StruveH

Notations

Traditional name

Struve function **H**

Traditional notation

$\boldsymbol{H}_\nu(z)$

Mathematica StandardForm notation

StruveH[ν, z]

Primary definition

03.09.02.0001.01

$$\boldsymbol{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2k}$$

Specific values

Specialized values

For fixed ν

03.09.03.0001.01

$$\boldsymbol{H}_\nu(0) = 0 \;/;\; \mathrm{Re}(\nu) > -1$$

03.09.03.0002.01

$$\boldsymbol{H}_\nu(0) = \tilde{\infty} \;/;\; \mathrm{Re}(\nu) < -1$$

03.09.03.0003.01

$$\boldsymbol{H}_\nu(0) = ¿ \;/;\; \mathrm{Re}(\nu) = -1$$

For fixed z

Explicit rational ν

03.09.03.0008.01

$$\boldsymbol{H}_{-\frac{11}{2}}(z) = \frac{\sqrt{\frac{2}{\pi}}\left(z\left(z^4-105z^2+945\right)\cos(z)-15\left(z^4-28z^2+63\right)\sin(z)\right)}{z^{11/2}}$$

03.09.03.0009.01

$$\mathbf{H}_{-\frac{9}{2}}(z) == \frac{\sqrt{\frac{2}{\pi}} \left(5 z \left(2 z^2 - 21\right) \cos(z) + \left(z^4 - 45 z^2 + 105\right) \sin(z)\right)}{z^{9/2}}$$

03.09.03.0010.01

$$\mathbf{H}_{-\frac{7}{2}}(z) == -\frac{\sqrt{\frac{2}{\pi}} \left(z \left(z^2 - 15\right) \cos(z) + 3 \left(5 - 2 z^2\right) \sin(z)\right)}{z^{7/2}}$$

03.09.03.0011.01

$$\mathbf{H}_{-\frac{5}{2}}(z) == -\frac{\sqrt{\frac{2}{\pi}} \left(3 z \cos(z) + \left(z^2 - 3\right) \sin(z)\right)}{z^{5/2}}$$

03.09.03.0012.01

$$\mathbf{H}_{-\frac{3}{2}}(z) == \frac{\sqrt{\frac{2}{\pi}} (z \cos(z) - \sin(z))}{z^{3/2}}$$

03.09.03.0005.01

$$\mathbf{H}_{-\frac{1}{2}}(z) == \sqrt{\frac{2}{\pi z}} \sin(z)$$

03.09.03.0004.01

$$\mathbf{H}_{\frac{1}{2}}(z) == \sqrt{\frac{2}{\pi z}} (1 - \cos(z))$$

03.09.03.0013.01

$$\mathbf{H}_{\frac{3}{2}}(z) == \frac{z^2 - 2 \sin(z) z - 2 \cos(z) + 2}{\sqrt{2 \pi} z^{3/2}}$$

03.09.03.0014.01

$$\mathbf{H}_{\frac{5}{2}}(z) == \frac{z^4 + 4 z^2 - 24 \sin(z) z + 8 \left(z^2 - 3\right) \cos(z) + 24}{4 \sqrt{2 \pi} z^{5/2}}$$

03.09.03.0015.01

$$\mathbf{H}_{\frac{7}{2}}(z) == \frac{z^6 + 6 z^4 + 72 z^2 + 48 \left(z^2 - 15\right) \sin(z) z + 144 \left(2 z^2 - 5\right) \cos(z) + 720}{24 \sqrt{2 \pi} z^{7/2}}$$

03.09.03.0016.01

$$\mathbf{H}_{\frac{9}{2}}(z) == \frac{z^8 + 8 z^6 + 144 z^4 + 2880 z^2 + 1920 \left(2 z^2 - 21\right) \sin(z) z - 384 \left(z^4 - 45 z^2 + 105\right) \cos(z) + 40\,320}{192 \sqrt{2 \pi} z^{9/2}}$$

03.09.03.0017.01

$$\mathbf{H}_{\frac{11}{2}}(z) == \frac{1}{1920 \sqrt{2 \pi} z^{11/2}} \left(z^{10} + 10 z^8 + 240 z^6 + 7200 z^4 + 201\,600 z^2 - 3840 \left(z^4 - 105 z^2 + 945\right) \sin(z) z - 57\,600 \left(z^4 - 28 z^2 + 63\right) \cos(z) + 3\,628\,800\right)$$

Symbolic rational ν

03.09.03.0006.01

$$\boldsymbol{H}_\nu(z) = \frac{(-1)^{\nu+\frac{1}{2}}}{\sqrt{z}} \sqrt{\frac{2}{\pi}} \left(\sin\left(\frac{1}{2}\pi\left(\nu+\frac{1}{2}\right)+z\right) \sum_{j=0}^{\left\lfloor -\frac{1}{4}(2\nu+1)\right\rfloor} \frac{(-1)^j \left(2j-\nu-\frac{1}{2}\right)!}{(2j)! \left(-2j-\nu-\frac{1}{2}\right)! (2z)^{2j}} + \cos\left(\frac{1}{2}\pi\left(\nu+\frac{1}{2}\right)+z\right) \sum_{j=0}^{\left\lfloor -\frac{1}{4}(2\nu+3)\right\rfloor} \frac{(-1)^j \left(2j-\nu+\frac{1}{2}\right)! (2z)^{-2j-1}}{(2j+1)! \left(-2j-\nu-\frac{3}{2}\right)!} \right) /; -\frac{1}{2}-\nu \in \mathbb{N}$$

03.09.03.0007.01

$$\boldsymbol{H}_\nu(z) = \frac{1}{\left(\nu-\frac{1}{2}\right)! \sqrt{\pi}} \left(\frac{z}{2}\right)^{\nu-1} \sum_{k=0}^{\nu-\frac{1}{2}} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(-\frac{z^2}{4}\right)^{-k} + \frac{\sqrt{\frac{2}{\pi}} (-1)^{\nu+\frac{1}{2}}}{\sqrt{z}} \left(\sin\left(\frac{1}{2}\pi\left(\nu+\frac{1}{2}\right)+z\right) \sum_{k=0}^{\left\lfloor \frac{1}{4}(2\nu-1)\right\rfloor} \frac{(-1)^k \left(2k+\nu-\frac{1}{2}\right)!}{(2k)! \left(-2k+\nu-\frac{1}{2}\right)! (2z)^{2k}} + \cos\left(\frac{1}{2}\pi\left(\nu+\frac{1}{2}\right)+z\right) \sum_{k=0}^{\left\lfloor \frac{1}{4}(2\nu-3)\right\rfloor} \frac{(-1)^k \left(2k+\nu+\frac{1}{2}\right)! (2z)^{-2k-1}}{(2k+1)! \left(-2k+\nu-\frac{3}{2}\right)!} \right) /; \nu-\frac{1}{2} \in \mathbb{Z}$$

Values at fixed points

03.09.03.0018.01

$$\boldsymbol{H}_{-1}(0) = \frac{2}{\pi}$$

General characteristics

Domain and analyticity

$\boldsymbol{H}_\nu(z)$ is an analytical function of ν and z which is defined over $\mathbb{C}^2$.

03.09.04.0001.01

$$(\nu * z) \longrightarrow \boldsymbol{H}_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

03.09.04.0002.01

$$\boldsymbol{H}_\nu(-z) = -(-z)^\nu z^{-\nu} \boldsymbol{H}_\nu(z)$$

Mirror symmetry

03.09.04.0003.02

$$\boldsymbol{H}_{\overline{\nu}}(\overline{z}) = \overline{\boldsymbol{H}_\nu(z)} /; z \notin (-\infty, 0)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν, the function $\boldsymbol{H}_\nu(z)$ has an essential singularity at $z == \tilde{\infty}$. At the same time, the point $z == \tilde{\infty}$ is a branch point for generic ν.

03.09.04.0004.01

$$\mathcal{S}ing_z(\boldsymbol{H}_\nu(z)) == \{\{\tilde{\infty}.\infty\}\}$$

With respect to ν

For fixed z, the function $\boldsymbol{H}_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

03.09.04.0005.01

$$\mathcal{S}ing_\nu(\boldsymbol{H}_\nu(z)) == \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to z

For fixed noninteger ν, the function $\boldsymbol{H}_\nu(z)$ has two branch points: $z == 0$, $z == \tilde{\infty}$. At the same time, the point $z == \tilde{\infty}$ is an essential singularity.

For integer ν, the function $\boldsymbol{H}_\nu(z)$ does not have branch points.

03.09.04.0006.01

$$\mathcal{BP}_z(\boldsymbol{H}_\nu(z)) == \{0, \tilde{\infty}\} /; \nu \notin \mathbb{Z}$$

03.09.04.0007.01

$$\mathcal{BP}_z(\boldsymbol{H}_\nu(z)) == \{\} /; \nu \in \mathbb{Z}$$

03.09.04.0008.01

$$\mathcal{R}_z(\boldsymbol{H}_\nu(z), 0) == \log /; \nu \notin \mathbb{Q}$$

03.09.04.0009.01

$$\mathcal{R}_z\left(\boldsymbol{H}_{\frac{p}{q}}(z), 0\right) == q /; p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1$$

03.09.04.0010.01

$$\mathcal{R}_z(\boldsymbol{H}_\nu(z), \tilde{\infty}) == \log /; \nu \notin \mathbb{Q}$$

03.09.04.0011.01

$$\mathcal{R}_z\left(\boldsymbol{H}_{\frac{p}{q}}(z), \tilde{\infty}\right) == q /; p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1$$

With respect to ν

For fixed z, the function $\boldsymbol{H}_\nu(z)$ does not have branch points.

03.09.04.0012.01

$$\mathcal{BP}_\nu(\boldsymbol{H}_\nu(z)) == \{\}$$

Branch cuts

With respect to z

When ν is an integer, $\mathbf{H}_\nu(z)$ is an entire function of z. For fixed noninteger ν, it has one infinitely long branch cut. For fixed noninteger ν, the function $\mathbf{H}_\nu(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

03.09.04.0013.01

$$\mathcal{BC}_z(\mathbf{H}_\nu(z)) == \{\{(-\infty, 0), -i\}\} /; \nu \notin \mathbb{Z}$$

03.09.04.0014.01

$$\mathcal{BC}_z(\mathbf{H}_\nu(z)) == \{\} /; \nu \in \mathbb{Z}$$

03.09.04.0015.01

$$\lim_{\epsilon \to +0} \mathbf{H}_\nu(x + i\,\epsilon) == \mathbf{H}_\nu(x) /; x < 0$$

03.09.04.0016.01

$$\lim_{\epsilon \to +0} \mathbf{H}_\nu(x - i\,\epsilon) == -e^{-i\pi\nu}\,\mathbf{H}_\nu(-x) /; x < 0$$

With respect to ν

For fixed z, the function $\mathbf{H}_\nu(z)$ is an entire function of ν and does not have branch cuts.

03.09.04.0017.01

$$\mathcal{BC}_\nu(\mathbf{H}_\nu(z)) == \{\}$$

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

03.09.06.0016.01

$$\mathbf{H}_\nu(z) \propto \left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} \left(\mathbf{H}_\nu(z_0) + \left(\mathbf{H}_{\nu-1}(z_0) - \frac{\nu}{z_0}\mathbf{H}_\nu(z_0)\right)(z-z_0) + \frac{1}{2z_0^2}\left(\left(\frac{2^{1-\nu}z_0^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - \mathbf{H}_{\nu-1}(z_0)\right)z_0 + \mathbf{H}_\nu(z_0)\left(\nu^2+\nu-z_0^2\right)\right)(z-z_0)^2 + \ldots\right) /; (z \to z_0)$$

03.09.06.0017.01

$$\mathbf{H}_\nu(z) \propto \left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} \left(\mathbf{H}_\nu(z_0) + \left(\mathbf{H}_{\nu-1}(z_0) - \frac{\nu}{z_0}\mathbf{H}_\nu(z_0)\right)(z-z_0) + \frac{1}{2z_0^2}\left(\left(\frac{2^{1-\nu}z_0^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - \mathbf{H}_{\nu-1}(z_0)\right)z_0 + \mathbf{H}_\nu(z_0)\left(\nu^2+\nu-z_0^2\right)\right)(z-z_0)^2 + O\left((z-z_0)^3\right)\right)$$

03.09.06.0018.01

$$\mathbf{H}_\nu(z) = \sqrt{\pi}\,\Gamma(\nu+2)\left(\frac{z_0}{4}\right)^{\nu+1}\left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} \sum_{k=0}^{\infty}\frac{2^k z_0^{-k}}{k!}\,{}_3\tilde{F}_4\left(1, \frac{\nu}{2}+1, \frac{\nu+3}{2}; \frac{3}{2}, \frac{\nu-k}{2}+1, \frac{1}{2}(-k+\nu+3), \nu+\frac{3}{2}; -\frac{z_0^2}{4}\right)(z-z_0)^k$$

03.09.06.0019.01

$$\mathbf{H}_\nu(z) = \left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} \sum_{k=0}^{\infty}\left(\frac{z_0^{-k}}{k!}\sum_{m=0}^{k}(-1)^{k+m}\binom{k}{m}(-\nu)_{k-m}\sum_{p=0}^{m}\frac{(-1)^{p-1}\,2^{2p-m}\,(-m)_{2(m-p)}\,(\nu)_p}{(m-p)!}\left(\frac{1}{2}z_0\sum_{j=0}^{p-1}\frac{(-j+p-1)!}{j!\,(-2j+p-1)!\,(-p-\nu+1)_j\,(\nu)_{j+1}}\left(\frac{z_0^2}{4}\right)^j \mathbf{H}_{\nu-1}(z_0) - \sum_{j=0}^{p}\frac{(p-j)!}{j!\,(p-2j)!\,(-p-\nu+1)_j\,(\nu)_j}\left(\frac{z_0^2}{4}\right)^j \mathbf{H}_\nu(z_0)\right) + \frac{2^{-\nu} z_0^{-k+\nu+1}}{\sqrt{\pi}\,k!\,\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{i=1}^{k-1}\sum_{m=0}^{i}(-1)^{i+m}\binom{i}{m}(-\nu)_{i-m}\sum_{p=0}^{m}\frac{(-1)^{p-1}\,2^{2p-m}\,(-m)_{2(m-p)}\,(\nu)_p}{(m-p)!}\sum_{j=0}^{p-1}\frac{2^{-2j}\,(-j+p-1)!\,(2j-k+\nu+2)_{-i+k-1}\,z_0^{2j}}{j!\,(-2j+p-1)!\,(-p-\nu+1)_j\,(\nu)_{j+1}}\right)(z-z_0)^k$$

