$$

$$\mathcal{BC}_z\left(z\sqrt{1/z^2}\right) = \{\{(-i\,\infty, 0), -1\}, \{(0, i\,\infty), 1\}\}$$

$$\mathcal{BC}_z\left(\sqrt{z}\,\sqrt{1/z}\right) = \{\{(-\infty, 0), \{\}\}\}$$

An expression of the form $\sqrt{z^2}/z$, $z\sqrt{1/z^2}$ are called differential-algebraic constants because their derivative vanishes generically everywhere as a complex function (but not as a generalized function).

History

– Babylonians (1900 BC) gave the first approximate evaluations of $\sqrt{2}$ and $\sqrt{\ }$ from other natural numbers
– H. Briggs (1617)
– I. Newton (1665) derived an infinite series expression for $1+z$; the notation $\sqrt{z}$ was introduced in 1675

The function sqrt is encountered often in mathematics and the natural sciences.

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