WeierstrassP

Notations

Traditional name

Weierstrass elliptic function

Traditional notation

$\wp(z; g_2, g_3)$

Mathematica StandardForm notation

`WeierstrassP[z, {g2, g3}]`

Primary definition

09.13.02.0001.01

$$\wp(z; g_2, g_3) == \frac{1}{z^2} + \sum_{\substack{m, n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(z - 2\, m\, \omega_1 - 2\, n\, \omega_3)^2} - \frac{1}{(2\, m\, \omega_1 + 2\, n\, \omega_3)^2} \;/;\; \{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

Special notations for this file:

09.13.02.0002.01

$$\{\omega_1, \omega_3\} == \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.13.02.0003.01

$$\{\omega_1, \omega_2, \omega_3\} == \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\}$$

09.13.02.0004.01

$$e_n == \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.13.02.0005.01

$$\eta_n == \zeta(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.13.02.0006.01

$$q == \exp\left(\frac{\pi\, i\, \omega_3}{\omega_1}\right)$$

Specific values

Specialized values

For fixed *z*

Degenerate case:

09.13.03.0001.01

$$\wp(z;\,0,\,0) = \frac{1}{z^2}$$

09.13.03.0002.01

$$\wp(z;\,3,\,1) = \frac{3}{2}\cot^2\left(\sqrt{\frac{3}{2}}\,z\right) + 1$$

For fixed $\{g_2, g_3\}$

Values at quarter-periods

09.13.03.0003.01

$$\wp\left(\frac{\omega_i}{2};\,g_2,\,g_3\right) = e_i + \epsilon_{i,j}\,\epsilon_{i,k}\sqrt{e_i - e_j}\,\sqrt{e_i - e_k}\ /;\ \{i,\,j,\,k\} \in \{1,\,2,\,3\} \bigwedge i \neq j \neq k \bigwedge \epsilon_{\alpha,\beta} = \operatorname{sgn}\left(\frac{\pi}{2} - \left|\arg\left(\frac{\sigma_\beta(\omega_\alpha;\,g_2,\,g_3)}{\sigma(\omega_\alpha;\,g_2,\,g_3)}\right)\right|\right)$$

Values at half-periods

09.13.03.0004.01

$$\{\wp(\omega_1;\,g_2,\,g_3),\,\wp(\omega_2;\,g_2,\,g_3),\,\wp(\omega_3;\,g_2,\,g_3)\} = \{e_1(g_2,\,g_3),\,e_2(g_2,\,g_3),\,e_3(g_2,\,g_3)\}$$

Values at poles

09.13.03.0005.01

$$\wp(2\,m\,\omega_1 + 2\,n\,\omega_3;\,g_2,\,g_3) = \tilde{\infty}\ /;\ \{m,\,n\} \in \mathbb{Z}$$

Values at fixed points

Equianharmonic case $\{g_2, g_3\} = \{0, 1\}$

09.13.03.0006.01

$$\wp(\omega_1;\,0,\,1) = \frac{1}{\sqrt[3]{4}}$$

09.13.03.0007.01

$$\wp(\omega_2;\,0,\,1) = 4^{-1/3}\,e^{4\pi i/3}$$

09.13.03.0008.01

$$\wp(\omega_3;\,0,\,1) = 4^{-1/3}\,e^{2\pi i/3}$$

09.13.03.0009.01

$$\left(\wp\left(\frac{\omega_i}{2};\,0,\,1\right) - e_i\right)^2 = (e_i - e_j)\,(e_i - e_k)\ /;\ \{i,\,j,\,k\} \in \{1,\,2,\,3\} \bigwedge i \neq j \neq k \bigwedge \{g_2,\,g_3\} = \{0,\,1\}$$

Lemniscatic case $\{g_2, g_3\} = \{1, 0\}$

09.13.03.0010.01

$$\wp(\omega_1;\,1,\,0) = \frac{1}{2}$$

09.13.03.0011.01

$$\wp(\omega_2; 1, 0) == 0$$

09.13.03.0012.01

$$\wp(\omega_3; 1, 0) == -\frac{1}{2}$$

Values at infinities

09.13.03.0013.01

$$\wp(z; g_2(\omega_1, \tilde{\infty}), g_3(\omega_1, \tilde{\infty})) == \left(\frac{\pi}{2\,\omega_1}\right)^2 \left(\frac{1}{\sin^2\left(\frac{\pi z}{2\,\omega_1}\right)} - \frac{1}{3}\right)$$

09.13.03.0014.01

$$\wp(z; g_2(\tilde{\infty}, \omega_3), g_3(\tilde{\infty}, \omega_3)) == \left(\frac{\pi}{2\,\omega_3}\right)^2 \left(\frac{1}{\sin^2\left(\frac{\pi z}{2\,\omega_3}\right)} - \frac{1}{3}\right)$$

09.13.03.0015.01

$$\wp(z; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) == \frac{1}{z^2}$$

09.13.03.0016.01

$$\wp(\tilde{\infty}; g_2(\tilde{\infty}, \tilde{\infty}), g_3(\tilde{\infty}, \tilde{\infty})) == 0$$

09.13.03.0017.01

$$\wp(\omega_1; g_2, g_3) == \frac{3\,g_3}{g_2} \;/;\; \omega_3 == \tilde{\infty}$$

09.13.03.0018.01

$$\wp(-\omega_1 - \omega_3; g_2, g_3) == -\frac{3\,g_3}{g_2} \;/;\; \omega_3 == \tilde{\infty}$$

09.13.03.0019.01

$$\wp(\omega_3; g_2, g_3) == -\frac{3\,g_3}{g_2} \;/;\; \omega_3 == \tilde{\infty}$$

General characteristics

Domain and analyticity

$\wp(z; g_2, g_3)$ is an analytical function of z, g_2, and g_3, which is defined in $\mathbb{C}^3$.

09.13.04.0001.01

$$(z * \{g_2 * g_3\}) \longrightarrow \wp(z; g_2, g_3) :: (\mathbb{C} \otimes \{\mathbb{C} \otimes \mathbb{C}\}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\wp(z; g_2, g_3)$ is an even function with respect to z.

09.13.04.0002.01

$\wp(z; g_2, g_3) == \wp(-z; g_2, g_3)$

Mirror symmetry

09.13.04.0003.01

$\wp(\bar{z}; \overline{g_2}, \overline{g_3}) == \overline{\wp(z; g_2, g_3)}$

Periodicity

$\wp(z; g_2, g_3)$ is a doubly periodic function with respect to z with periods $2\,\omega_1$ and $2\,\omega_3$.