03.09.06.0020.01

$$\mathbf{H}_\nu(z) \propto \left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor}\mathbf{H}_\nu(z_0)\,(1+O(z-z_0))$$

Expansions on branch cuts

For the function itself

03.09.06.0021.01

$$\mathbf{H}_\nu(z) \propto e^{2\nu\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left(\mathbf{H}_\nu(x) + \left(\mathbf{H}_{\nu-1}(x) - \frac{\nu}{x}\mathbf{H}_\nu(x)\right)(z-x) + \frac{1}{2x^2}\left(\left(\frac{2^{1-\nu}x^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - \mathbf{H}_{\nu-1}(x)\right)x + \mathbf{H}_\nu(x)\left(\nu^2+\nu-x^2\right)\right)(z-x)^2 + \ldots\right) /; (z \to x) \land x \in \mathbb{R} \land x < 0$$

03.09.06.0022.01

$$\mathbf{H}_\nu(z) \propto e^{2\nu\pi i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor}\left(\mathbf{H}_\nu(x) + \left(\mathbf{H}_{\nu-1}(x) - \frac{\nu}{x}\mathbf{H}_\nu(x)\right)(z-x) + \frac{1}{2x^2}\left(\left(\frac{2^{1-\nu}x^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - \mathbf{H}_{\nu-1}(x)\right)x + \mathbf{H}_\nu(x)\left(\nu^2+\nu-x^2\right)\right)(z-x)^2 + O\left((z-x)^3\right)\right) /; x \in \mathbb{R} \land x < 0$$

03.09.06.0023.01

$$\mathbf{H}_{\nu}(z) = \sqrt{\pi}\, \Gamma(\nu+2) \left(\frac{x}{4}\right)^{\nu+1} e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \frac{2^k x^{-k}}{k!}\, {}_3\tilde{F}_4\left(1, \frac{\nu}{2}+1, \frac{\nu+3}{2}; \frac{3}{2}, \frac{\nu-k}{2}+1, \frac{1}{2}(-k+\nu+3), \nu+\frac{3}{2}; -\frac{x^2}{4}\right) (z-x)^k \;/;\; x \in \mathbb{R} \land x < 0$$

03.09.06.0024.01

$$\mathbf{H}_{\nu}(z) = e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \sum_{k=0}^{\infty} \left(\frac{x^{-k}}{k!} \sum_{m=0}^{k} (-1)^{k+m} \binom{k}{m} (-\nu)_{k-m} \sum_{p=0}^{m} \frac{(-1)^{p-1} 2^{2p-m} (-m)_{2(m-p)} (\nu)_p}{(m-p)!} \left(\frac{x}{2} \sum_{j=0}^{p-1} \frac{(-j+p-1)!}{j!\,(-2j+p-1)!\,(-p-\nu+1)_j\,(\nu)_{j+1}} \left(\frac{x^2}{4}\right)^j \mathbf{H}_{\nu-1}(x) - \sum_{j=0}^{p} \frac{(p-j)!}{j!\,(p-2j)!\,(-p-\nu+1)_j\,(\nu)_j} \left(\frac{x^2}{4}\right)^j \mathbf{H}_{\nu}(x) \right) + \frac{2^{-\nu} x^{-k+\nu+1}}{\sqrt{\pi}\, k!\, \Gamma\left(\nu+\frac{1}{2}\right)} \sum_{i=1}^{k-1} \sum_{m=0}^{i} (-1)^{i+m} \binom{i}{m} (-\nu)_{i-m} \sum_{p=0}^{m} \frac{(-1)^{p-1} 2^{2p-m} (-m)_{2(m-p)} (\nu)_p}{(m-p)!} \sum_{j=0}^{p-1} \frac{2^{-2j} (-j+p-1)!\,(2j-k+\nu+2)_{-i+k-1}\, x^{2j}}{j!\,(-2j+p-1)!\,(-p-\nu+1)_j\,(\nu)_{j+1}} \right) (z-x)^k \;/;\; x \in \mathbb{R} \land x < 0$$

03.09.06.0025.01

$$\mathbf{H}_{\nu}(z) \propto e^{2\nu\pi i \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} \mathbf{H}_{\nu}(x)\,(1+O(z-x)) \;/;\; x \in \mathbb{R} \land x < 0$$

Expansions at $z = 0$

For the function itself

General case

03.09.06.0001.02

$$\mathbf{H}_{\nu}(z) \propto \frac{2^{-\nu} z^{\nu+1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} \left(1 - \frac{z^2}{3(2\nu+3)} + \frac{z^4}{15(2\nu+3)(2\nu+5)} - \ldots\right) \;/;\; (z \to 0)$$

03.09.06.0026.01

$$\mathbf{H}_{\nu}(z) \propto \frac{2^{-\nu} z^{\nu+1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} \left(1 - \frac{z^2}{3(2\nu+3)} + \frac{z^4}{15(2\nu+3)(2\nu+5)} - O\left(z^6\right)\right)$$

03.09.06.0002.01

$$\mathbf{H}_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma\left(k+\frac{3}{2}\right) \Gamma\left(k+\nu+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2k}$$

03.09.06.0027.01

$$\mathbf{H}_{\nu}(z) = \frac{2}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{4^k \left(\frac{3}{2}\right)_k \left(\nu+\frac{3}{2}\right)_k}$$

03.09.06.0028.01

$$\boldsymbol{H}_\nu(z) = \frac{2}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{\nu+1}{}_1F_2\left(1;\frac{3}{2},\nu+\frac{3}{2};-\frac{z^2}{4}\right)$$

03.09.06.0003.01

$$\boldsymbol{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1}{}_1\tilde{F}_2\left(1;\frac{3}{2},\nu+\frac{3}{2};-\frac{z^2}{4}\right)$$

03.09.06.0004.02

$$\boldsymbol{H}_\nu(z) \propto \frac{2^{-\nu}z^{\nu+1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}+O\left(z^{\nu+3}\right)\,/;\,-\nu-\frac{3}{2}\notin\mathbb{N}$$

03.09.06.0029.01

$$\boldsymbol{H}_\nu(z) = F_\infty(z,\nu)\,/;\left(\left(F_n(z,\nu) = \left(\frac{z}{2}\right)^{\nu+1}\sum_{k=0}^{n}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)} = \boldsymbol{H}_\nu(z)+\frac{(-1)^n}{\Gamma\left(n+\frac{5}{2}\right)\Gamma\left(n+\nu+\frac{5}{2}\right)}\left(\frac{z}{2}\right)^{2n+\nu+3}{}_1F_2\left(1;n+\frac{5}{2},n+\nu+\frac{5}{2};-\frac{z^2}{4}\right)\right)\bigwedge n\in\mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

03.09.06.0030.01

$$\boldsymbol{H}_\nu(z) \propto \frac{(-1)^{-\nu-\frac{1}{2}}}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu}\left(1-\frac{z^2}{4(1-\nu)}+\frac{z^4}{32(1-\nu)(2-\nu)}-\ldots\right)/;\,(z\to 0)\bigwedge -\nu-\frac{3}{2}\in\mathbb{N}^+$$

03.09.06.0031.01

$$\boldsymbol{H}_\nu(z) \propto \frac{(-1)^{-\nu-\frac{1}{2}}}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu}\left(1-\frac{z^2}{4(1-\nu)}+\frac{z^4}{32(1-\nu)(2-\nu)}-O\left(z^6\right)\right)/;\,-\nu-\frac{3}{2}\in\mathbb{N}^+$$

03.09.06.0032.01

$$\boldsymbol{H}_\nu(z) = (-1)^{-\nu-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k-\nu}}{\Gamma(k-\nu+1)\,k!}\,/;\,-\nu-\frac{3}{2}\in\mathbb{N}$$

03.09.06.0033.01

$$\boldsymbol{H}_\nu(z) = \frac{(-1)^{-\nu-\frac{1}{2}}}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu}\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{4^k(1-\nu)_k\,k!}\,/;\,-\nu-\frac{3}{2}\in\mathbb{N}$$

03.09.06.0034.01

$$\boldsymbol{H}_\nu(z) = \frac{(-1)^{-\nu-\frac{1}{2}}}{\Gamma(1-\nu)}\left(\frac{z}{2}\right)^{-\nu}{}_0F_1\left(;1-\nu;-\frac{z^2}{4}\right)/;\,-\nu-\frac{3}{2}\in\mathbb{N}$$

03.09.06.0035.01

$$\boldsymbol{H}_\nu(z) = (-1)^{-\nu-\frac{1}{2}}\left(\frac{z}{2}\right)^{-\nu}{}_0\tilde{F}_1\left(;1-\nu;-\frac{z^2}{4}\right)/;\,-\nu-\frac{3}{2}\in\mathbb{N}$$

03.09.06.0036.01

$$\mathbf{H}_\nu(z) = (-1)^{-\nu-\frac{1}{2}} J_{-\nu}(z) \,/;\, -\nu - \frac{3}{2} \in \mathbb{N}$$

03.09.06.0005.02

$$\mathbf{H}_\nu(z) \propto \frac{(-1)^{-\frac{1}{2}-\nu} 2^\nu z^{-\nu}}{\text{Gamma}[1-\nu]} + O\left(z^{2-\nu}\right) \,/;\, -\nu - \frac{3}{2} \in \mathbb{N}$$

Asymptotic series expansions

Expansions inside Stokes sectors

Expansions containing $z \to \infty$

In exponential form ||| In exponential form

03.09.06.0037.01

$$\mathbf{H}_\nu(z) \propto \frac{1}{\sqrt{2\pi z}} \left(e^{-i z + \frac{1}{4}(2\nu+3)\pi i}\left(1 - \frac{i\left(4\nu^2-1\right)}{8z} - \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \ldots\right) + e^{i z - \frac{1}{4}(2\nu+3)\pi i}\left(1 + \frac{i\left(4\nu^2-1\right)}{8z} - \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \ldots\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1 + \frac{2\nu-1}{z^2} + \frac{3\left(4\nu^2-8\nu+3\right)}{z^4} + \ldots\right) \,/;\, |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

03.09.06.0038.01

$$\mathbf{H}_\nu(z) \propto \frac{1}{\sqrt{2\pi z}} \left(e^{-i z + \frac{2\nu+3}{4}\pi i}\left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(\frac{i}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) + e^{i z - \frac{2\nu+3}{4}\pi i}\left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(-\frac{i}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(-\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) \,/;\, |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

03.09.06.0039.01

$$\mathbf{H}_\nu(z) \propto \frac{1}{\sqrt{2\pi z}} \left(e^{-i z + \frac{2\nu+3}{4}\pi i}\, {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;; \frac{i}{2z}\right) + e^{i z - \frac{2\nu+3}{4}\pi i}\, {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;; -\frac{i}{2z}\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\, {}_3F_0\left(\frac{1}{2}, \frac{1}{2}-\nu, 1;; -\frac{4}{z^2}\right) \,/;\, |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

03.09.06.0040.01

$$\mathbf{H}_\nu(z) \propto \frac{1}{\sqrt{2\pi z}} \left(e^{-i z + \frac{2\nu+3}{4}\pi i}\left(1 + O\left(\frac{1}{z}\right)\right) + e^{i z - \frac{2\nu+3}{4}\pi i}\left(1 + O\left(\frac{1}{z}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1 + O\left(\frac{1}{z^2}\right)\right) \,/;\, |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

In trigonometric form ||| In trigonometric form

03.09.06.0041.01

$$\boldsymbol{H}_\nu(z) \propto \sqrt{\frac{2}{\pi z}} \left(\sin\left(z - \frac{(2\nu+1)\pi}{4}\right)\left(1 - \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \frac{256\nu^8 - 5376\nu^6 + 31584\nu^4 - 51664\nu^2 + 11025}{98304 z^4} + \ldots\right) + \frac{4\nu^2-1}{8z} \cos\left(z - \frac{(2\nu+1)\pi}{4}\right)\left(1 - \frac{16\nu^4 - 136\nu^2 + 225}{384 z^2} + \frac{256\nu^8 - 10496\nu^6 + 137824\nu^4 - 656784\nu^2 + 893025}{491520 z^4} + \ldots\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + \frac{2\nu-1}{z^2} + \frac{3\left(4\nu^2-8\nu+3\right)}{z^4} + \ldots\right) /; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

03.09.06.0042.01

$$\boldsymbol{H}_\nu(z) \propto \sqrt{\frac{2}{\pi z}} \left(\sin\left(z - \frac{(2\nu+1)\pi}{4}\right)\left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) + \frac{4\nu^2-1}{8z} \cos\left(z - \frac{(2\nu+1)\pi}{4}\right)\left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(-\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

03.09.06.0043.01

$$\boldsymbol{H}_\nu(z) \propto \sqrt{\frac{2}{\pi z}} \left(\sin\left(z - \frac{2\nu+1}{4}\pi\right) {}_4F_1\left(\frac{1-2\nu}{4}, \frac{3-2\nu}{4}, \frac{2\nu+1}{4}, \frac{2\nu+3}{4}; \frac{1}{2}; -\frac{1}{z^2}\right) + \frac{4\nu^2-1}{8z} \cos\left(z - \frac{2\nu+1}{4}\pi\right) {}_4F_1\left(\frac{3-2\nu}{4}, \frac{5-2\nu}{4}, \frac{2\nu+3}{4}, \frac{2\nu+5}{4}; \frac{3}{2}; -\frac{1}{z^2}\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \, {}_3F_0\left(\frac{1}{2}, \frac{1}{2}-\nu, 1;; -\frac{4}{z^2}\right) /; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

03.09.06.0044.01

$$\boldsymbol{H}_\nu(z) \propto \sqrt{\frac{2}{\pi z}} \left(\sin\left(z - \frac{2\nu+1}{4}\pi\right)\left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{4\nu^2-1}{8z} \cos\left(z - \frac{2\nu+1}{4}\pi\right)\left(1 + O\left(\frac{1}{z^2}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

Containing Bessel functions

03.09.06.0045.01

$$\boldsymbol{H}_\nu(z) - Y_\nu(z) \propto \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + \frac{2\nu-1}{z^2} + \frac{3\left(4\nu^2-8\nu+3\right)}{z^4} + \ldots\right) /; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

03.09.06.0046.01

$$\boldsymbol{H}_\nu(z) - Y_\nu(z) \propto \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(-\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

03.09.06.0047.01

$$\mathbf{H}_\nu(z) - Y_\nu(z) \propto \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)}\, {}_3F_0\left(\frac{1}{2}, \frac{1}{2} - \nu, 1;; -\frac{4}{z^2}\right) /; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

03.09.06.0048.01

$$\mathbf{H}_\nu(z) - Y_\nu(z) \propto \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; |\arg(z)| < \pi \bigwedge (|z| \to \infty)$$

03.09.06.0049.01

$$\mathbf{H}_\nu(z) + i\, J_\nu(z) \propto \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(1 + \frac{2\nu - 1}{z^2} + \frac{3\left(4\nu^2 - 8\nu + 3\right)}{z^4} + \ldots\right) /; \left|\frac{\nu}{z}\right| > 1 \bigwedge (|z| \to \infty)$$

03.09.06.0050.01

$$\mathbf{H}_\nu(z) + i\, J_\nu(z) \propto \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2} - \nu\right)_k \left(-\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; \left|\frac{\nu}{z}\right| > 1 \bigwedge (|z| \to \infty)$$

03.09.06.0051.01

$$\mathbf{H}_\nu(z) + i\, J_\nu(z) \propto \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)}\, {}_3F_0\left(\frac{1}{2}, \frac{1}{2} - \nu, 1;; -\frac{4}{z^2}\right) /; \left|\frac{\nu}{z}\right| > 1 \bigwedge (|z| \to \infty)$$

03.09.06.0052.01

$$\mathbf{H}_\nu(z) + i\, J_\nu(z) \propto \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; \left|\frac{\nu}{z}\right| > 1 \bigwedge (|z| \to \infty)$$

Expansions containing $z \to -\infty$

In exponential form ||| In exponential form

03.09.06.0053.01

$$\mathbf{H}_\nu(z) \propto \frac{(-1)^{1+\nu}}{\sqrt{-2\pi z}} \left(e^{i z + \frac{1}{4}(2\nu+3)\pi i} \left(1 + \frac{i\left(4\nu^2 - 1\right)}{8z} - \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \ldots\right) + e^{-i z - \frac{1}{4}(2\nu+3)\pi i} \left(1 - \frac{i\left(4\nu^2 - 1\right)}{8z} - \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \ldots\right)\right) + \frac{2^{1-\nu} (-1)^{\nu+1} (-z)^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(1 + \frac{2\nu - 1}{z^2} + \frac{3\left(4\nu^2 - 8\nu + 3\right)}{z^4} + \ldots\right) /; 0 < \arg(z) \leq \pi \bigwedge (|z| \to \infty)$$

03.09.06.0054.01

$$\mathbf{H}_\nu(z) \propto \frac{(-1)^{\nu+1}}{\sqrt{-2\pi z}} \left(e^{-i z - \frac{2\nu+3}{4}\pi i} \left(\sum_{k=0}^{n} \frac{\left(\nu + \frac{1}{2}\right)_k \left(\frac{1}{2} - \nu\right)_k}{k!} \left(\frac{i}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) + e^{i z + \frac{2\nu+3}{4}\pi i} \left(\sum_{k=0}^{n} \frac{\left(\nu + \frac{1}{2}\right)_k \left(\frac{1}{2} - \nu\right)_k}{k!} \left(-\frac{i}{2z}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) + \frac{2^{1-\nu} (-1)^{\nu+1} (-z)^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2} - \nu\right)_k \left(-\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; 0 < \arg(z) \leq \pi \bigwedge (|z| \to \infty)$$

03.09.06.0055.01

$$\mathbf{H}_\nu(z) \propto \frac{(-1)^{\nu+1}}{\sqrt{-2\pi z}}\left(e^{-i z-\frac{2\nu+3}{4}\pi i}\,{}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;;\frac{i}{2z}\right)+e^{i z+\frac{2\nu+3}{4}\pi i}\,{}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;;-\frac{i}{2z}\right)\right)+ \frac{2^{1-\nu}(-1)^{\nu+1}(-z)^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\,{}_3F_0\left(\frac{1}{2}, \frac{1}{2}-\nu, 1;;-\frac{4}{z^2}\right)\,/;\, 0<\arg(z)\leq\pi \bigwedge (|z|\to\infty)$$

03.09.06.0056.01

$$\mathbf{H}_\nu(z) \propto \frac{(-1)^{\nu+1}}{\sqrt{-2\pi z}}\left(e^{-i z-\frac{2\nu+3}{4}\pi i}\left(1+O\left(\frac{1}{z}\right)\right)+e^{i z+\frac{2\nu+3}{4}\pi i}\left(1+O\left(\frac{1}{z}\right)\right)\right)+\frac{2^{1-\nu}(-1)^{\nu+1}(-z)^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1+O\left(\frac{1}{z^2}\right)\right)/;\; 0<\arg(z)\leq\pi \bigwedge (|z|\to\infty)$$

In trigonometric form ||| In trigonometric form

03.09.06.0057.01

$$\mathbf{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^\nu(-z)^{-\nu-\frac{1}{2}} \left(\sin\left(z+\frac{2\nu+1}{4}\pi\right)\left(1-\frac{16\nu^4-40\nu^2+9}{128 z^2}+\frac{256\nu^8-5376\nu^6+31584\nu^4-51664\nu^2+11025}{98304 z^4}+\ldots\right)-\frac{1-4\nu^2}{8z}\cos\left(z+\frac{2\nu+1}{4}\pi\right)\left(1-\frac{16\nu^4-136\nu^2+225}{384 z^2}+\frac{256\nu^8-10496\nu^6+137824\nu^4-656784\nu^2+893025}{491520 z^4}+\ldots\right)\right)+ \frac{2^{1-\nu}z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1+\frac{2\nu-1}{z^2}+\frac{3\left(4\nu^2-8\nu+3\right)}{z^4}+\ldots\right)/;\; |\arg(-z)|<\pi\; (|z|\to\infty)$$

03.09.06.0058.01

$$\mathbf{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\,(-z)^{-\nu-\frac{1}{2}} z^\nu\left(\sin\left(z+\frac{\pi(2\nu+1)}{4}\right)\left(\sum_{k=0}^n\frac{\left(\frac{1}{4}(1-2\nu)\right)_k\left(\frac{1}{4}(3-2\nu)\right)_k\left(\frac{1}{4}(2\nu+1)\right)_k\left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!}\left(-\frac{1}{z^2}\right)^k+O\left(\frac{1}{z^{2n+2}}\right)\right)- \frac{1-4\nu^2}{8z}\cos\left(z+\frac{\pi(2\nu+1)}{4}\right)\left(\sum_{k=0}^n\frac{\left(\frac{1}{4}(3-2\nu)\right)_k\left(\frac{1}{4}(5-2\nu)\right)_k\left(\frac{1}{4}(2\nu+3)\right)_k\left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!}\left(-\frac{1}{z^2}\right)^k+O\left(\frac{1}{z^{2n+2}}\right)\right)\right)+ \frac{2^{1-\nu}z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\sum_{k=0}^n\left(\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k\left(-\frac{4}{z^2}\right)^k+O\left(\frac{1}{z^{2n+2}}\right)\right)/;\; |\arg(-z)|<\pi\; (|z|\to\infty)$$

03.09.06.0006.02

$$\mathbf{H}_\nu(z) \propto \frac{2^{1-\nu}z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\,{}_3F_0\left(\frac{1}{2}, 1, \frac{1}{2}-\nu;;-\frac{4}{z^2}\right)+ \sqrt{\frac{2}{\pi}}\,(-z)^{-\nu-\frac{1}{2}} z^\nu\left(\sin\left(z+\frac{2\nu+1}{4}\pi\right)\,{}_4F_1\left(\frac{1-2\nu}{4}, \frac{3-2\nu}{4}, \frac{2\nu+1}{4}, \frac{2\nu+3}{4}; \frac{1}{2}; -\frac{1}{z^2}\right)- \frac{1-4\nu^2}{8z}\cos\left(z+\frac{2\nu+1}{4}\pi\right)\,{}_4F_1\left(\frac{3-2\nu}{4}, \frac{5-2\nu}{4}, \frac{2\nu+3}{4}, \frac{2\nu+5}{4}; \frac{3}{2}; -\frac{1}{z^2}\right)\right)/;\; |\arg(-z)|<\pi\; (|z|\to\infty)$$

03.09.06.0007.02

$$\mathbf{H}_\nu(z) \propto \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1+O\left(\frac{1}{z^2}\right)\right)+ \sqrt{\frac{2}{\pi}}\,(-z)^{-\nu-\frac{1}{2}} z^\nu \left(\sin\left(z+\frac{2\nu+1}{4}\pi\right)\left(1+O\left(\frac{1}{z^2}\right)\right)-\frac{1-4\nu^2}{8z}\cos\left(z+\frac{2\nu+1}{4}\pi\right)\left(1+O\left(\frac{1}{z^2}\right)\right)\right) /;\ |\arg(-z)|<\pi\ (|z|\to\infty)$$

The general formulas

03.09.06.0008.01

$$\mathbf{H}_\nu(z) \propto \left(\frac{z}{2}\right)^{\nu+1} \mathcal{A}_{\tilde{F}}\left(\begin{matrix}1;\\ \frac{3}{2},\ \nu+\frac{3}{2};\end{matrix}\ \left\{-\frac{z^2}{4},\ \tilde{\infty},\ \infty\right\}\right) /;\ (|z|\to\infty)$$

03.09.06.0009.01

$$\mathbf{H}_\nu(z) \propto \left(\frac{z}{2}\right)^{\nu+1} \left(\mathcal{A}_{\tilde{F}}^{(\text{power})}\left(\begin{matrix}1;\\ \frac{3}{2},\ \nu+\frac{3}{2};\end{matrix}\ \left\{-\frac{z^2}{4},\ \tilde{\infty},\ \infty\right\}\right)+\mathcal{A}_{\tilde{F}}^{(\text{trig})}\left(\begin{matrix}1;\\ \frac{3}{2},\ \nu+\frac{3}{2};\end{matrix}\ \left\{-\frac{z^2}{4},\ \tilde{\infty},\ \infty\right\}\right)\right) /;\ (|z|\to\infty)$$

Expansions for any *z* in exponential form

Using exponential function with branch cut-containing arguments

03.09.06.0059.01

$$\mathbf{H}_\nu(z) \propto \frac{z^{\nu+1}}{\sqrt{2\pi}} \left(z^2\right)^{-\frac{1}{4}(2\nu+3)} \left(e^{-i\sqrt{z^2}+\frac{1}{4}(2\nu+3)\pi i}\left(1-\frac{i\left(4\nu^2-1\right)}{8\sqrt{z^2}}-\frac{16\nu^4-40\nu^2+9}{128 z^2}+\ldots\right)+ e^{i\sqrt{z^2}-\frac{1}{4}(2\nu+3)\pi i}\left(1+\frac{i\left(4\nu^2-1\right)}{8\sqrt{z^2}}-\frac{16\nu^4-40\nu^2+9}{128 z^2}+\ldots\right)\right)+ \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1+\frac{2\nu-1}{z^2}+\frac{3\left(4\nu^2-8\nu+3\right)}{z^4}+\ldots\right) /;\ (|z|\to\infty)$$

03.09.06.0060.01

$$\mathbf{H}_\nu(z) \propto \frac{1}{\sqrt{2\pi}} z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(e^{-i\sqrt{z^2}+\frac{2\nu+3}{4}\pi i}\left(\sum_{k=0}^{n}\frac{\left(\nu+\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k}{k!}\left(\frac{i}{2\sqrt{z^2}}\right)^k+O\left(\frac{1}{z^{n+1}}\right)\right)+ e^{i\sqrt{z^2}-\frac{2\nu+3}{4}\pi i}\left(\sum_{k=0}^{n}\frac{\left(\nu+\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k}{k!}\left(-\frac{i}{2\sqrt{z^2}}\right)^k+O\left(\frac{1}{z^{n+1}}\right)\right)\right)+ \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\sum_{k=0}^{n}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k\left(-\frac{4}{z^2}\right)^k+O\left(\frac{1}{z^{2n+2}}\right)\right) /;\ (|z|\to\infty)$$

03.09.06.0061.01

$$\mathbf{H}_\nu(z) \propto \frac{z^{\nu+1}}{\sqrt{2\pi}} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(e^{-i\sqrt{z^2} + \frac{2\nu+3}{4}\pi i} \, {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;\, ;\, \frac{i}{2\sqrt{z^2}}\right) + e^{i\sqrt{z^2} - \frac{2\nu+3}{4}\pi i} \, {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;\, ;\, -\frac{i}{2\sqrt{z^2}}\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \, {}_3F_0\left(\frac{1}{2}, \frac{1}{2}-\nu, 1;\, ;\, -\frac{4}{z^2}\right) /; (|z| \to \infty)$$

03.09.06.0062.01

$$\mathbf{H}_\nu(z) \propto \frac{1}{\sqrt{2\pi}} z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(e^{-i\sqrt{z^2} + \frac{2\nu+3}{4}\pi i} \left(1 + O\left(\frac{1}{z}\right)\right) + e^{i\sqrt{z^2} - \frac{2\nu+3}{4}\pi i} \left(1 + O\left(\frac{1}{z}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

Using exponential function with branch cut-free arguments

03.09.06.0063.01

$$\mathbf{H}_\nu(z) \propto \frac{1}{2\sqrt{2\pi}} z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(e^{-\frac{2\nu+3}{4} i\pi} \left(e^{iz} \left(\frac{\sqrt{z^2}}{z} + 1\right) + e^{-iz} \left(1 - \frac{\sqrt{z^2}}{z}\right)\right) \left(1 + \frac{i\left(4\nu^2-1\right)}{8\sqrt{z^2}} - \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \ldots\right) + e^{\frac{2\nu+3}{4} i\pi} \left(e^{-iz} \left(\frac{\sqrt{z^2}}{z} + 1\right) + e^{iz} \left(1 - \frac{\sqrt{z^2}}{z}\right)\right) \left(1 - \frac{i\left(4\nu^2-1\right)}{8\sqrt{z^2}} - \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \ldots\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + \frac{2\nu-1}{z^2} + \frac{3\left(4\nu^2 - 8\nu + 3\right)}{z^4} + \ldots\right) /; (|z| \to \infty)$$

03.09.06.0064.01

$$\mathbf{H}_\nu(z) \propto \frac{1}{2\sqrt{2\pi}} z^{\nu+1} \left(z^2\right)^{-\frac{3+2\nu}{4}} \left(e^{\frac{3+2\nu}{4} i\pi} \left(e^{-iz} \left(\frac{\sqrt{z^2}}{z} + 1\right) + e^{iz} \left(1 - \frac{\sqrt{z^2}}{z}\right)\right) \left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(\frac{i}{2\sqrt{z^2}}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right) + e^{-\frac{3+2\nu}{4} i\pi} \left(e^{iz} \left(\frac{\sqrt{z^2}}{z} + 1\right) + e^{-iz} \left(1 - \frac{\sqrt{z^2}}{z}\right)\right) \left(\sum_{k=0}^{n} \frac{\left(\nu+\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k}{k!} \left(-\frac{i}{2\sqrt{z^2}}\right)^k + O\left(\frac{1}{z^{n+1}}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(-\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; (|z| \to \infty)$$