09.13.04.0004.01

$\wp(z + 2\,m\,\omega_1 + 2\,n\,\omega_3; g_2, g_3) == \wp(z; g_2, g_3)\ /;\ \{m, n\} \in \mathbb{Z}$

Transformation of half-periods

09.13.04.0005.01

$\wp(z; g_2(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3), g_3(a\,\omega_1 + b\,\omega_3, c\,\omega_1 + d\,\omega_3)) == \wp(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))\ /;$
$\{a, b, c, d\} \in \mathbb{Z} \bigwedge a\,d - b\,c == \pm 1$

Homogeneity

09.13.04.0006.01

$$\wp(z\,t; g_2, g_3) == \frac{1}{t^2}\,\wp\left(z; g_2\,t^4, g_3\,t^6\right)\ /;\ t \in \mathbb{R}$$

09.13.04.0007.01

$$\wp(\lambda\,z; g_2(\lambda\,\omega_1, \lambda\,\omega_3), g_3(\lambda\,\omega_1, \lambda\,\omega_3)) == \wp\left(\lambda\,z; \frac{g_2(\omega_1, \omega_3)}{\lambda^4}, \frac{g_3(\omega_1, \omega_3)}{\lambda^6}\right)$$

09.13.04.0008.01

$$\wp(\lambda\,z; g_2(\lambda\,\omega_1, \lambda\,\omega_3), g_3(\lambda\,\omega_1, \lambda\,\omega_3)) == \frac{1}{\lambda^2}\,\wp(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$$

Poles and essential singularities

With respect to *z*

For fixed g_2, g_3, the function $\wp(z; g_2, g_3)$ has an infinite set of singular points:

a) $z == 2\,m\,\omega_1(g_2, g_3) + 2\,n\,\omega_3(g_2, g_3)$, $\{m, n\} \in \mathbb{Z}$, are the poles of order two with residues 0;

b) $z == \tilde{\infty}$ is an essential singular point.

09.13.04.0009.01

$\mathcal{S}ing_z(\wp(z; g_2, g_3)) == \{\{\{2\,m\,\omega_1 + 2\,n\,\omega_3, 2\}\ /;\ \{m, n\} \in \mathbb{Z}\}, \{\tilde{\infty}, \infty\}\}$

09.13.04.0010.01

$\mathrm{res}_z(\wp(z; g_2, g_3))\,(2\,m\,\omega_1 + 2\,n\,\omega_3) == 0\ /;\ \{m, n\} \in \mathbb{Z}$

Branch points

With respect to *z*

For fixed g_2, g_3, the function $\wp(z; g_2, g_3)$ does not have branch points.

09.13.04.0011.01

$$\mathcal{BP}_z(\wp(z; g_2, g_3)) == \{\}$$

Branch cuts

With respect to z

For fixed g_2, g_3, the function $\wp(z; g_2, g_3)$ does not have branch cuts.

09.13.04.0012.01

$$\mathcal{BC}_z(\wp(z; g_2, g_3)) == \{\}$$

Series representations

Generalized power series

Expansions at z == 0

09.13.06.0013.01

$$\wp(z; g_2, g_3) \propto \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \ldots \;/;\; (z \to 0)$$

09.13.06.0014.01

$$\wp(z; g_2, g_3) \propto \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + O\left(z^6\right)$$

09.13.06.0001.01

$$\wp(z; g_2, g_3) == \frac{1}{z^2} + \sum_{k=2}^{\infty} a_k z^{2k-2} \;/;\; a_2 == \frac{g_2}{20} \bigwedge a_3 == \frac{g_3}{28} \bigwedge a_k == \frac{3}{(2k+1)(k-3)} \sum_{l=2}^{k-2} a_l a_{k-l}$$

09.13.06.0002.01

$$\wp(z; g_2, g_3) == \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) \sum_{\substack{m,n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2m\omega_1 + 2n\omega_3)^{2k+2}} z^{2k}$$

09.13.06.0015.01

$$\wp(z; g_2, g_3) \propto \frac{1}{z^2} \left(1 + O\left(z^4\right)\right)$$

q-series

09.13.06.0003.01

$$\wp(z; g_2, g_3) == -\frac{\eta_1}{\omega_1} + \left(\frac{\pi}{2\omega_1}\right)^2 \csc^2\left(\frac{\pi z}{2\omega_1}\right) - \frac{2\pi^2}{\omega_1^2} \sum_{k=1}^{\infty} \frac{k q^{2k}}{1 - q^{2k}} \cos\left(\frac{k\pi z}{\omega_1}\right)$$

09.13.06.0004.01

$$\wp(z + \omega_1; g_2, g_3) == -\frac{\eta_1}{\omega_1} + \left(\frac{\pi}{2\omega_1}\right)^2 \sec^2\left(\frac{\pi z}{2\omega_1}\right) - \frac{2\pi^2}{\omega_1^2} \sum_{k=1}^{\infty} (-1)^k \frac{k q^{2k}}{1 - q^{2k}} \cos\left(\frac{k\pi z}{\omega_1}\right)$$

09.13.06.0005.01

$$\wp(z + \omega_2; g_2, g_3) == -\frac{\eta_1}{\omega_1} - \frac{2\pi^2}{\omega_1^2} \sum_{k=1}^{\infty} (-1)^k \frac{k q^k}{1 - q^{2k}} \cos\left(\frac{k\pi z}{\omega_1}\right)$$

09.13.06.0006.01

$$\wp(z+\omega_3; g_2, g_3) = -\frac{\eta_1}{\omega_1} - \frac{2\pi^2}{\omega_1^2} \sum_{k=1}^{\infty} \frac{k\, q^k}{1-q^{2k}} \cos\left(\frac{k\pi z}{\omega_1}\right)$$

Other series representations

09.13.06.0007.01

$$\wp(z; g_2, g_3) = \frac{1}{z^2} + \sum_{\substack{m,n=-\infty \\ \{m,n\}\neq\{0,0\}}}^{\infty} \frac{1}{(z-2m\omega_1-2n\omega_3)^2} - \frac{1}{(2m\omega_1+2n\omega_3)^2}$$

09.13.06.0008.01

$$\wp(z; g_2, g_3) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(z+2m\omega_1+2n\omega_3)^2} - \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m\omega_1+n\omega_3)^2} - \frac{\pi^2}{12\,\omega_1^2}$$

09.13.06.0009.01

$$\wp(z; g_2, g_3) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(z+2m\omega_1+2n\omega_3)^2} - \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(m\omega_1+n\omega_3)^2} - \frac{\pi^2}{12\,\omega_3^2}$$

09.13.06.0010.01

$$\wp(z; g_2, g_3)^2 = \frac{g_2}{12} + \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{1}{(z-2m\omega_1-2n\omega_3)^4}$$

09.13.06.0011.01

$$\wp(z; g_2, g_3) = -\frac{\eta_i}{\omega_i} + \frac{\pi^2}{4\,\omega_i^2} \sum_{n=-\infty}^{\infty} \csc^2\left(\frac{\pi\left(z-2n\omega_j\right)}{2\,\omega_i}\right) /;\ \{i, j\} \in \{1, 2, 3\} \land i \neq j$$