03.09.06.0065.01

$$\mathbf{H}_\nu(z) \propto \frac{1}{2\sqrt{2\pi}} z^{\nu+1} \left(z^2\right)^{-\frac{3+2\nu}{4}} \left(e^{\frac{3+2\nu}{4} i\pi} \left(e^{-iz} \left(\frac{\sqrt{z^2}}{z} + 1\right) + e^{iz} \left(1 - \frac{\sqrt{z^2}}{z}\right)\right) {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;\, ;\, \frac{i}{2\sqrt{z^2}}\right) + e^{-\frac{3+2\nu}{4} i\pi} \left(e^{iz} \left(\frac{\sqrt{z^2}}{z} + 1\right) + e^{-iz} \left(1 - \frac{\sqrt{z^2}}{z}\right)\right) {}_2F_0\left(\nu+\frac{1}{2}, \frac{1}{2}-\nu;\, ;\, -\frac{i}{2\sqrt{z^2}}\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \, {}_3F_0\left(\frac{1}{2}, \frac{1}{2}-\nu, 1;\, ;\, -\frac{4}{z^2}\right) /; (|z| \to \infty)$$

03.09.06.0066.01

$$\mathbf{H}_\nu(z) \propto \frac{1}{2\sqrt{2\pi}} z^{\nu+1} \left(z^2\right)^{-\frac{3+2\nu}{4}} \left(e^{\frac{3+2\nu}{4} i\pi} \left(e^{-iz}\left(\frac{\sqrt{z^2}}{z}+1\right) + e^{iz}\left(1-\frac{\sqrt{z^2}}{z}\right)\right)\left(1+O\left(\frac{1}{z}\right)\right) + e^{-\frac{3+2\nu}{4} i\pi} \left(e^{iz}\left(\frac{\sqrt{z^2}}{z}+1\right) + e^{-iz}\left(1-\frac{\sqrt{z^2}}{z}\right)\right)\left(1+O\left(\frac{1}{z}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1+O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

Expansions for any z in trigonometric form

Using trigonometric functions with branch cut-containing arguments

03.09.06.0067.01

$$\mathbf{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}} z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(\sin\left(\sqrt{z^2} - \frac{(2\nu+1)\pi}{4}\right)\left(1 - \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \frac{256\nu^8 - 5376\nu^6 + 31584\nu^4 - 51664\nu^2 + 11025}{98304 z^4} + \ldots\right) + \frac{4\nu^2-1}{8\sqrt{z^2}} \cos\left(\sqrt{z^2} - \frac{(2\nu+1)\pi}{4}\right) \left(1 - \frac{16\nu^4 - 136\nu^2 + 225}{384 z^2} + \frac{256\nu^8 - 10496\nu^6 + 137824\nu^4 - 656784\nu^2 + 893025}{491520 z^4} + \ldots\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(1 + \frac{2\nu-1}{z^2} + \frac{3\left(4\nu^2 - 8\nu + 3\right)}{z^4} + \ldots\right) /; (|z| \to \infty)$$

03.09.06.0068.01

$$\mathbf{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}} \left(z^2\right)^{-\frac{1}{4}(2\nu+3)} z^{\nu+1} \left(\sin\left(\sqrt{z^2} - \frac{(2\nu+1)\pi}{4}\right)\left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!}\left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) + \frac{4\nu^2-1}{8\sqrt{z^2}} \cos\left(\sqrt{z^2} - \frac{(2\nu+1)\pi}{4}\right)\left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!}\left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(-\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right)\right) /; (|z| \to \infty)$$

03.09.06.0010.01

$$\boldsymbol{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(\sin\left(\sqrt{z^2} - \frac{2\nu+1}{4}\pi\right) {}_4F_1\left(\frac{1-2\nu}{4}, \frac{3-2\nu}{4}, \frac{2\nu+1}{4}, \frac{2\nu+3}{4}; \frac{1}{2}; -\frac{1}{z^2}\right) + \frac{4\nu^2-1}{8\sqrt{z^2}} \cos\left(\sqrt{z^2} - \frac{2\nu+1}{4}\pi\right) {}_4F_1\left(\frac{3-2\nu}{4}, \frac{5-2\nu}{4}, \frac{2\nu+3}{4}, \frac{2\nu+5}{4}; \frac{3}{2}; -\frac{1}{z^2}\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} {}_3F_0\left(\frac{1}{2}, \frac{1}{2}-\nu, 1;; -\frac{4}{z^2}\right) /; (|z| \to \infty)$$

03.09.06.0011.01

$$\boldsymbol{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(\sin\left(\sqrt{z^2} - \frac{2\nu+1}{4}\pi\right)\left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{4\nu^2-1}{8\sqrt{z^2}} \cos\left(\sqrt{z^2} - \frac{2\nu+1}{4}\pi\right)\left(1 + O\left(\frac{1}{z^2}\right)\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

Using trigonometric functions with branch cut-free arguments

03.09.06.0069.01

$$\boldsymbol{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(\left(\frac{z}{\sqrt{z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \sin(z) - \sin\left(\frac{2\nu+1}{4}\pi\right) \cos(z)\right) \left(1 - \frac{16\nu^4 - 40\nu^2 + 9}{128 z^2} + \frac{256\nu^8 - 5376\nu^6 + 31\,584\nu^4 - 51\,664\nu^2 + 11\,025}{98\,304 z^4} + \ldots\right) + \frac{4\nu^2-1}{8} \left(\frac{1}{\sqrt{z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \cos(z) + \frac{1}{z} \sin\left(\frac{2\nu+1}{4}\pi\right) \sin(z)\right) \left(1 - \frac{16\nu^4 - 136\nu^2 + 225}{384 z^2} + \frac{256\nu^8 - 10\,496\nu^6 + 137\,824\nu^4 - 656\,784\nu^2 + 893\,025}{491\,520 z^4} + \ldots\right)\right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\, \Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + \frac{2\nu-1}{z^2} + \frac{3\left(4\nu^2 - 8\nu + 3\right)}{z^4} + \ldots\right) /; (|z| \to \infty)$$

03.09.06.0070.01

$$\mathbf{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(\left(\frac{z}{\sqrt{z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \sin(z) - \sin\left(\frac{2\nu+1}{4}\pi\right)\cos(z) \right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(1-2\nu)\right)_k \left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(2\nu+1)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k}{\left(\frac{1}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right) \right) + \frac{4\nu^2-1}{8} \left(\frac{1}{\sqrt{z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right)\cos(z) + \frac{1}{z}\sin\left(\frac{2\nu+1}{4}\pi\right)\sin(z) \right) \left(\sum_{k=0}^{n} \frac{\left(\frac{1}{4}(3-2\nu)\right)_k \left(\frac{1}{4}(5-2\nu)\right)_k \left(\frac{1}{4}(2\nu+3)\right)_k \left(\frac{1}{4}(2\nu+5)\right)_k}{\left(\frac{3}{2}\right)_k k!} \left(-\frac{1}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right) \right) \right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \left(\sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}-\nu\right)_k \left(-\frac{4}{z^2}\right)^k + O\left(\frac{1}{z^{2n+2}}\right) \right) /; (|z| \to \infty)$$

03.09.06.0071.01

$$\mathbf{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(\left(\frac{z}{\sqrt{z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \sin(z) - \sin\left(\frac{2\nu+1}{4}\pi\right)\cos(z) \right) {}_4F_1\left(\frac{1}{4}(1-2\nu), \frac{1}{4}(3-2\nu), \frac{1}{4}(2\nu+1), \frac{1}{4}(2\nu+3); \frac{1}{2}; -\frac{1}{z^2}\right) + \frac{4\nu^2-1}{8} \left(\frac{1}{\sqrt{z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right)\cos(z) + \frac{1}{z}\sin\left(\frac{2\nu+1}{4}\pi\right)\sin(z) \right) {}_4F_1\left(\frac{1}{4}(3-2\nu), \frac{1}{4}(5-2\nu), \frac{1}{4}(2\nu+3), \frac{1}{4}(2\nu+5); \frac{3}{2}; -\frac{1}{z^2}\right) \right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} {}_3F_0\left(1, \frac{1}{2}, \frac{1}{2}-\nu; ; -\frac{4}{z^2}\right) /; (|z| \to \infty)$$

03.09.06.0072.01

$$\mathbf{H}_\nu(z) \propto \sqrt{\frac{2}{\pi}}\, z^{\nu+1} \left(z^2\right)^{-\frac{2\nu+3}{4}} \left(\left(\frac{z}{\sqrt{z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right) \sin(z) - \sin\left(\frac{2\nu+1}{4}\pi\right)\cos(z) \right) \left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{4\nu^2-1}{8} \left(\frac{1}{\sqrt{z^2}} \cos\left(\frac{2\nu+1}{4}\pi\right)\cos(z) + \frac{1}{z}\sin\left(\frac{2\nu+1}{4}\pi\right)\sin(z) \right) \left(1 + O\left(\frac{1}{z^2}\right)\right) \right) + \frac{2^{1-\nu} z^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

Residue representations

03.09.06.0012.01

$$\mathbf{H}_\nu(z) = z^{\nu+1} \left(z^2\right)^{-\frac{\nu+1}{2}} \sum_{j=0}^{\infty} \operatorname{res}_s \left(\frac{\Gamma\left(\frac{1-\nu}{2} - s\right) \left(\frac{z^2}{4}\right)^{-s}}{\Gamma\left(1+\frac{\nu}{2}-s\right)\Gamma\left(1-\frac{\nu}{2}-s\right)} \Gamma\left(\frac{\nu+1}{2}+s\right) \right) \left(-\frac{\nu+1}{2} - j\right)$$

03.09.06.0013.01

$$\mathbf{H}_\nu(z) = \sum_{j=0}^{\infty} \operatorname{res}_s\left(\frac{\Gamma\left(\frac{1-\nu}{2}-s\right)\left(\frac{z}{2}\right)^{-2s}}{\Gamma\left(1+\frac{\nu}{2}-s\right)\Gamma\left(1-\frac{\nu}{2}-s\right)}\Gamma\left(\frac{\nu+1}{2}+s\right)\right)\left(-\frac{\nu+1}{2}-j\right)$$

Other series representations

03.09.06.0014.01

$$\mathbf{H}_0(z) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{J_{2k+1}(z)}{2k+1}$$

03.09.06.0015.01

$$\mathbf{H}_1(z) = \frac{2}{\pi}(1-J_0(z)) + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{J_{2k}(z)}{4k^2-1}$$

Integral representations

On the real axis

Of the direct function

03.09.07.0001.01

$$\mathbf{H}_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \sin(t\,z)\,dt \;/;\; \operatorname{Re}(\nu) > -\frac{1}{2}$$

03.09.07.0002.01

$$\mathbf{H}_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}} \sin^{2\nu}(t)\sin(z\cos(t))\,dt \;/;\; \operatorname{Re}(\nu) > -\frac{1}{2}$$

03.09.07.0003.01

$$\mathbf{H}_\nu(z) = Y_\nu(z) + \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\infty} e^{-t z}\left(t^2+1\right)^{\nu-\frac{1}{2}}\,dt \;/;\; \operatorname{Re}(z) > 0$$

Contour integral representations

03.09.07.0004.01

$$\mathbf{H}_\nu(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\,\Gamma(1-s)}{\Gamma\left(\frac{3}{2}-s\right)\Gamma\left(\nu+\frac{3}{2}-s\right)}\left(\frac{x}{2}\right)^{-2s+\nu+1} ds \;/;\; 0 < \gamma < \min\left(1, \frac{\operatorname{Re}(\nu)}{2}+\frac{5}{4}\right) \bigwedge x > 0$$

03.09.07.0005.01

$$\mathbf{H}_\nu(z) = z^{\nu+1}\left(z^2\right)^{-\frac{\nu+1}{2}}\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma\left(\frac{\nu+1}{2}+s\right)\Gamma\left(\frac{1-\nu}{2}-s\right)}{\Gamma\left(1+\frac{\nu}{2}-s\right)\Gamma\left(1-\frac{\nu}{2}-s\right)}\left(\frac{z^2}{4}\right)^{-s} ds$$

03.09.07.0006.01

$$\mathbf{H}_\nu(z) = \frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\Gamma\left(\frac{\nu+1}{2}+s\right)\Gamma\left(\frac{1-\nu}{2}-s\right)}{\Gamma\left(1+\frac{\nu}{2}-s\right)\Gamma\left(1-\frac{\nu}{2}-s\right)}\left(\frac{z}{2}\right)^{-2s} ds$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

03.09.13.0001.01

$$w''(z)\,z^2 + w'(z)\,z + \left(z^2 - \nu^2\right) w(z) = \frac{4}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \text{ /; } w(z) = c_1 J_\nu(z) + c_2 Y_\nu(z) + \boldsymbol{H}_\nu(z)$$

03.09.13.0002.01

$$W_z(J_\nu(z),\,Y_\nu(z)) = \frac{2}{\pi z}$$

03.09.13.0003.01

$$w''(z)\,z^2 + w'(z)\,z + \left(z^2 - \nu^2\right) w(z) = \frac{4}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \text{ /; } w(z) = c_1 J_\nu(z) + c_2 J_{-\nu}(z) + \boldsymbol{H}_\nu(z) \land \nu \notin \mathbb{Z}$$

03.09.13.0004.01

$$W_z(J_\nu(z),\,J_{-\nu}(z)) = -\frac{2\sin(\pi\nu)}{\pi z}$$

03.09.13.0005.01

$$z^3 w^{(3)}(z) + (2-\nu)\,z^2 w''(z) + \left(z^2 - \nu(\nu+1)\right) z\, w'(z) + \left(\nu^3 + \nu^2 - z^2\nu + z^2\right) w(z) = 0 \text{ /; } w(z) = c_1 \boldsymbol{H}_\nu(z) + c_2 J_\nu(z) + c_3 Y_\nu(z)$$

03.09.13.0006.01

$$W_z(\boldsymbol{H}_\nu(z),\,J_\nu(z),\,Y_\nu(z)) = \frac{2^{2-\nu} z^{\nu-2}}{\pi^{3/2}\,\Gamma\left(\nu + \frac{1}{2}\right)}$$

03.09.13.0007.01

$$w^{(3)}(z) - \left(\frac{(\nu-2)\,g'(z)}{g(z)} + \frac{3\,g''(z)}{g'(z)}\right) w''(z) + \left(-\frac{\nu(\nu+1)\,g'(z)^2}{g(z)^2} + g'(z)^2 + \frac{3\,g''(z)^2}{g'(z)^2} + \frac{(\nu-2)\,g''(z)}{g(z)} - \frac{g^{(3)}(z)}{g'(z)}\right) w'(z) + \frac{\left(\nu^2(\nu+1) - (\nu-1)\,g(z)^2\right) g'(z)^3}{g(z)^3}\, w(z) = 0 \text{ /; } w(z) = c_1 \boldsymbol{H}_\nu(g(z)) + c_2 J_\nu(g(z)) + c_3 Y_\nu(g(z))$$

03.09.13.0008.01

$$W_z(\boldsymbol{H}_\nu(g(z)),\,J_\nu(g(z)),\,Y_\nu(g(z))) = \frac{2^{2-\nu} g(z)^{\nu-2} g'(z)^3}{\pi^{3/2}\,\Gamma\left(\nu + \frac{1}{2}\right)}$$

03.09.13.0009.01

$$w^{(3)}(z) - \left(\frac{(\nu-2)\,g'(z)}{g(z)} + \frac{3\,h'(z)}{h(z)} + \frac{3\,g''(z)}{g'(z)}\right) w''(z) + \left(-\frac{\nu(\nu+1)\,g'(z)^2}{g(z)^2} + g'(z)^2 + \frac{2(\nu-2)\,h'(z)\,g'(z)}{g(z)\,h(z)} + \frac{6\,h'(z)^2}{h(z)^2} + \frac{3\,g''(z)^2}{g'(z)^2} + \frac{6\,h'(z)\,g''(z)}{h(z)\,g'(z)} + \frac{(\nu-2)\,g''(z)}{g(z)} - \frac{3\,h''(z)}{h(z)} - \frac{g^{(3)}(z)}{g'(z)}\right) w'(z) + \left(-\frac{(\nu-1)\,g'(z)^3}{g(z)} + \frac{\nu^2(\nu+1)\,g'(z)^3}{g(z)^3} + \frac{\nu(\nu+1)\,h'(z)\,g'(z)^2}{g(z)^2\,h(z)} - \frac{2(\nu-2)\,h'(z)^2\,g'(z)}{g(z)\,h(z)^2} + \frac{6\,h'(z)\,h''(z)}{h(z)^2} + \frac{3\,g''(z)\,h''(z) + h'(z)\,g^{(3)}(z)}{h(z)\,g'(z)} - \frac{h'(z)\,g'(z)^2 + h^{(3)}(z)}{h(z)} - \frac{(\nu-2)\,(h'(z)\,g''(z) - g'(z)\,h''(z))}{g(z)\,h(z)} - \frac{6\,h'(z)^3}{h(z)^3} - \frac{6\,h'(z)^2\,g''(z)}{h(z)^2\,g'(z)} - \frac{3\,h'(z)\,g''(z)^2}{h(z)\,g'(z)^2}\right) w(z) = 0 \;/;\; w(z) = c_1\,h(z)\,\boldsymbol{H}_\nu(g(z))\,c_1 + c_2\,J_\nu(g(z))\,h(z) + c_3\,Y_\nu(g(z))\,h(z)$$

03.09.13.0010.01

$$W_z(h(z)\,\boldsymbol{H}_\nu(g(z)),\,h(z)\,J_\nu(g(z)),\,h(z)\,Y_\nu(g(z))) = \frac{2^{2-\nu}\,g(z)^{\nu-2}\,h(z)^3\,g'(z)^3}{\pi^{3/2}\,\Gamma\left(\nu+\frac{1}{2}\right)}$$

03.09.13.0011.01

$$z^3\,w^{(3)}(z) - (\nu\,r + r + 3\,s - 3)\,z^2\,w''(z) + \left(\left(a^2\,z^{2r} - \nu^2\right)r^2 + (2\,s-1)(\nu+1)\,r + 3\,(s-1)\,s + 1\right) z\,w'(z) + \left(\left(\nu^2(\nu+1) - a^2\,z^{2r}(\nu-1)\right) r^3 + s\left(\nu^2 - a^2\,z^{2r}\right) r^2 - s^2(\nu+1)\,r - s^3\right) w(z) = 0 \;/;\; w(z) = c_1\,z^s\,\boldsymbol{H}_\nu(a\,z^r) + c_2\,z^s\,J_\nu(a\,z^r) + c_3\,z^s\,Y_\nu(a\,z^r)$$

03.09.13.0012.01

$$W_z(z^s\,\boldsymbol{H}_\nu(a\,z^r),\,z^s\,J_\nu(a\,z^r),\,z^s\,Y_\nu(a\,z^r)) = \frac{2^{2-\nu}\,a\,r^3\,z^{r+3s-3}\,(a\,z^r)^\nu}{\pi^{3/2}\,\Gamma\left(\nu+\frac{1}{2}\right)}$$

03.09.13.0013.01

$$w^{(3)}(z) - ((\nu+1)\log(r) + 3\log(s))\,w''(z) + \left(\left(a^2\,r^{2z} - \nu^2\right)\log^2(r) + 2\,(\nu+1)\log(s)\log(r) + 3\log^2(s)\right) w'(z) + \left(\left(\nu^2(\nu+1) - a^2\,r^{2z}(\nu-1)\right)\log^3(r) - \left(a^2\,r^{2z} - \nu^2\right)\log(s)\log^2(r) - (\nu+1)\log^2(s)\log(r) - \log^3(s)\right) w(z) = 0 \;/;\; w(z) = c_1\,s^z\,\boldsymbol{H}_\nu(a\,r^z) + c_2\,s^z\,J_\nu(a\,r^z) + c_3\,s^z\,Y_\nu(a\,r^z)$$

03.09.13.0014.01

$$W_z(s^z\,\boldsymbol{H}_\nu(a\,r^z),\,s^z\,J_\nu(a\,r^z),\,s^z\,Y_\nu(a\,r^z)) = \frac{2^{2-\nu}\,(a\,r^z)^{\nu+1}\,s^{3z}\,\log^3(r)}{\pi^{3/2}\,\Gamma\left(\nu+\frac{1}{2}\right)}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

03.09.16.0001.01

$$\boldsymbol{H}_\nu(-z) = -(-z)^\nu\,z^{-\nu}\,\boldsymbol{H}_\nu(z)$$

03.09.16.0002.01

$$\boldsymbol{H}_\nu(i\,z) = i\,(i\,z)^\nu\,z^{-\nu}\,\boldsymbol{L}_\nu(z)$$

03.09.16.0003.01

$$\mathbf{H}_\nu(-i\,z) = -i\,(-i\,z)^\nu\,z^{-\nu}\,\mathbf{L}_\nu(z)$$

03.09.16.0004.01

$$\mathbf{H}_\nu\left(\sqrt{z^2}\right) = z^{-\nu-1}\left(z^2\right)^{\frac{\nu+1}{2}}\,\mathbf{H}_\nu(z)$$

03.09.16.0005.01

$$\mathbf{H}_\nu(c\,(d\,z^n)^m) = \frac{(c\,(d\,z^n)^m)^{\nu+1}}{(c\,d^m\,z^{m\,n})^{\nu+1}}\,\mathbf{H}_\nu(c\,d^m\,z^{m\,n})\;/;\;2\,m\in\mathbb{Z}$$

Identities

Recurrence identities

Consecutive neighbors

03.09.17.0001.01

$$\mathbf{H}_\nu(z) = \frac{2\,(\nu+1)}{z}\,\mathbf{H}_{\nu+1}(z) - \mathbf{H}_{\nu+2}(z) + \frac{2^{-\nu-1}\,z^{\nu+1}}{\sqrt{\pi}\;\Gamma\left(\nu+\frac{5}{2}\right)}$$

03.09.17.0002.01

$$\mathbf{H}_\nu(z) = \frac{2\,(\nu-1)}{z}\,\mathbf{H}_{\nu-1}(z) - \mathbf{H}_{\nu-2}(z) + \frac{2^{1-\nu}\,z^{\nu-1}}{\sqrt{\pi}\;\Gamma\left(\nu+\frac{1}{2}\right)}$$

Distant neighbors

Increasing

03.09.17.0012.01

$$\mathbf{H}_\nu(z) = 2^{n-1}\,z^{-n}\,(\nu+1)_{n-1}\left(2\,(n+\nu)\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(n-k)!}{k!\,(n-2\,k)!\,(-n-\nu)_k\,(\nu+1)_k}\left(\frac{z^2}{4}\right)^k\mathbf{H}_{n+\nu}(z) - z\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{(n-k-1)!}{k!\,(n-2\,k-1)!\,(1-n-\nu)_k\,(\nu+1)_k}\left(\frac{z^2}{4}\right)^k\mathbf{H}_{n+\nu+1}(z)\right) + \frac{1}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{\nu+1}\sum_{j=0}^{n-1}\frac{(\nu+1)_j}{\Gamma\left(j+\nu+\frac{5}{2}\right)}\sum_{k=0}^{\left\lfloor\frac{j}{2}\right\rfloor}\frac{(j-k)!}{k!\,(j-2\,k)!\,(-j-\nu)_k\,(\nu+1)_k}\left(\frac{z^2}{4}\right)^k\;/;\;n\in\mathbb{N}$$

Brychkov Yu.A. (2005)

03.09.17.0013.01

$$\mathbf{H}_\nu(z) = 2^{n-1} z^{-n} (\nu+1)_{n-1} \left(2(n+\nu)\, {}_3F_4\!\left(1, \frac{1-n}{2}, -\frac{n}{2}; 1, -n, -n-\nu, \nu+1; -z^2\right) \mathbf{H}_{n+\nu}(z) - z\, {}_3F_4\!\left(1, \frac{1-n}{2}, 1-\frac{n}{2}; 1, 1-n, 1-n-\nu, \nu+1; -z^2\right) \mathbf{H}_{n+\nu+1}(z)\right) + \frac{2^{-\nu-1} z^{\nu+1}}{\sqrt{\pi}} \sum_{j=0}^{n-1} \frac{(\nu+1)_j}{\Gamma\!\left(j+\nu+\frac{5}{2}\right)}\, {}_3F_4\!\left(1, \frac{1-j}{2}, -\frac{j}{2}; 1, -j, -j-\nu, \nu+1; -z^2\right) /; n \in \mathbb{N}$$

Brychkov Yu.A. (2005)

03.09.17.0004.01

$$\mathbf{H}_\nu(z) = \frac{2^{-\nu-2} (4\nu+7) z^{\nu+1}}{\sqrt{\pi}\, \Gamma\!\left(\nu+\frac{7}{2}\right)} + \frac{\left(4\left(\nu^2+3\nu+2\right)-z^2\right) \mathbf{H}_{\nu+2}(z)}{z^2} - \frac{2(\nu+1)\, \mathbf{H}_{\nu+3}(z)}{z}$$

03.09.17.0005.01

$$\mathbf{H}_\nu(z) = -\frac{2^{-\nu-3} \left(z^2-12\nu^2-54\nu-57\right) z^{\nu+1}}{\sqrt{\pi}\, \Gamma\!\left(\nu+\frac{9}{2}\right)} + \frac{4(\nu+2)\left(-z^2+2\nu^2+8\nu+6\right) \mathbf{H}_{\nu+3}(z)}{z^3} + \left(1-\frac{4(\nu+1)(\nu+2)}{z^2}\right) \mathbf{H}_{\nu+4}(z)$$

03.09.17.0006.01

$$\mathbf{H}_\nu(z) = -\frac{2^{-\nu-4} \left(-32\nu^3-264\nu^2+6z^2\nu-688\nu+17z^2-561\right) z^{\nu+1}}{\sqrt{\pi}\, \Gamma\!\left(\nu+\frac{11}{2}\right)} + \left(\frac{8(\nu+4)\left(-z^2+2\nu^2+8\nu+6\right)(\nu+2)}{z^4} - \frac{4(\nu+1)(\nu+2)}{z^2} + 1\right) \mathbf{H}_{\nu+4}(z) - \frac{4(\nu+2)\left(-z^2+2\nu^2+8\nu+6\right) \mathbf{H}_{\nu+5}(z)}{z^3}$$

03.09.17.0007.01

$$\mathbf{H}_\nu(z) = \frac{2(\nu+3)\left(3z^4-16\left(\nu^2+6\nu+8\right)z^2+16\left(\nu^4+12\nu^3+49\nu^2+78\nu+40\right)\right) \mathbf{H}_{\nu+5}(z)}{z^5} - \left(\frac{8(\nu+4)\left(-z^2+2\nu^2+8\nu+6\right)(\nu+2)}{z^4} - \frac{4(\nu+1)(\nu+2)}{z^2} + 1\right) \mathbf{H}_{\nu+6}(z) + \frac{2^{-\nu-5} z^{\nu+1} \left(z^4-24\nu^2z^2-160\nu z^2-259z^2+80\nu^4+1040\nu^3+4840\nu^2+9490\nu+6555\right)}{\sqrt{\pi}\, \Gamma\!\left(\nu+\frac{13}{2}\right)}$$

03.09.17.0014.01

$$\mathbf{H}_\nu(z) = C_n(\nu, z)\, \mathbf{H}_{\nu+n}(z) - C_{n-1}(\nu, z)\, \mathbf{H}_{\nu+n+1}(z) + \frac{1}{\sqrt{\pi}} \sum_{j=0}^{n-1} \frac{1}{\Gamma\!\left(j+\nu+\frac{5}{2}\right)} \left(\frac{z}{2}\right)^{j+\nu+1} C_j(\nu, z) /;$$

$$C_0(\nu, z) = 1 \bigwedge C_1(\nu, z) = \frac{2(\nu+1)}{z} \bigwedge C_n(\nu, z) = \frac{2(n+\nu)}{z} C_{n-1}(\nu, z) - C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

03.09.17.0015.01

$$\mathbf{H}_\nu(z) = C_n(\nu, z)\, \mathbf{H}_{n+\nu}(z) - C_{n-1}(\nu, z)\, \mathbf{H}_{n+\nu+1}(z) + \frac{1}{\sqrt{\pi}\, \Gamma\!\left(\nu+\frac{5}{2}\right)} \left(\frac{z}{2}\right)^{\nu+1} \sum_{j=0}^{n-1} \frac{(\nu+1)_j}{\left(\nu+\frac{5}{2}\right)_j}\, {}_2F_3\!\left(\frac{1-j}{2}, -\frac{j}{2}; \nu+1, -j, -j-\nu; -z^2\right) /;$$

$$C_n(\nu, z) = 2^n z^{-n} (\nu+1)_n\, {}_2F_3\!\left(\frac{1-n}{2}, -\frac{n}{2}; \nu+1, -n, -n-\nu; -z^2\right) \bigwedge n \in \mathbb{N}^+$$