09.13.06.0012.01

$$\wp(z; g_2, g_3) = \left(\frac{\pi}{2\,\omega_1}\right)^2 \left(-\frac{1}{3} + \sum_{k=-\infty}^{\infty} \csc^2\left(\frac{\pi(z-2k\omega_3)}{2\,\omega_1}\right) - 2\sum_{k=1}^{\infty} \csc^2\left(\frac{\pi k\,\omega_3}{\omega_1}\right)\right)$$

Integral representations

On the real axis

Of the direct function

09.13.07.0001.01

$$\wp(z; g_2, g_3) = \frac{1}{z^2} + \frac{1}{4}\int_0^{\infty} t\left(\left(\left(\cosh(t\,\omega_3) + e^{-\frac{t}{2}\omega_3}\sinh\left(\frac{t\,\omega_3}{2}\right)\right)\left(\cosh\left(\frac{t\,z}{2}\right)-1\right)\right)\Big/\left(\sinh\left(\frac{1}{2}t(\omega_1-\omega_3)\right)\sinh\left(\frac{1}{2}t(\omega_1+\omega_3)\right)\right) + \left(e^{\frac{i\,t}{2}\omega_3}\left(1-\cos\left(\frac{t\,z}{2}\right)\right)\cos\left(\frac{t\,\omega_3}{2}\right)\right)\Big/\left(\sin\left(\frac{1}{2}t(\omega_1-\omega_3)\right)\sin\left(\frac{1}{2}t(\omega_1+\omega_3)\right)\right)\right)dt$$

Involving related functions

09.13.07.0002.01

$$\wp(z; g_2, g_3) = \frac{1}{z^2} + \int_0^z \left(\wp'(t; g_2, g_3) + \frac{2}{t^3}\right) dt$$

09.13.07.0003.01

$$\wp(z; g_2, g_3) == w \,/;\, z == \int_{\infty}^{w} \frac{1}{\sqrt{4\,t^3 - g_2\,t - g_3}}\,dt \bigwedge w \in \mathbb{R}$$

09.13.07.0004.01

$$z == \int_{\wp(z;g_2,g_3)}^{\infty} \frac{dt}{\sqrt{4\,t^3 - g_2\,t - g_3}}$$

Product representations

09.13.08.0001.01

$$\wp(z; g_2, g_3) == e_i + \frac{\pi^2}{4\,\omega_i^2} \cot^2\left(\frac{\pi\,z}{2\,\omega_i}\right) \prod_{k=1}^{\infty} \tan^4\left(\frac{k\,\pi\,\omega_j}{\omega_i}\right) \left(\frac{\cos^2\left(\frac{k\,\pi\,\omega_j}{\omega_i}\right) - \sin^2\left(\frac{\pi\,z}{2\,\omega_i}\right)}{\sin^2\left(\frac{k\,\pi\,\omega_j}{\omega_i}\right) - \sin^2\left(\frac{\pi\,z}{2\,\omega_i}\right)}\right)^2 \,/;\, \{i, j\} \in \{1, 2, 3\} \bigwedge i \neq j$$

09.13.08.0002.01

$$\wp(z; g_2, g_3) == e_1 + \frac{\pi^2}{4\,\omega_1^2} \cot^2\left(\frac{\pi\,z}{2\,\omega_1}\right) \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 + q^{2n}}\right)^4 \left(\frac{1 + 2\,q^{2n} \cos\left(\frac{\pi\,z}{\omega_1}\right) + q^{4n}}{1 - 2\,q^{2n} \cos\left(\frac{\pi\,z}{\omega_1}\right) + q^{4n}}\right)^2$$

09.13.08.0003.01

$$\wp(z; g_2, g_3) == e_2 + \frac{\pi^2}{4\,\omega_1^2} \csc^2\left(\frac{\pi\,z}{2\,\omega_1}\right) \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 + q^{2n-1}}\right)^4 \left(\frac{1 + 2\,q^{2n-1} \cos\left(\frac{\pi\,z}{\omega_1}\right) + q^{4n-2}}{1 - 2\,q^{2n} \cos\left(\frac{\pi\,z}{\omega_1}\right) + q^{4n}}\right)^2$$

09.13.08.0004.01

$$\wp(z; g_2, g_3) == e_3 + \frac{\pi^2}{4\,\omega_1^2} \csc^2\left(\frac{\pi\,z}{2\,\omega_1}\right) \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n}}{1 - q^{2n-1}}\right)^4 \left(\frac{1 - 2\,q^{2n-1} \cos\left(\frac{\pi\,z}{\omega_1}\right) + q^{4n-2}}{1 - 2\,q^{2n} \cos\left(\frac{\pi\,z}{\omega_1}\right) + q^{4n}}\right)^2$$

Differential equations

Ordinary nonlinear differential equations

Every solution of the following nonlinear equation is of the form $\wp(z + a; g_2, g_3)$ for some $\alpha \in \mathbb{C}$.

09.13.13.0001.01

$$w'(z)^2 - 4\,w(z)^3 + g_2\,w(z) + g_3 \,/;\, w(z) == \wp(z + a; g_2, g_3)$$

09.13.13.0002.01

$$w''(z) - 6\,w(z)^2 + \frac{g_2}{2} == 0 \,/;\, w(z) == \wp(a + z; g_2, g_3)$$

09.13.13.0003.01

$$w^{(3)}(z) - 12\,w(z)\,w'(z) == 0 \,/;\, w(z) == \wp(a + z; g_2, g_3)$$

09.13.13.0004.01

$$\wp'(z; g_2, g_3)^2 == 4\,(\wp(z; g_2, g_3) - \wp(\omega_1; g_2, g_3))\,(\wp(z; g_2, g_3) - \wp(\omega_2; g_2, g_3))\,(\wp(z; g_2, g_3) - \wp(\omega_3; g_2, g_3))$$

09.13.13.0005.01

$$w'(z)^4 - \frac{128}{3}(a+w(z))^2 (b+w(z))^3 == 0 \;/;\; w(z) == 6\,\wp\left(z;\, \frac{1}{3}(-2)(a-b),\, 0\right)^2 - b$$

Partial differential equations

09.13.13.0006.01

$$z\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial z} - 4\,g_2\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial g_2} - 6\,g_3\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial g_3} + 2\,\wp(z;\, g_2,\, g_3) == 0$$

09.13.13.0007.01

$$12\,g_3\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial g_2} + \frac{2}{3}\,g_2^2\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial g_3} - 2\,\zeta(z;\, g_2,\, g_3)\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial z} == \frac{2}{3}\,\frac{\partial^2 \wp(z;\, g_2,\, g_3)}{\partial z^2} - \frac{g_2}{3}$$

09.13.13.0008.01

$$\omega_1\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial \omega_1} + \omega_3\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial \omega_3} + z\,\wp(z;\, g_2,\, g_3) == -2\,\wp(z;\, g_2,\, g_3)$$

09.13.13.0009.01

$$\eta_1\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial \omega_1} + \eta_3\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial \omega_3} + \zeta(z;\, g_2,\, g_3)\,\frac{\partial \wp(z;\, g_2,\, g_3)}{\partial z} == -\frac{1}{3}\,\frac{\partial^2 \wp(z;\, g_2,\, g_3)}{\partial x^2} + \frac{g_2}{6}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.13.16.0001.01

$$\wp(i\,z;\, g_2,\, g_3) == -\wp(z;\, g_2,\, -g_3)$$

Addition formulas

Translation by half-periods

09.13.16.0002.01

$$\wp(z \pm \omega_j;\, g_2,\, g_3) == e_j + \frac{(e_j - e_k)(e_j - e_l)}{\wp(z;\, g_2,\, g_3) - e_j} \;/;\; \{i,\, j,\, k\} \in \{1,\, 2,\, 3\} \bigwedge i \neq j \neq k$$

09.13.16.0003.01

$$\wp(z_1 + z_2;\, g_2,\, g_3) == \frac{1}{4}\left(\frac{\wp'(z_1;\, g_2,\, g_3) - \wp'(z_2;\, g_2,\, g_3)}{\wp(z_1;\, g_2,\, g_3) - \wp(z_2;\, g_2,\, g_3)}\right)^2 - \wp(z_1;\, g_2,\, g_3) - \wp(z_2;\, g_2,\, g_3)$$

09.13.16.0004.01

$$\wp(z_1 + z_2;\, g_2,\, g_3) == \Big(-\wp'(z_1;\, g_2,\, g_3)\,\wp'(z_2;\, g_2,\, g_3) + 2\,\wp(z_1;\, g_2,\, g_3)\,\wp(z_2;\, g_2,\, g_3)\,(\wp(z_1;\, g_2,\, g_3) + \wp(z_2;\, g_2,\, g_3)) - \frac{1}{2}\,g_2\,(\wp(z_1;\, g_2,\, g_3) + \wp(z_2;\, g_2,\, g_3)) - g_3\Big) \Big/ \left(2\,(\wp(z_1;\, g_2,\, g_3) - \wp(z_2;\, g_2,\, g_3))^2\right)$$

09.13.16.0005.01

$$\wp(z_1+z_2;g_2,g_3)=\left(2\left(\wp(z_1;g_2,g_3)\,\wp(z_2;g_2,g_3)-\frac{g_2}{4}\right)(\wp(z_1;g_2,g_3)+\wp(z_2;g_2,g_3))-g_3-\wp'(z_1;g_2,g_3)\,\wp'(z_2;g_2,g_3)\right)\Big/\left(2\,(\wp(z_1;g_2,g_3)-\wp(z_2;g_2,g_3))^2\right)$$

09.13.16.0006.01

$$\wp(z_1\pm z_2;g_2,g_3)=\left((\wp(z_1;g_2,g_3)+\wp(z_2;g_2,g_3))\left(2\,\wp(z_1;g_2,g_3)\,\wp(z_2;g_2,g_3)-\frac{g_2}{2}\right)-(g_3\mp\wp'(z_1;g_2,g_3)\,\wp'(z_2;g_2,g_3))\right)\Big/\left(2\,(\wp(z_1;g_2,g_3)-\wp(z_2;g_2,g_3))^2\right)$$

09.13.16.0007.01

$$\wp(z_1\pm a;g_2,g_3)=\wp(z_1;g_2,g_3)-\frac{1}{2}\frac{\partial}{\partial z_1}\frac{\wp'(z_1;g_2,g_3)\pm\wp'(z_2;g_2,g_3)}{\wp(z_1;g_2,g_3)-\wp(z_2;g_2,g_3)}$$

09.13.16.0008.01

$$\wp(z_1\pm z_2;g_2,g_3)=-\wp(z_1;g_2,g_3)-\wp(z_2;g_2,g_3)+\frac{1}{4}\left(\frac{\wp'(z_1;g_2,g_3)\mp\wp'(z_2;g_2,g_3)}{\wp(z_1;g_2,g_3)-\wp(z_2;g_2,g_3)}\right)^2$$

09.13.16.0009.01

$$\wp(z_1+z_2;g_2,g_3)=\wp(z_2;g_2,g_3)+\sum_{m,n=-\infty}^{\infty}\left(\frac{1}{(z_2+z_1-2\,m\,\omega_1-2\,n\,\omega_3)^2}-\frac{1}{(z_2-2\,m\,\omega_1-2\,n\,\omega_3)^2}\right)$$

09.13.16.0010.01

$$\wp(z_1+z_2;g_2,g_3)+\wp(z_1-z_2;g_2,g_3)=2\,\wp(z_1;g_2,g_3)+\left(\wp'(z_1;g_2,g_3)^2-\wp''(z_1;g_2,g_3)\,(\wp(z_1;g_2,g_3)-\wp(z_2;g_2,g_3))\right)\Big/(\wp(z_1;g_2,g_3)-\wp(z_2;g_2,g_3))^2$$

09.13.16.0011.01

$$\wp(z_1+z_2;g_2,g_3)-\wp(z_1-z_2;g_2,g_3)=-\frac{\wp'(z_1;g_2,g_3)\,\wp'(z_2;g_2,g_3)}{(\wp(z_1;g_2,g_3)-\wp(z_2;g_2,g_3))^2}$$

09.13.16.0012.01

$$\wp(z_1+z_2;g_2,g_3)\,\wp(z_1-z_2;g_2,g_3)=\left(\left(\wp(z_1;g_2,g_3)\,\wp(z_2;g_2,g_3)+\frac{g_2}{4}\right)^2+g_3\,(\wp(z_1;g_2,g_3)+\wp(z_2;g_2,g_3))\right)\Big/(\wp(z_1;g_2,g_3)-\wp(z_2;g_2,g_3))^2$$

Half-angle formulas

09.13.16.0013.01

$$\wp\left(\frac{z}{2};g_2,g_3\right)=\wp(z;g_2,g_3)+\epsilon_2\,\epsilon_3\sqrt{\wp(z;g_2,g_3)-e_2}\,\sqrt{\wp(z;g_2,g_3)-e_3}+\epsilon_3\,\epsilon_1\sqrt{\wp(z;g_2,g_3)-e_3}\,\sqrt{\wp(z;g_2,g_3)-e_1}+\epsilon_1\,\epsilon_2\sqrt{\wp(z;g_2,g_3)-e_1}\,\sqrt{\wp(z;g_2,g_3)-e_2}\;/;\;\epsilon_n=\operatorname{sgn}\left(\frac{\pi}{2}-\left|\arg\left(\frac{\sigma_n(z;g_2,g_3)}{\sigma(z;g_2,g_3)}\right)\right|\right)\bigwedge n\in\{1,2,3\}$$