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Decreasing

03.09.17.0016.01

$$\mathbf{H}_\nu(z) = 2^{n-1}\,(-z)^{-n}\,(1-\nu)_{n-1} \left(2\,(n-\nu)\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor} \frac{(n-k)!}{k!\,(n-2\,k)!\,(1-\nu)_k\,(\nu-n)_k}\left(\frac{z^2}{4}\right)^k \mathbf{H}_{\nu-n}(z) + z\sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor} \frac{(n-k-1)!}{k!\,(n-2\,k-1)!\,(1-\nu)_k\,(\nu-n+1)_k}\left(\frac{z^2}{4}\right)^k \mathbf{H}_{\nu-n-1}(z)\right) + \frac{1}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{\nu-1}\sum_{j=0}^{n-1}\frac{(1-\nu)_j}{\Gamma\left(\nu-j+\frac{1}{2}\right)\left(-\frac{z^2}{4}\right)^j}\sum_{k=0}^{\left\lfloor \frac{j}{2}\right\rfloor}\frac{(j-k)!\left(\frac{z^2}{4}\right)^k}{k!\,(j-2\,k)!\,(1-\nu)_k\,(\nu-j)_k}\ /;\ n\in\mathbb{N}$$

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03.09.17.0017.01

$$\mathbf{H}_\nu(z) = 2^{n-1}\,(1-\nu)_{n-1}\,(-z)^{-n}\left(z\,{}_3F_4\!\left(1, \frac{1-n}{2}, 1-\frac{n}{2}; 1, 1-n, 1-\nu, \nu-n+1; -z^2\right)\mathbf{H}_{-n+\nu-1}(z) + 2\,(n-\nu)\,{}_3F_4\!\left(1, \frac{1-n}{2}, -\frac{n}{2}; 1, -n, 1-\nu, \nu-n; -z^2\right)\mathbf{H}_{\nu-n}(z)\right) + \frac{2^{1-\nu}\,z^{\nu-1}}{\sqrt{\pi}}\sum_{j=0}^{n-1}\frac{4^j\,\left(-z^2\right)^{-j}\,(1-\nu)_j}{\Gamma\left(\nu-j+\frac{1}{2}\right)}\,{}_3F_4\!\left(1, \frac{1-j}{2}, -\frac{j}{2}; 1, -j, 1-\nu, \nu-j; -z^2\right)\ /;\ n\in\mathbb{N}$$

Brychkov Yu.A. (2005)

03.09.17.0008.01

$$\mathbf{H}_\nu(z) = \frac{2^{1-\nu}\left(z^2+4\,\nu^2-6\,\nu+2\right)z^{\nu-3}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - \left(1-\frac{4\,(\nu-2)\,(\nu-1)}{z^2}\right)\mathbf{H}_{\nu-2}(z) - \frac{2\,(\nu-1)\,\mathbf{H}_{\nu-3}(z)}{z}$$

03.09.17.0009.01

$$\mathbf{H}_\nu(z) = \frac{2^{1-\nu}\left(z^4+2\,\nu\,z^2-z^2+16\,\nu^4-80\,\nu^3+140\,\nu^2-100\,\nu+24\right)z^{\nu-5}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} + \left(1-\frac{4\,(\nu-2)\,(\nu-1)}{z^2}\right)\mathbf{H}_{\nu-4}(z) + \frac{4\,(\nu-2)\left(-z^2+2\,\nu^2-8\,\nu+6\right)\mathbf{H}_{\nu-3}(z)}{z^3}$$

03.09.17.0010.01

$$\mathbf{H}_\nu(z) = \left(-\frac{4\,(\nu-2)\,(\nu-1)}{z^2} + \frac{8\,(\nu-4)\,(\nu-2)\left(-z^2+2\,\nu^2-8\,\nu+6\right)}{z^4} + 1\right)\mathbf{H}_{\nu-4}(z) - \frac{4\,(\nu-2)\left(-z^2+2\,\nu^2-8\,\nu+6\right)\mathbf{H}_{\nu-5}(z)}{z^3} + \frac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(2^{1-\nu}\,z^{\nu-7}\left(z^6+2\,\nu\,z^4-z^4-16\,\nu^4\,z^2+128\,\nu^3\,z^2-332\,\nu^2\,z^2+328\,\nu\,z^2-96\,z^2+64\,\nu^6-672\,\nu^5+2800\,\nu^4-5880\,\nu^3+6496\,\nu^2-3528\,\nu+720\right)\right)$$

03.09.17.0011.01

$$\mathbf{H}_\nu(z) == -\left(-\frac{4(\nu-2)(\nu-1)}{z^2} + \frac{8(\nu-4)(\nu-2)\left(-z^2+2\nu^2-8\nu+6\right)}{z^4} + 1\right)\mathbf{H}_{\nu-6}(z) + \frac{2(\nu-3)\left(3z^4-16\left(\nu^2-6\nu+8\right)z^2+16\left(\nu^4-12\nu^3+49\nu^2-78\nu+40\right)\right)\mathbf{H}_{\nu-5}(z)}{z^5} + \frac{1}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(2^{1-\nu}z^{\nu-9}\left(z^8+2\nu z^6-z^6+12\nu^2z^4-24\nu z^4+9z^4-128\nu^6z^2+1824\nu^5z^2 - 10160\nu^4z^2+28200\nu^3z^2-40652\nu^2z^2+28116\nu z^2-6840z^2+256\nu^8-4608\nu^7+ 34944\nu^6-145152\nu^5+359184\nu^4-538272\nu^3+472496\nu^2-219168\nu+40320\right)\right)$$

03.09.17.0018.01

$$\mathbf{H}_\nu(z) == C_n(\nu, z)\,\mathbf{H}_{\nu-n}(z) - C_{n-1}(\nu, z)\,\mathbf{H}_{\nu-n-1}(z) + \frac{1}{\sqrt{\pi}}\sum_{j=0}^{n-1}\frac{1}{\Gamma\left(\nu+\frac{1}{2}-j\right)}\left(\frac{z}{2}\right)^{\nu-j-1}C_j(\nu, z)\,/;$$
$$C_0(\nu, z) == 1 \bigwedge C_1(\nu, z) == \frac{2(\nu-1)}{z} \bigwedge C_n(\nu, z) == \frac{2(\nu-n)}{z}C_{n-1}(\nu, z) - C_{n-2}(\nu, z) \bigwedge n \in \mathbb{N}^+$$

03.09.17.0019.01

$$\mathbf{H}_\nu(z) = C_n(\nu, z)\,\mathbf{H}_{\nu-n}(z) - C_{n-1}(\nu, z)\,\mathbf{H}_{\nu-n-1}(z) + \frac{1}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{\nu-1}\sum_{j=0}^{n-1}\frac{(1-\nu)_j}{\Gamma\left(\nu-j+\frac{1}{2}\right)\left(-\frac{z^2}{4}\right)^j}\,{}_2F_3\left(\frac{1-j}{2}, -\frac{j}{2}; 1-\nu, -j, \nu-j; -z^2\right)/;$$
$$C_n(\nu, z) = (-2)^n z^{-n}(1-\nu)_n\,{}_2F_3\left(\frac{1-n}{2}, -\frac{n}{2}; 1-\nu, -n, \nu-n; -z^2\right)/; n \in \mathbb{N}^+$$

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Functional identities

Relations between contiguous functions

03.09.17.0003.01

$$\mathbf{H}_\nu(z) == \frac{z}{2\nu}\left(\mathbf{H}_{\nu-1}(z) + \mathbf{H}_{\nu+1}(z)\right) - \frac{2^{-\nu-1}z^{\nu+1}}{\sqrt{\pi}\,\nu\,\Gamma\left(\nu+\frac{3}{2}\right)}$$

Differentiation

Low-order differentiation

With respect to ν

03.09.20.0001.01

$$\mathbf{H}_\nu^{(1,0)}(z) == \log\left(\frac{z}{2}\right)\mathbf{H}_\nu(z) - \left(\frac{z}{2}\right)^{\nu+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}\psi\left(k+\nu+\frac{3}{2}\right)\left(\frac{z}{2}\right)^{2k}$$

03.09.20.0014.01

$$\boldsymbol{H}_n^{(1,0)}(z) = -\frac{1}{2}\pi J_n(z) + \frac{2^{n-1}}{z^n \pi} G_{2,4}^{3,2}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ n, \frac{1}{2}, \frac{1}{2}, 0 \end{array}\right) + \frac{1}{\pi}\sum_{k=0}^{n-1} \frac{1}{\left(\frac{1}{2}\right)_{n-k}} \left(\frac{1}{2}\right)_k \left(\frac{z}{2}\right)^{-2k+n-1} \left(\log\left(\frac{z}{2}\right) - \psi\left(n-k+\frac{1}{2}\right)\right) + \frac{1}{2} n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!\,(n-k)} \left(\frac{z}{2}\right)^{k-n} \boldsymbol{H}_{-k}(z) \;/;\; n \in \mathbb{N}$$

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03.09.20.0015.01

$$\boldsymbol{H}_{-n}^{(1,0)}(z) = \frac{(-1)^{n+1}\pi}{2} J_n(z) + \frac{(-1)^n 2^{n-1}}{z^n \pi} G_{2,4}^{3,2}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ n, \frac{1}{2}, \frac{1}{2}, 0 \end{array}\right) - \frac{n!}{2}\sum_{k=0}^{n-1} \frac{1}{k!\,(n-k)} \left(-\frac{z}{2}\right)^{k-n} \boldsymbol{H}_{-k}(z) \;/;\; n \in \mathbb{N}$$

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03.09.20.0016.01

$$\boldsymbol{H}_{n+\frac{1}{2}}^{(1,0)}(z) = \frac{1}{n!\sqrt{\pi}} \log\left(\frac{z}{2}\right)\left(\frac{z}{2}\right)^{n-\frac{1}{2}} {}_3F_0\left(-n, \frac{1}{2}, 1;; -\frac{4}{z^2}\right) - \frac{1}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{n-\frac{1}{2}} \sum_{k=0}^{n-1} \frac{\psi(-k+n+1)}{(n-k)!} \left(\frac{2}{z}\right)^{2k} \left(\frac{1}{2}\right)_k - \frac{n!\sqrt{\pi}}{2}\left(\frac{z}{2}\right)^{\frac{1}{2}-n} \sum_{k=0}^{n-1} \frac{1}{k!\,(n-k)} \left(\frac{z}{2}\right)^k \sum_{p=0}^{n-k-1} \frac{\left(\frac{z}{2}\right)^p}{p!} \left((-1)^{p+1} J_{k+\frac{1}{2}}(z)\left(2 J_{\frac{1}{2}-p}(z) - 2^{p+\frac{1}{2}} J_{\frac{1}{2}-p}(2z)\right) - (-1)^k J_{-k-\frac{1}{2}}(z)\left(2 J_{p-\frac{1}{2}}(z) - 2^{p+\frac{1}{2}} J_{p-\frac{1}{2}}(2z)\right)\right) + \frac{1}{2\pi}\Gamma\left(n+\frac{1}{2}\right)\left(\log(4) + \psi\left(\frac{1}{2}-n\right) + 3\gamma\right)\left(\frac{z}{2}\right)^{-n-\frac{1}{2}} - \frac{n!}{2\sqrt{\pi}}\left(\frac{z}{2}\right)^{-n-\frac{1}{2}} \sum_{k=0}^{n-1} \frac{\left(\frac{1}{2}\right)_k}{k!\,(n-k)} + (-1)^n J_{-n-\frac{1}{2}}(z)\,(\text{Ci}(2z) - 2\,\text{Ci}(z)) + J_{n+\frac{1}{2}}(z)\,(\text{Si}(2z) - 2\,\text{Si}(z)) - \frac{n!}{2}\left(\frac{2}{z}\right)^n \sum_{k=0}^{n-1} \frac{1}{(n-k)!\,k!}\left(-\frac{z}{2}\right)^k J_{-k-\frac{1}{2}}(z) \;/;\; n \in \mathbb{N}$$

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03.09.20.0017.01

$$\boldsymbol{H}_{-n-\frac{1}{2}}^{(1,0)}(z) = (-1)^n J_{n+\frac{1}{2}}(z)\,(2\,\text{Ci}(z) - \text{Ci}(2z)) + J_{-n-\frac{1}{2}}(z)\,(\text{Si}(2z) - 2\,\text{Si}(z)) - \frac{1}{2}((-1)^n n!) \sum_{k=0}^{n-1} \frac{1}{k!\,(n-k)}\left(\frac{z}{2}\right)^{k-n} J_{k+\frac{1}{2}}(z) + \frac{n!\sqrt{\pi}}{2} \sum_{k=1}^{n} \frac{2^k}{(n-k)!\,k} \sum_{p=0}^{k-1} \frac{z^{p-k+\frac{1}{2}}}{p!} \left((-1)^n \left(J_{p-\frac{1}{2}}(2z) - 2^{\frac{1}{2}-p} J_{p-\frac{1}{2}}(z)\right) J_{n-k+\frac{1}{2}}(z) - (-1)^{k+p}\left(J_{\frac{1}{2}-p}(2z) - 2^{\frac{1}{2}-p} J_{\frac{1}{2}-p}(z)\right) J_{k-n-\frac{1}{2}}(z)\right) \;/;\; n \in \mathbb{N}$$

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03.09.20.0002.01

$$\boldsymbol{H}_\nu^{(1,0)}(z) = \frac{2^{-\nu} z^{\nu+3}}{3\sqrt{\pi}\,(2\nu+3)\,\Gamma\left(\nu+\frac{5}{2}\right)} F_{3\times 0\times 1}^{1\times 1\times 2}\left(\begin{array}{c} 2;\, 1;\, 1, \nu+\frac{3}{2}; \\ 2, \frac{5}{2}, \nu+\frac{5}{2};;\, \nu+\frac{5}{2}; \end{array} -\frac{z^2}{4}, -\frac{z^2}{4}\right) + \left(\log(z) - \log(2) - \psi\left(\nu+\frac{3}{2}\right)\right)\boldsymbol{H}_\nu(z)$$

With respect to z

03.09.20.0003.01

$$\frac{\partial \boldsymbol{H}_\nu(z)}{\partial z} = \boldsymbol{H}_{\nu-1}(z) - \frac{\nu}{z}\boldsymbol{H}_\nu(z)$$

03.09.20.0004.01

$$\frac{\partial \boldsymbol{H}_\nu(z)}{\partial z} = \frac{2^{-\nu} z^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} - \boldsymbol{H}_{\nu+1}(z) + \frac{\nu}{z}\boldsymbol{H}_\nu(z)$$