Multiple arguments

Argument involving numeric multiples of variable

Double angle formulas

09.13.16.0014.01

$$\wp(2\,z;\,g_2,\,g_3) = \frac{\left(\wp(z;\,g_2,\,g_3)^2 + \frac{g_2}{4}\right)^2 + 2\,g_3\,\wp(z;\,g_2,\,g_3)}{4\,\wp(z;\,g_2,\,g_3)^3 - g_2\,\wp(z;\,g_2,\,g_3) - g_3}$$

09.13.16.0015.01

$$\wp(2\,z;\,g_2,\,g_3) = -2\,\wp(z;\,g_2,\,g_3) + \frac{1}{4}\left(\frac{\wp''(z;\,g_2,\,g_3)}{\wp'(z;\,g_2,\,g_3)}\right)^2$$

09.13.16.0016.01

$$4\,\wp(2\,z;\,g_2,\,g_3) = \wp(z;\,g_2,\,g_3) + \wp(z-\omega_1;\,g_2,\,g_3) + \wp(z-\omega_2;\,g_2,\,g_3) + \wp(z-\omega_3;\,g_2,\,g_3)$$

09.13.16.0017.01

$$\wp(2\,z;\,g_2,\,g_3) = \frac{\left(\wp(z;\,g_2,\,g_3)^2 + \frac{g_2}{4}\right)^2 + 2\,g_3\,\wp(z;\,g_2,\,g_3)}{\wp'(z;\,g_2,\,g_3)^2}$$

Argument involving symbolic multiples of variable

Multiple angle formulas:

09.13.16.0018.01

$$\wp(n\,z;\,g_2,\,g_3) = \frac{1}{n^2}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\wp\left(z - \frac{2\,j\,\omega_1 + 2\,k\,\omega_3}{n};\,g_2,\,g_3\right) /;\, n \in \mathbb{N}^+$$

09.13.16.0019.01

$$\wp(n\,z;\,g_2,\,g_3) = \wp(z;\,g_2,\,g_3) - \frac{\psi_{n+1}\,\psi_{n-1}}{\psi_n^2} /;\, n \in \mathbb{N}^+ \bigwedge \psi_1 = 1 \bigwedge \psi_2 = -\wp'(z;\,g_2,\,g_3) \bigwedge$$
$$\psi_3 = 3\,\wp(z;\,g_2,\,g_3)^4 - \frac{3}{2}\,g_2\,\wp(z;\,g_2,\,g_3)^2 - 3\,g_3\,\wp(z;\,g_2,\,g_3) - \frac{g_2^2}{16} \bigwedge \psi_4 = \wp'(z;\,g_2,\,g_3)$$
$$\left(-2\,\wp(z;\,g_2,\,g_3)^6 + \frac{5\,g_2}{2}\,\wp(z;\,g_2,\,g_3)^4 + 10\,g_3\,\wp(z;\,g_2,\,g_3)^3 + \frac{5\,g_2^2}{8}\,\wp(z;\,g_2,\,g_3)^2 + \frac{g_2\,g_3}{2}\,\wp(z;\,g_2,\,g_3) + g_3^2 - \frac{g_2^3}{32}\right) \bigwedge$$
$$\left(\psi_n = -\frac{1}{\wp'(z;\,g_2,\,g_3)}\,\psi_{\frac{n}{2}}\left(\psi_{\frac{n}{2}+2}\,\psi_{\frac{n}{2}-1}^2 - \psi_{\frac{n}{2}-2}\,\psi_{\frac{n}{2}+1}^2\right) /;\, \frac{n}{2} \in \mathbb{N}\right) \bigwedge \left(\psi_n = \psi_{\frac{n-1}{2}+2}\,\psi_{\frac{n-1}{2}}^3 - \psi_{\frac{n-1}{2}-1}\,\psi_{\frac{n-1}{2}+1}^3 /;\, \frac{n-1}{2} \in \mathbb{N}\right)$$

Determinants involving derivatives

09.13.16.0020.01

$$\begin{vmatrix} 1 & \wp(z_1;\,g_2,\,g_3) & \wp'(z_1;\,g_2,\,g_3) \\ 1 & \wp(z_2;\,g_2,\,g_3) & \wp'(z_2;\,g_2,\,g_3) \\ 1 & \wp(z_3;\,g_2,\,g_3) & \wp'(z_3;\,g_2,\,g_3) \end{vmatrix} =$$
$$-(2\,\sigma(z_2-z_3;\,g_2,\,g_3)\,\sigma(z_1-z_2;\,g_2,\,g_3)\,\sigma(z_1-z_3;\,g_2,\,g_3)\,\sigma(z_1+z_2+z_3;\,g_2,\,g_3)) \Big/ \left(\sigma(z_1;\,g_2,\,g_3)^3\,\sigma(z_2;\,g_2,\,g_3)^3\,\sigma(z_3;\,g_2,\,g_3)^3\right)$$

09.13.16.0021.01

$$\begin{vmatrix} 1 & \wp(z_0; g_2, g_3) & \wp'(z_0; g_2, g_3) & \ldots & \wp^{(n-1)}(z_0; g_2, g_3) \\ 1 & \wp(z_1; g_2, g_3) & \wp'(z_1; g_2, g_3) & \ldots & \wp^{(n-1)}(z_1; g_2, g_3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \wp(z_n; g_2, g_3) & \wp'(z_n; g_2, g_3) & \ldots & \wp^{(n-1)}(z_n; g_2, g_3) \end{vmatrix} = (-1)^n \sigma\left(\sum_{i=0}^{n} z_i; g_2, g_3\right) \left(\prod_{k=0}^{n} \frac{k!}{\sigma(z_k; g_2, g_3)^{n+1}}\right) \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \sigma(z_j - z_i; g_2, g_3) \;/;\; n \in \mathbb{N}^+$$

Related transformations

Halving half-period

09.13.16.0022.01

$$\wp\left(z; g_2\left(\frac{\omega_1}{2}, \omega_2\right), g_3\left(\frac{\omega_1}{2}, \omega_2\right)\right) = \left(\wp(z; g_2, g_3)^2 - e_1 \wp(z; g_2, g_3) + (e_1 - e_2)(e_1 - e_3)\right) / (\wp(z; g_2, g_3) - e_1)$$

09.13.16.0023.01

$$\wp\left(z; g_2\left(\frac{\omega_1}{2}, \omega_3\right), g_3\left(\frac{\omega_1}{2}, \omega_3\right)\right) = \wp(z; g_2, g_3) + \wp(z + \omega_1; g_2, g_3) - e_1$$

Third of half-period

09.13.16.0024.01

$$\wp\left(z; g_2\left(\frac{\omega_1}{3}, \omega_3\right), g_3\left(\frac{\omega_1}{3}, \omega_3\right)\right) = \wp\left(z + \frac{2\omega_1}{3}; g_2, g_3\right) - 2\wp\left(\frac{2\omega_1}{3}; g_2, g_3\right) + \wp\left(z + \frac{4\omega_1}{3}; g_2, g_3\right) + \wp(z; g_2, g_3)$$

General fractions of half-periods

09.13.16.0025.01

$$\wp\left(z; g_2\left(\frac{\omega_1}{2n+1}, \omega_2\right), g_3\left(\frac{\omega_1}{2n+1}, \omega_2\right)\right) = \sum_{k=-n}^{n} \frac{1 - \delta_{k,0}}{4} \left(\frac{\wp'\left(\frac{2k\omega_1}{2n+1}; g_2, g_3\right)^2 + \wp'(z; g_2, g_3)^2}{\left(\wp\left(\frac{2k\omega_1}{2n+1}; g_2, g_3\right)^2 - \wp(z; g_2, g_3)^2\right)^2} - 2\wp\left(\frac{2k\omega_1}{2n+1}; g_2, g_3\right) - \wp(z; g_2, g_3)\right) + \wp(z; g_2, g_3) \;/;\; n \in \mathbb{N}$$

09.13.16.0026.01

$$\wp\left(z; g_2\left(\frac{\omega_1}{n}, \omega_3\right), g_3\left(\frac{\omega_1}{n}, \omega_3\right)\right) = \sum_{k=1}^{n-1} \left(\wp\left(z + \frac{2k\omega_1}{n}; g_2, g_3\right) - \wp\left(\frac{2k\omega_1}{n}; g_2, g_3\right)\right) + \wp(z; g_2, g_3) \;/;\; n \in \mathbb{N}^+$$

Other constructions

09.13.16.0027.01

$$\frac{\wp'(z_1; g_2, g_3) - \wp'(z_2; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)} = \frac{1}{\wp'(z_1; g_2, g_3)} \left(2(\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3))(\wp(z_1 + z_2; g_2, g_3) - \wp(z_1; g_2, g_3)) + \wp''(z_1; g_2, g_3)\right)$$

09.13.16.0028.01

$$\begin{vmatrix} 1 & \wp(z_1; g_2, g_3) & \wp'(z_1; g_2, g_3) \\ 1 & \wp(z_2; g_2, g_3) & \wp'(z_2; g_2, g_3) \\ 1 & \wp(z_1 + z_2; g_2, g_3) & -\wp'(z_1 + z_2; g_2, g_3) \end{vmatrix} = 0$$

Identities

Functional identities

Expressions involving $a + b + c == 0 \bmod(2\,\omega_1,\ 2\,\omega_3)$

09.13.17.0001.01

$$\frac{\wp'(a;\,g_2,\,g_3) - \wp'(b;\,g_2,\,g_3)}{\wp(a;\,g_2,\,g_3) - \wp(b;\,g_2,\,g_3)} == \frac{\wp'(b;\,g_2,\,g_3) - \wp'(c;\,g_2,\,g_3)}{\wp(b;\,g_2,\,g_3) - \wp(c;\,g_2,\,g_3)} \;/;\; a + b + c == 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m,\,n\} \in \mathbb{Z}$$

09.13.17.0002.01

$$\frac{\wp'(a;\,g_2,\,g_3) - \wp'(b;\,g_2,\,g_3)}{\wp(a;\,g_2,\,g_3) - \wp(b;\,g_2,\,g_3)} == \frac{\wp'(c;\,g_2,\,g_3) - \wp'(a;\,g_2,\,g_3)}{\wp(c;\,g_2,\,g_3) - \wp(a;\,g_2,\,g_3)} \;/;\; a + b + c == 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m,\,n\} \in \mathbb{Z}$$

09.13.17.0003.01

$$(\wp(b;\,g_2,\,g_3)\,\wp'(c;\,g_2,\,g_3) - \wp(c;\,g_2,\,g_3)\,\wp'(b;\,g_2,\,g_3))\,/\,(\wp(b;\,g_2,\,g_3) - \wp(c;\,g_2,\,g_3)) == (\wp(c;\,g_2,\,g_3)\,\wp'(a;\,g_2,\,g_3) - \wp(a;\,g_2,\,g_3)\,\wp'(c;\,g_2,\,g_3))\,/\,(\wp(c;\,g_2,\,g_3) - \wp(a;\,g_2,\,g_3)) \;/;\; a + b + c == 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m,\,n\} \in \mathbb{Z}$$

09.13.17.0004.01

$$(\wp(b;\,g_2,\,g_3)\,\wp'(c;\,g_2,\,g_3) - \wp(c;\,g_2,\,g_3)\,\wp'(b;\,g_2,\,g_3))\,/\,(\wp(b;\,g_2,\,g_3) - \wp(c;\,g_2,\,g_3)) == (\wp(a;\,g_2,\,g_3)\,\wp'(b;\,g_2,\,g_3) - \wp(b;\,g_2,\,g_3)\,\wp'(a;\,g_2,\,g_3))\,/\,(\wp(a;\,g_2,\,g_3) - \wp(b;\,g_2,\,g_3)) \;/;\; a + b + c == 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m,\,n\} \in \mathbb{Z}$$

09.13.17.0005.01

$$\left(\wp(a;\,g_2,\,g_3)\,\wp(b;\,g_2,\,g_3) + \wp(a;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) + \wp(b;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) + \frac{g_2}{4}\right)^2 == (4\,\wp(a;\,g_2,\,g_3)\,\wp(b;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) - g_3)\,(\wp(a;\,g_2,\,g_3) + \wp(b;\,g_2,\,g_3) + \wp(c;\,g_2,\,g_3)) \;/;\; a \pm b \pm c == 2\,m\,\omega_1 + 2\,n\,\omega_3 \bigwedge \{m,\,n\} \in \mathbb{Z}$$

Generalization for arbitrary a, b, c, d

09.13.17.0006.01

$$(4\,\wp(a;\,g_2,\,g_3)\,\wp(b;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) - g_3)\,(\wp(a;\,g_2,\,g_3) + \wp(b;\,g_2,\,g_3) + \wp(c;\,g_2,\,g_3)) - \left(\wp(a;\,g_2,\,g_3)\,\wp(b;\,g_2,\,g_3) + \wp(a;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) + \wp(b;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) + \frac{g_2}{4}\right)^2 == (\sigma(a + b + c;\,g_2,\,g_3)\,\sigma(a + b - c;\,g_2,\,g_3)\,\sigma(a - b + c;\,g_2,\,g_3)\,\sigma(-a + b + c;\,g_2,\,g_3))\Big/\left(\sigma(a;\,g_2,\,g_3)^4\,\sigma(b;\,g_2,\,g_3)^4\,\sigma(c;\,g_2,\,g_3)^4\right)$$