03.09.20.0005.01

$$\frac{\partial \boldsymbol{H}_\nu(z)}{\partial z} = \frac{1}{2}\left(\frac{2^{-\nu} z^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} + \boldsymbol{H}_{\nu-1}(z) - \boldsymbol{H}_{\nu+1}(z)\right)$$

03.09.20.0006.01

$$\frac{\partial^2 \boldsymbol{H}_\nu(z)}{\partial z^2} = \frac{1}{z^2}\left(\boldsymbol{H}_{\nu-2}(z)\,z^2 + (z - 2\,z\,\nu)\,\boldsymbol{H}_{\nu-1}(z) + \nu\,(\nu+1)\,\boldsymbol{H}_\nu(z)\right)$$

03.09.20.0007.01

$$\frac{\partial^2 \boldsymbol{H}_\nu(z)}{\partial z^2} = \frac{1}{4}\left(\boldsymbol{H}_{\nu-2}(z) + \boldsymbol{H}_{\nu+2}(z) - 2\,\boldsymbol{H}_\nu(z)\right) + \frac{2^{-\nu-1}\left(8\,\nu^2 + 14\,\nu + 3 - z^2\right) z^{\nu-1}}{\sqrt{\pi}\,(4\,\nu\,(\nu+2)+3)\,\Gamma\left(\nu+\frac{1}{2}\right)}$$

03.09.20.0008.01

$$\frac{\partial\left(z^\nu\,\boldsymbol{H}_\nu(z)\right)}{\partial z} = z^\nu\,\boldsymbol{H}_{\nu-1}(z)$$

03.09.20.0009.01

$$\frac{\partial\left(z^{-\nu}\,\boldsymbol{H}_\nu(z)\right)}{\partial z} = \frac{2^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} - z^{-\nu}\,\boldsymbol{H}_{\nu+1}(z)$$

Symbolic differentiation

With respect to z

03.09.20.0018.01

$$\frac{\partial^n \boldsymbol{H}_\nu(z)}{\partial z^n} = \frac{n!}{\left(-\frac{z}{2}\right)^n}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^k}{2^{2k}\,k!\,(n-2k)!}\sum_{p=0}^{n-k}\binom{n-k}{p}\left(\frac{\nu}{2}\right)_{-k+n-p}\left(-\frac{z}{2}\right)^p \boldsymbol{H}_{\nu-p}(z)\;/;\;n\in\mathbb{N}$$

Brychkov Yu.A. (2005)

03.09.20.0019.01

$$\frac{\partial^n \boldsymbol{H}_\nu(z)}{\partial z^n} = \frac{n!}{\left(-\frac{z}{2}\right)^n}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^k}{2^{2k}\,k!\,(n-2k)!}\sum_{p=0}^{n-k}\binom{n-k}{p}\left(-\frac{\nu}{2}\right)_{-k+n-p}\left(\left(\frac{z}{2}\right)^p \boldsymbol{H}_{p+\nu}(z) - \frac{1}{\pi}\left(\frac{z}{2}\right)^{2p+\nu-1}\sum_{r=0}^{p-1}\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(p-r+\nu+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^{-2r}\right)/;$$

$n\in\mathbb{N}$

Brychkov Yu.A. (2005)

03.09.20.0020.01

$$\frac{\partial^n \boldsymbol{H}_\nu(z)}{\partial z^n} = z^{-n} \sum_{m=0}^{n} (-1)^{m+n} \binom{n}{m} (-\nu)_{n-m} \sum_{k=0}^{m} \frac{(-1)^{k-1}\, 2^{2k-m}\, (-m)_{2(m-k)}\, (\nu)_k}{(m-k)!} \left(\frac{z}{2} \sum_{j=0}^{k-1} \frac{(k-j-1)!\left(\frac{z^2}{4}\right)^j}{j!\,(k-2j-1)!\,(1-k-\nu)_j\,(\nu)_{j+1}} \boldsymbol{H}_{\nu-1}(z) - \sum_{j=0}^{k} \frac{(k-j)!\left(\frac{z^2}{4}\right)^j}{j!\,(k-2j)!\,(1-k-\nu)_j\,(\nu)_j} \boldsymbol{H}_\nu(z) \right) + \frac{2^{-\nu}\, z^{-n+\nu+1}}{\sqrt{\pi}\;\Gamma\left(\nu+\frac{1}{2}\right)} \sum_{i=1}^{n-1} \sum_{m=0}^{i} (-1)^{i+m} \binom{i}{m} (-\nu)_{i-m} \sum_{k=0}^{m} \frac{(-1)^{k-1}\, 2^{2k-m}\, (-m)_{2(m-k)}\, (\nu)_k}{(m-k)!} \sum_{j=0}^{k-1} \frac{2^{-2j}\,(k-j-1)!\,(2j-n+\nu+2)_{n-i-1}\, z^{2j}}{j!\,(k-2j-1)!\,(1-k-\nu)_j\,(\nu)_{j+1}} \;/;\; n \in \mathbb{N}$$

03.09.20.0010.02

$$\frac{\partial^n \boldsymbol{H}_\nu(z)}{\partial z^n} = 2^{n-2\nu-2} \sqrt{\pi}\; z^{\nu-n+1}\, \Gamma(\nu+2)\, {}_3\tilde{F}_4\left(1, \frac{\nu}{2}+1, \frac{\nu+3}{2}; \frac{3}{2}, \frac{\nu-n}{2}+1, \frac{\nu-n+3}{2}, \nu+\frac{3}{2}; -\frac{z^2}{4}\right) /;\; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

03.09.20.0011.01

$$\frac{\partial^\alpha \boldsymbol{H}_\nu(z)}{\partial z^\alpha} = 2^{\alpha-2\nu-2} \sqrt{\pi}\; z^{1-\alpha+\nu}\, \Gamma(\nu+2)\, {}_3\tilde{F}_4\left(1, \frac{\nu}{2}+1, \frac{\nu+3}{2}; \frac{3}{2}, \frac{\nu-\alpha}{2}+1, \frac{3+\nu-\alpha}{2}, \nu+\frac{3}{2}; -\frac{z^2}{4}\right) /;\; -\nu \notin \mathbb{N}^+$$

03.09.20.0012.01

$$\frac{\partial^\alpha \boldsymbol{H}_\nu(z)}{\partial z^\alpha} = (-1)^{-\left\lfloor \frac{\nu+1}{2} \right\rfloor} 2^{\alpha-2(\nu+1)+4\left\lfloor \frac{\nu+1}{2} \right\rfloor} \sqrt{\pi}\; \Gamma\left(\nu-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+2\right) {}_3\tilde{F}_4\left(1, \frac{1}{2}\left(\nu-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+2\right), \frac{1}{2}\left(\nu-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+3\right); \frac{3}{2}-\left\lfloor \frac{\nu+1}{2} \right\rfloor, \frac{1}{2}\left(\nu-\alpha-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+2\right), \frac{1}{2}\left(\nu-\alpha-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+3\right), \nu-\left\lfloor \frac{\nu+1}{2} \right\rfloor+\frac{3}{2}; -\frac{z^2}{4}\right) z^{\nu-\alpha-2\left\lfloor \frac{\nu+1}{2} \right\rfloor+1} + \sum_{k=0}^{-\left\lfloor \frac{\nu+3}{2} \right\rfloor} \frac{(-1)^{k+\nu}\, 2^{-2k-\nu-1}\, z^{2k-\alpha+\nu+1}\, (\log(z)+\psi(-2k-\nu-1)-\psi(2k-\alpha+\nu+2))}{(-2k-\nu-2)!\,\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)\Gamma(2k-\alpha+\nu+2)} /;\; -\nu \in \mathbb{N}^+$$

03.09.20.0013.01

$$\frac{\partial^\alpha \boldsymbol{H}_\nu(z)}{\partial z^\alpha} = \sum_{k=0}^{\infty} \frac{2^{-2k-\nu-1}\,(-1)^k\, \mathcal{FC}_{\exp}^{(\alpha)}(z, 2k+\nu+1)\, z^{2k-\alpha+\nu+1}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}$$

Integration

Indefinite integration

Involving only one direct function

03.09.21.0001.01

$$\int \boldsymbol{H}_\nu(z)\, dz = \frac{2^{-\nu}\, z^{\nu+2}}{\sqrt{\pi}\;(\nu+2)\,\Gamma\left(\nu+\frac{3}{2}\right)}\, {}_2F_3\left(1, \frac{\nu}{2}+1; \frac{3}{2}, \frac{\nu}{2}+2, \nu+\frac{3}{2}; -\frac{z^2}{4}\right)$$

Involving one direct function and elementary functions

Involving power function

03.09.21.0002.01

$$\int z^{\alpha-1}\,\boldsymbol{H}_\nu(z)\,dz = \frac{2^{-\nu}\,z^{\alpha+\nu+1}}{\sqrt{\pi}\,(\alpha+\nu+1)\,\Gamma\left(\nu+\frac{3}{2}\right)}\,{}_2F_3\left(1,\frac{\alpha+\nu+1}{2};\frac{3}{2},\frac{\alpha+\nu+3}{2},\nu+\frac{3}{2};-\frac{z^2}{4}\right)$$

03.09.21.0003.01

$$\int z^{1-\nu}\,\boldsymbol{H}_\nu(z)\,dz = \frac{2^{1-\nu}\,z}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - z^{1-\nu}\,\boldsymbol{H}_{\nu-1}(z)$$

03.09.21.0004.01

$$\int z^{1-\nu}\,\boldsymbol{H}_\nu(a\,z)\,dz = \frac{z^{1-\nu}}{a}\left(\frac{2^{1-\nu}\,(a\,z)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - \boldsymbol{H}_{\nu-1}(a\,z)\right)$$

03.09.21.0005.01

$$\int z^{\nu+1}\,\boldsymbol{H}_\nu(a\,z)\,dz = \frac{z^{\nu+1}}{a}\,\boldsymbol{H}_{\nu+1}(a\,z)$$

03.09.21.0006.01

$$\int \frac{\boldsymbol{H}_0(a\,z)}{z^2}\,dz = -\frac{1}{4}\,a\,G_{2,4}^{2,1}\left(\frac{a^2\,z^2}{4}\,\middle|\,\begin{array}{c}0,1\\0,0,-\frac{1}{2},-\frac{1}{2}\end{array}\right)$$

03.09.21.0007.01

$$\int z^n\,\boldsymbol{H}_\nu(a\,z)\,dz = 2^{-\nu-2}\,a\,z^{n+2}\,(a\,z)^\nu\,\Gamma\left(\frac{1}{2}(n+\nu+2)\right)\,{}_2\tilde{F}_3\left(1,\frac{1}{2}(n+\nu+2);\nu+\frac{3}{2},\frac{3}{2},\frac{1}{2}(n+\nu+4);-\frac{1}{4}\,a^2\,z^2\right)$$

03.09.21.0008.01

$$\int z^{\nu+3}\,\boldsymbol{H}_\nu(a\,z)\,dz = \frac{z^\nu\,(a\,z)^{-\nu}}{a^4}\left(2\,(\nu+1)\,\boldsymbol{H}_{\nu+2}(a\,z)\,(a\,z)^{\nu+2} - \boldsymbol{H}_{\nu+3}(a\,z)\,(a\,z)^{\nu+3} + \frac{2^{-\nu-1}\,(a\,z)^{2\nu+5}}{\sqrt{\pi}\,(2\,\nu+5)\,\Gamma\left(\nu+\frac{5}{2}\right)}\right)$$

Involving exponential function and a power function

03.09.21.0009.01

$$\int z^{-\nu}\,e^{i\,z}\,\boldsymbol{H}_\nu(z)\,dz = \frac{e^{i\,z}}{2\,\nu-1}\left(\frac{2^{1-\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} - z^{1-\nu}\,(i\,\boldsymbol{H}_{\nu-1}(z)+\boldsymbol{H}_\nu(z))\right)$$

03.09.21.0010.01

$$\int z^\nu\,e^{i\,z}\,\boldsymbol{H}_\nu(z)\,dz = \frac{z^\nu}{2\,\nu+1}\left(e^{i\,z}\,z\,(\boldsymbol{H}_\nu(z) - i\,\boldsymbol{H}_{\nu+1}(z)) - \frac{2^{-\nu}\,(-i\,z)^{-2\,\nu}\,z^\nu\,\Gamma(2\,\nu+2,-i\,z)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\right)$$

Involving functions of the direct function and elementary functions

Involving elementary functions of the direct function and elementary functions

Involving products of the direct function and a power function

03.09.21.0011.01

$$\int \frac{\boldsymbol{H}_\mu(z)\,\boldsymbol{H}_\nu(z)}{z}\,dz = \frac{1}{2(\mu-\nu)(\mu+\nu)}\Bigg(2\,\boldsymbol{H}_{\mu-1}(z)\,\boldsymbol{H}_\nu(z)\,z + \frac{1}{\sqrt{\pi}\,\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}\left(2^{-\mu-\nu}\,z^{\mu+\nu+2}\,\Gamma\left(\frac{1}{2}(\mu+\nu+2)\right)\left(\Gamma\left(\mu+\frac{1}{2}\right){}_2\tilde{F}_3\left(1,\,\frac{1}{2}(\mu+\nu+2);\,\mu+\frac{3}{2},\,\frac{3}{2},\,\frac{1}{2}(\mu+\nu+4);\,-\frac{z^2}{4}\right) - \Gamma\left(\nu+\frac{1}{2}\right){}_2\tilde{F}_3\left(1,\,\frac{1}{2}(\mu+\nu+2);\,\nu+\frac{3}{2},\,\frac{3}{2},\,\frac{1}{2}(\mu+\nu+4);\,-\frac{z^2}{4}\right)\right)\right) - 2\,\boldsymbol{H}_\mu(z)\,(z\,\boldsymbol{H}_{\nu-1}(z)+(\mu-\nu)\,\boldsymbol{H}_\nu(z))\Bigg)$$

03.09.21.0012.01

$$\int z^{1-\mu-\nu}\,\boldsymbol{H}_\mu(z)\,\boldsymbol{H}_\nu(z)\,dz = -\frac{1}{2(\mu+\nu-1)}\Bigg(z^{-\mu-\nu+2}\left(\boldsymbol{H}_{\mu-1}(z)\,\boldsymbol{H}_{\nu-1}(z)+\boldsymbol{H}_\mu(z)\,\boldsymbol{H}_\nu(z)\right) - \frac{2^{-\mu-\nu+1}\,z^2}{\pi\,\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}\left({}_2F_3\left(1,\,1;\,\frac{3}{2},\,2,\,\mu+\frac{1}{2};\,-\frac{z^2}{4}\right)+{}_2F_3\left(1,\,1;\,\frac{3}{2},\,2,\,\nu+\frac{1}{2};\,-\frac{z^2}{4}\right)\right)\Bigg)$$