09.13.17.0007.01

$$(4\,\wp(a;\,g_2,\,g_3)\,\wp(b;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) - g_3)\,(\wp(a;\,g_2,\,g_3) + \wp(b;\,g_2,\,g_3) + \wp(c;\,g_2,\,g_3)) - \left(\wp(a;\,g_2,\,g_3)\,\wp(b;\,g_2,\,g_3) + \wp(a;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) + \wp(b;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) + \frac{g_2}{4}\right)^2 == -(\wp(b;\,g_2,\,g_3) - \wp(c;\,g_2,\,g_3))^2\,(\wp(a;\,g_2,\,g_3) - \wp(b + c;\,g_2,\,g_3))\,(\wp(a;\,g_2,\,g_3) - \wp(b - c;\,g_2,\,g_3))$$

09.13.17.0008.01

$$(4\,\wp(a;\,g_2,\,g_3)\,\wp(b;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) - g_3)\,(\wp(a;\,g_2,\,g_3) + \wp(b;\,g_2,\,g_3) + \wp(c;\,g_2,\,g_3)) - \left(\wp(a;\,g_2,\,g_3)\,\wp(b;\,g_2,\,g_3) + \wp(a;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) + \wp(b;\,g_2,\,g_3)\,\wp(c;\,g_2,\,g_3) + \frac{g_2}{4}\right)^2 == -(\wp(c;\,g_2,\,g_3) - \wp(a;\,g_2,\,g_3))^2\,(\wp(b;\,g_2,\,g_3) - \wp(a + c;\,g_2,\,g_3))\,(\wp(b;\,g_2,\,g_3) - \wp(c - a;\,g_2,\,g_3))$$

09.13.17.0009.01

$$(4\wp(a;g_2,g_3)\wp(b;g_2,g_3)\wp(c;g_2,g_3)-g_3)(\wp(a;g_2,g_3)+\wp(b;g_2,g_3)+\wp(c;g_2,g_3))-\left(\wp(a;g_2,g_3)\wp(b;g_2,g_3)+\wp(a;g_2,g_3)\wp(c;g_2,g_3)+\wp(b;g_2,g_3)\wp(c;g_2,g_3)+\frac{g_2}{4}\right)^2 = -(\wp(a;g_2,g_3)-\wp(b;g_2,g_3))^2(\wp(c;g_2,g_3)-\wp(a+b;g_2,g_3))(\wp(c;g_2,g_3)-\wp(a-b;g_2,g_3))$$

09.13.17.0010.01

$$\frac{\wp'(a;g_2,g_3)-\wp'(b;g_2,g_3)}{\wp(a;g_2,g_3)-\wp(b;g_2,g_3)}+\frac{\wp'(c;g_2,g_3)-\wp'(d;g_2,g_3)}{\wp(c;g_2,g_3)-\wp(d;g_2,g_3)}+\frac{\wp'(a+b;g_2,g_3)-\wp'(c+d;g_2,g_3)}{\wp(a+b;g_2,g_3)-\wp(c+d;g_2,g_3)} = \frac{\wp'(a;g_2,g_3)-\wp'(c;g_2,g_3)}{\wp(a;g_2,g_3)-\wp(c;g_2,g_3)}+\frac{\wp'(b;g_2,g_3)-\wp'(d;g_2,g_3)}{\wp(b;g_2,g_3)-\wp(d;g_2,g_3)}+\frac{\wp'(a+c;g_2,g_3)-\wp'(b+d;g_2,g_3)}{\wp(a+c;g_2,g_3)-\wp(b+d;g_2,g_3)}$$

Differentiation

Low-order differentiation

With respect to z

09.13.20.0001.01

$$\frac{\partial\wp(z;g_2,g_3)}{\partial z} = \wp'(z;g_2,g_3)$$

09.13.20.0002.01

$$\frac{\partial^2\wp(z;g_2,g_3)}{\partial z^2} = 6\wp(z;g_2,g_3)^2-\frac{g_2}{2}$$

09.13.20.0003.01

$$\frac{\partial^3\wp(z;g_2,g_3)}{\partial z^3} = 12\wp(z;g_2,g_3)\wp'(z;g_2,g_3)$$

09.13.20.0004.01

$$\frac{\partial^4\wp(z;g_2,g_3)}{\partial z^4} = 120\wp(z;g_2,g_3)^3-18g_2\wp(z;g_2,g_3)-12g_3$$

With respect to g_2

09.13.20.0005.01

$$\frac{\partial\wp(z;g_2,g_3)}{\partial g_2} = \frac{1}{4(g_2^3-27g_3^2)}\left(2\wp(z;g_2,g_3)g_2^2+6g_3g_2-36g_3\wp(z;g_2,g_3)^2+\wp'(z;g_2,g_3)\left(g_2^2z-18g_3\zeta(z;g_2,g_3)\right)\right)$$

09.13.20.0006.01

$$\frac{\partial^2\wp(z;g_2,g_3)}{\partial g_2^2} = -1\Big/\left(32\left(g_2^3-27g_3^2\right)^2\right)\Big(\left(z^2g_2^5-2\left(6z^2\wp(z;g_2,g_3)^2-4\wp(z;g_2,g_3)+z\wp'(z;g_2,g_3)\right)g_2^4-36g_3(z\zeta(z;g_2,g_3)-2)g_2^3+216g_3\,g_2^2\left(2(z\zeta(z;g_2,g_3)-1)\wp(z;g_2,g_3)^2+z\wp'(z;g_2,g_3)\wp(z;g_2,g_3)-\wp'(z;g_2,g_3)\zeta(z;g_2,g_3)\right)+54g_2g_3^2\left(6\zeta(z;g_2,g_3)^2+32\wp(z;g_2,g_3)+7z\wp'(z;g_2,g_3)\right)+324g_3^2\left(-16\wp(z;g_2,g_3)^3-12\zeta(z;g_2,g_3)^2\wp(z;g_2,g_3)^2-12\wp'(z;g_2,g_3)\zeta(z;g_2,g_3)\wp(z;g_2,g_3)+\wp'(z;g_2,g_3)^2+4g_3\right)\Big)$$

With respect to g_3

09.13.20.0007.01

$$\frac{\partial \wp(z; g_2, g_3)}{\partial g_3} = \frac{1}{2\left(g_2^3 - 27 g_3^2\right)} \left(12 \wp(z; g_2, g_3)^2 g_2 - 18 g_3 \wp(z; g_2, g_3) - 2 g_2^2 + (6 g_2 \zeta(z; g_2, g_3) - g_3 z) \wp'(z; g_2, g_3)\right)$$