03.09.21.0013.01

$$\int z^{\mu+\nu+1}\,\boldsymbol{H}_\mu(z)\,\boldsymbol{H}_\nu(z)\,dz = \frac{1}{2(\mu+\nu+1)}\Bigg(z^{\mu+\nu+2}\left(\boldsymbol{H}_\mu(z)\,\boldsymbol{H}_\nu(z)+\boldsymbol{H}_{\mu+1}(z)\,\boldsymbol{H}_{\nu+1}(z)\right) - \frac{2^{-\mu-\nu-1}\,z^{2(\mu+\nu+2)}}{\pi(\mu+\nu+2)\,\Gamma\left(\mu+\frac{3}{2}\right)\Gamma\left(\nu+\frac{3}{2}\right)}\left({}_2F_3\left(1,\,\mu+\nu+2;\,\frac{3}{2},\,\mu+\frac{3}{2},\,\mu+\nu+3;\,-\frac{z^2}{4}\right)+{}_2F_3\left(1,\,\mu+\nu+2;\,\frac{3}{2},\,\nu+\frac{3}{2},\,\mu+\nu+3;\,-\frac{z^2}{4}\right)\right)\Bigg)$$

Involving direct function and Bessel-, Airy-, Struve-type functions

Involving Bessel functions

Involving Bessel J and power

03.09.21.0014.01

$$\int \frac{J_\nu(z)\,\boldsymbol{H}_\mu(z)}{z}\,dz = \frac{1}{(\mu-\nu)(\mu+\nu)}\left(J_\nu(z)\,\boldsymbol{H}_{\mu-1}(z)\,z - \frac{1}{\sqrt{\pi}\,\Gamma\left(\mu+\frac{1}{2}\right)}\left(2^{-\mu-\nu}\,z^{\mu+\nu+1}\,\Gamma\left(\frac{1}{2}(\mu+\nu+1)\right){}_1\tilde{F}_2\left(\frac{1}{2}(\mu+\nu+1);\,\nu+1,\,\frac{1}{2}(\mu+\nu+3);\,-\frac{z^2}{4}\right)\right) - (z\,J_{\nu-1}(z)+(\mu-\nu)\,J_\nu(z))\,\boldsymbol{H}_\mu(z)\right)$$

03.09.21.0015.01

$$\int z^{1-\mu-\nu} J_\nu(z)\, \boldsymbol{H}_\mu(z)\, dz = -\frac{1}{2(\mu+\nu-1)} \left(z^{-\mu-\nu+2} \left(J_{\nu-1}(z)\, \boldsymbol{H}_{\mu-1}(z) + J_\nu(z)\, \boldsymbol{H}_\mu(z) \right) - \frac{2^{1-\mu-\nu} z}{\Gamma\left(\mu+\frac{1}{2}\right)}\, {}_1\tilde{F}_2\left(\frac{1}{2}; \nu, \frac{3}{2}; -\frac{z^2}{4} \right) \right)$$

03.09.21.0016.01

$$\int z^{\mu+\nu+1} J_\nu(z)\, \boldsymbol{H}_\mu(z)\, dz =$$

$$\frac{z^{\mu+\nu+2}}{4(\mu+\nu+1)} \left(-\frac{2^{-\mu-\nu} z^{\mu+\nu+1}\, \Gamma\left(\mu+\nu+\frac{3}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\mu+\frac{3}{2}\right)}\, {}_1\tilde{F}_2\left(\mu+\nu+\frac{3}{2}; \nu+1, \mu+\nu+\frac{5}{2}; -\frac{z^2}{4} \right) + 2 J_\nu(z)\, \boldsymbol{H}_\mu(z) + 2 J_{\nu+1}(z)\, \boldsymbol{H}_{\mu+1}(z) \right)$$

Definite integration

For the direct function itself

03.09.21.0017.01

$$\int_0^\infty \boldsymbol{H}_\nu(t)\, dt = -\cot\left(\frac{\pi\nu}{2}\right) /;\ \mathrm{Re}(\nu) > -2$$

03.09.21.0018.01

$$\int_0^\infty t^{\alpha-1} \boldsymbol{H}_\nu(t)\, dt = \frac{2^{\alpha-1} \pi \sec\left(\frac{1}{2}\pi(\alpha+\nu)\right)}{\Gamma\left(1-\frac{\alpha}{2}-\frac{\nu}{2}\right) \Gamma\left(\frac{1}{2}(2-\alpha+\nu)\right)} /;\ \mathrm{Re}(\alpha+\nu) > -1 \bigwedge \mathrm{Re}(\alpha) < \frac{3}{2}$$

03.09.21.0019.01

$$\int_0^\infty \frac{\boldsymbol{H}_0(t)}{t}\, dt = \frac{\pi}{2}$$

Involving the direct function

03.09.21.0020.01

$$\int_0^\infty t^{\alpha-1} J_\nu(a\,t)\, \boldsymbol{H}_\mu(b\,t)\, dt = \frac{2^\alpha a^{-\alpha-\mu-1} b^{\mu+1}}{\sqrt{\pi}\, \Gamma\left(\mu+\frac{3}{2}\right) \Gamma\left(\frac{1}{2}(-\alpha-\mu+\nu+1)\right)} \Gamma\left(\frac{1}{2}(\alpha+\mu+\nu+1)\right)$$

$${}_3F_2\left(1, \frac{1}{2}(\alpha+\mu+\nu+1), \frac{1}{2}(\alpha+\mu-\nu+1); \mu+\frac{3}{2}, \frac{3}{2}; \frac{b^2}{a^2}\right) /;\ a > 0 \land b > 0 \land \mathrm{Re}(\mu+\nu) > -1 \land \mathrm{Re}(\alpha) < 1$$

Integral transforms

Fourier cos transforms

03.09.22.0001.01

$$\mathcal{F}c_t[\boldsymbol{H}_\nu(t)](z) = -\frac{2^{\frac{1}{2}-\nu} z^{-\nu-2}\, \Gamma(\nu+2)}{\pi\, \Gamma\left(\nu+\frac{3}{2}\right)} \cos\left(\frac{\pi\nu}{2}\right) {}_3F_2\left(1, \frac{\nu+2}{2}, \frac{\nu+3}{2}; \nu+\frac{3}{2}, \frac{3}{2}; \frac{1}{z^2}\right) /;\ \mathrm{Re}(\nu) > -2$$

Fourier sin transforms

03.09.22.0002.01

$$\mathcal{F}s_t[\boldsymbol{H}_\nu(t)](z) = -\frac{2^{2^{\frac{1}{2}-\nu}} z^{-\nu-2}\,\Gamma(\nu+2)}{\pi\,\Gamma\left(\nu+\frac{3}{2}\right)} \sin\left(\frac{\pi\,\nu}{2}\right) {}_3F_2\left(1, \frac{\nu+3}{2}, \frac{\nu+2}{2}; \nu+\frac{3}{2}, \frac{3}{2}; \frac{1}{z^2}\right) /; \operatorname{Re}(\nu) > -3$$

Laplace transforms

03.09.22.0003.01

$$\mathcal{L}_t[\boldsymbol{H}_\nu(t)](z) = \frac{2^{-\nu} z^{-\nu-2}\,\Gamma(\nu+2)}{\sqrt{\pi}\;\Gamma\left(\nu+\frac{3}{2}\right)} {}_3F_2\left(1, \frac{\nu+3}{2}, \frac{\nu+2}{2}; \nu+\frac{3}{2}, \frac{3}{2}; -\frac{1}{z^2}\right) /; \operatorname{Re}(\nu) > -2$$

Mellin transforms

03.09.22.0004.01

$$\mathcal{M}_t[\boldsymbol{H}_\nu(t)](z) = \frac{2^{z-1}\,\pi}{\Gamma\left(-\frac{z}{2}-\frac{\nu}{2}+1\right)\Gamma\left(\frac{1}{2}(-z+\nu+2)\right)} \sec\left(\frac{1}{2}\pi(z+\nu)\right) /; \operatorname{Re}(z+\nu) > -1 \bigwedge \operatorname{Re}(z) < \frac{3}{2}$$

Hankel transforms

03.09.22.0005.01

$$\mathcal{H}_{t;\mu}[\boldsymbol{H}_\nu(t)](z) = \frac{2\sqrt{\frac{2}{\pi}}\; z^{-\nu-2}\,\Gamma\left(\frac{1}{4}(2\mu+2\nu+5)\right)}{\Gamma\left(\frac{1}{4}(2\mu-2\nu-1)\right)\Gamma\left(\nu+\frac{3}{2}\right)} {}_3F_2\left(1, \frac{1}{4}(2\mu+2\nu+5), \frac{1}{4}(-2\mu+2\nu+5); \nu+\frac{3}{2}, \frac{3}{2}; \frac{1}{z^2}\right) /;$$

$$\operatorname{Re}(\mu+\nu) > -\frac{5}{2}$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_p\tilde{F}_q$

03.09.26.0001.01

$$\boldsymbol{H}_\nu(z) = \left(\frac{z}{2}\right)^{\nu+1} {}_1\tilde{F}_2\left(1; \frac{3}{2}, \nu+\frac{3}{2}; -\frac{z^2}{4}\right)$$

03.09.26.0010.01

$$\boldsymbol{H}_\nu(z) = (-1)^{-\nu-\frac{1}{2}} \left(\frac{z}{2}\right)^{-\nu} {}_0\tilde{F}_1\left(; 1-\nu; -\frac{z^2}{4}\right) /; -\nu-\frac{3}{2} \in \mathbb{N}$$

Involving ${}_pF_q$

03.09.26.0002.01

$$\boldsymbol{H}_\nu(z) = \frac{z^{\nu+1}}{2^\nu\sqrt{\pi}\;\Gamma\left(\nu+\frac{3}{2}\right)} {}_1F_2\left(1; \frac{3}{2}, \nu+\frac{3}{2}; -\frac{z^2}{4}\right) /; -\nu-\frac{3}{2} \notin \mathbb{N}$$

03.09.26.0011.01

$$\mathbf{H}_\nu(z) = \frac{(-1)^{-\nu-\frac{1}{2}}}{\Gamma(1-\nu)} \left(\frac{z}{2}\right)^{-\nu} {}_0F_1\left(;1-\nu;-\frac{z^2}{4}\right) /; -\nu-\frac{3}{2} \in \mathbb{N}$$

Through Meijer *G*

Classical cases for the direct function itself

03.09.26.0003.01

$$\mathbf{H}_\nu(z) = z^{\nu+1} \left(z^2\right)^{-\frac{\nu+1}{2}} G_{1,3}^{1,1}\left(\frac{z^2}{4} \middle| \begin{matrix} \frac{\nu+1}{2} \\ \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

03.09.26.0004.01

$$\mathbf{H}_\nu(z) = G_{1,3}^{1,1}\left(\frac{z^2}{4} \middle| \begin{matrix} \frac{\nu+1}{2} \\ \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right) /; \operatorname{Re}(z) > 0$$

03.09.26.0005.01

$$\mathbf{H}_\nu\left(\sqrt{z}\right) = G_{1,3}^{1,1}\left(\frac{z}{4} \middle| \begin{matrix} \frac{\nu+1}{2} \\ \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

Classical cases involving Bessel *Y*

03.09.26.0006.01

$$Y_\nu\left(\sqrt{z}\right) - \mathbf{H}_\nu\left(\sqrt{z}\right) = -\frac{\cos(\nu\pi)}{\pi^2} G_{1,3}^{3,1}\left(\frac{z}{4} \middle| \begin{matrix} \frac{\nu+1}{2} \\ \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

Generalized cases for the direct function itself

03.09.26.0007.01

$$\mathbf{H}_\nu(z) = G_{1,3}^{1,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{matrix} \frac{\nu+1}{2} \\ \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

Generalized cases involving Bessel *Y*

03.09.26.0008.01

$$Y_\nu(z) - \mathbf{H}_\nu(z) = -\frac{\cos(\nu\pi)}{\pi^2} G_{1,3}^{3,1}\left(\frac{z}{2}, \frac{1}{2} \middle| \begin{matrix} \frac{\nu+1}{2} \\ \frac{\nu+1}{2}, -\frac{\nu}{2}, \frac{\nu}{2} \end{matrix}\right)$$

Through other functions

03.09.26.0009.01

$$\mathbf{H}_\nu(z) = z\csc(\pi\nu)\left(\frac{\sqrt{\pi}}{\Gamma(1-\nu)\,\Gamma\left(\nu+\frac{1}{2}\right)} J_\nu(z)\,{}_1F_2\left(\frac{1}{2}; \frac{3}{2}, 1-\nu; -\frac{z^2}{4}\right) - \frac{z^{2\nu}}{\Gamma(2(\nu+1))} J_{-\nu}(z)\,{}_1F_2\left(\nu+\frac{1}{2}; \nu+1, \nu+\frac{3}{2}; -\frac{z^2}{4}\right)\right)$$

Representations through equivalent functions

With related functions

03.09.27.0001.01

$$\mathbf{H}_\nu(i\,z) = i\,(i\,z)^\nu\,z^{-\nu}\,\mathbf{L}_\nu(z)$$

03.09.27.0002.01

$$\mathbf{H}_\nu(-i\,z) = -i\,(-i\,z)^\nu\,z^{-\nu}\,\mathbf{L}_\nu(z)$$

03.09.27.0003.01

$$\mathbf{H}_\nu(z) = Y_\nu(z) + \frac{1}{\left(\nu-\frac{1}{2}\right)!\,\sqrt{\pi}}\left(\frac{z}{2}\right)^{\nu-1}\sum_{k=0}^{\nu-\frac{1}{2}}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}-\nu\right)_k\left(-\frac{z^2}{4}\right)^{-k} /;\ \nu-\frac{1}{2}\in\mathbb{Z}$$

03.09.27.0004.01

$$\mathbf{H}_\nu(z) = (-1)^{\nu+\frac{1}{2}}\,J_{-\nu}(z) /;\ -\nu-\frac{1}{2}\in\mathbb{N}$$

Inequalities

03.09.29.0001.01

$$\mathbf{H}_\nu(x) \geq 0 /;\ x>0 \bigwedge \nu \geq \frac{1}{2}$$

Theorems

Struve H-Transformation

$$\hat{f}_\nu(y) = \int_0^\infty f(x)\sqrt{x\,y}\;\mathbf{H}_\nu(x\,y)\,dx \Leftrightarrow f(x) = \int_0^\infty \hat{f}_\nu(y)\sqrt{x\,y}\;Y_\nu(x\,y)\,dy /;\ \operatorname{Re}(\nu)\geq -\frac{1}{2}.$$

History

– H. Struve (1882)

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