09.13.20.0008.01

$$\frac{\partial^2 \wp(z; g_2, g_3)}{\partial g_3^2} = -\frac{1}{8\left(g_2^3 - 27 g_3^2\right)^2} \Big(3 \Big(2 g_2^3 \left(6 \zeta(z; g_2, g_3)^2 + 28 \wp(z; g_2, g_3) + 5 z \wp'(z; g_2, g_3)\right) - 12 g_2^2 \big(16 \wp(z; g_2, g_3)^3 + 12 \zeta(z; g_2, g_3)^2 \wp(z; g_2, g_3)^2 + 12 \wp'(z; g_2, g_3) \zeta(z; g_2, g_3) \wp(z; g_2, g_3) - \wp'(z; g_2, g_3)^2 + g_3 (3 z \zeta(z; g_2, g_3) - 10)\big) + 27 g_2 g_3 \left(g_3 z^2 + 8 \left(2 (z \zeta(z; g_2, g_3) - 1) \wp(z; g_2, g_3)^2 + z \wp'(z; g_2, g_3) \wp(z; g_2, g_3) - \wp'(z; g_2, g_3) \zeta(z; g_2, g_3)\right)\right) + 54 g_3^2 \left(-6 z^2 \wp(z; g_2, g_3)^2 + 8 \wp(z; g_2, g_3) + z \wp'(z; g_2, g_3)\right)\Big)\Big)$$

With respect to ω_1

09.13.20.0012.02

$$\frac{\partial \wp(z; g_2, g_3)}{\partial \omega_1} = -\frac{2 \omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\omega_3 \left(2 \wp(z; g_2, g_3)^2 + \zeta(z; g_2, g_3) \wp'(z; g_2, g_3) - \frac{g_2}{3}\right) - \eta_3 (2 \wp(z; g_2, g_3) + z \wp'(z; g_2, g_3))\right)$$

With respect to ω_3

09.13.20.0013.02

$$\frac{\partial \wp(z; g_2, g_3)}{\partial \omega_3} = \frac{2 \omega_1}{\pi \omega_3} \sqrt{-\frac{\omega_3^2}{\omega_1^2}} \left(\omega_1 \left(2 \wp(z; g_2, g_3)^2 + \zeta(z; g_2, g_3) \wp'(z; g_2, g_3) - \frac{g_2}{3}\right) - \eta_1 (2 \wp(z; g_2, g_3) + z \wp'(z; g_2, g_3))\right)$$

Symbolic differentiation

With respect to z

The n-th derivative of $\wp(z; g_2, g_3)$ can be expressed in terms of $\wp(z; g_2, g_3)$ and $\wp'(z; g_2, g_3)$ by using the first two equations above and the following recursive rule:

09.13.20.0009.01

$$\frac{\partial^n \wp(z; g_2, g_3)}{\partial z^n} = \frac{\partial}{\partial z} \frac{\partial^{n-1} \wp(z; g_2, g_3)}{\partial z^{n-1}} \;/;\; n \in \mathbb{N}^+$$

09.13.20.0010.01

$$\frac{\partial^k \wp(z; g_2, g_3)}{\partial z^k} = (-1)^k (k+1)! \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(z - 2 m \omega_1 - 2 n \omega_3)^{k+2}} \;/;\; k \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to z

09.13.20.0011.01

$$\frac{\partial^{\alpha} \wp(z; g_2, g_3)}{\partial z^{\alpha}} = \mathcal{F}C_{\exp}^{(\alpha)}(z, -2)\, z^{-\alpha-2} + 2\, z^{1-\alpha} \sum_{\substack{m, n=-\infty \\ \{m,n\} \neq \{0,0\}}}^{\infty} \frac{1}{(2\, m\, \omega_1 + 2\, n\, \omega_3)^3} \times$$
$$\, {}_2\tilde{F}_1\left(1, 3; 2-\alpha; \frac{z}{2\, m\, \omega_1 + 2\, n\, \omega_3}\right)$$

Integration

Indefinite integration

Involving only one direct function

09.13.21.0001.01

$$\int \wp(z; g_2, g_3)\, dz = -\zeta(z; g_2, g_3)$$

Involving functions of the direct function

Involving elementary functions of the direct function

Involving rational functions of the direct function

09.13.21.0002.01

$$\int \frac{1}{\wp(z; g_2, g_3) - \wp(a; g_2, g_3)}\, dz =$$
$$(\log(\sigma(z-a; g_2, g_3)) - \log(\sigma(a+z; g_2, g_3)) + 2\, z\, \zeta(a; g_2, g_3)) / \wp'(a; g_2, g_3) \;/;\; \wp'(a; g_2, g_3) \neq 0$$

09.13.21.0003.01

$$\int \frac{b + a\, \wp(z; g_2, g_3)}{d + c\, \wp(z; g_2, g_3)}\, dz = \frac{a\, z}{c} - \frac{a\, d - b\, c}{c^2\, \wp'(v; g_2, g_3)} (\log(\sigma(z+v; g_2, g_3)) - \log(\sigma(z-v; g_2, g_3)) - 2\, z\, \zeta(v; g_2, g_3)) \;/;$$
$$\wp(v; g_2, g_3) = -\frac{d}{c}$$

09.13.21.0004.01

$$\int \frac{dz}{\wp(z; g_2, g_3) - v} = \frac{1}{\wp'(v; g_2, g_3)} (\log(\sigma(z+v; g_2, g_3)) - \log(\sigma(z-v; g_2, g_3)) - 2\, z\, \zeta(v; g_2, g_3)) \;/;\; \wp(v; g_2, g_3) = -\frac{d}{c}$$

Involving powers of the direct function

09.13.21.0005.01

$$\int \wp(z; g_2, g_3)^2\, dz = \frac{1}{6} \wp'(z; g_2, g_3) + \frac{z\, g_2}{12}$$

09.13.21.0006.01

$$\int \wp(z; g_2, g_3)^3\, dz = \frac{1}{120} \frac{\partial^3 \wp(z; g_2, g_3)}{\partial z^3} - \frac{3}{20} g_2\, \zeta(z; g_2, g_3) + \frac{1}{10} z\, g_3$$

09.13.21.0007.01

$$\int \wp(z; g_2, g_3)^4 \, dz = \frac{\partial^5 \wp(z; g_2, g_3)}{\partial z^5} + \frac{1}{30} g_2 \wp'(z; g_2, g_3) - \frac{1}{7} g_3 \zeta(z; g_2, g_3) + \frac{5 z g_2^2}{336}$$

Summation

Finite summation

09.13.23.0001.01

$$\sum_{\substack{j, k=0 \\ \{j,k\} \neq \{0,0\}}}^{n-1} \wp\left(\frac{2 j \omega_1 + 2 k \omega_3}{n}; g_2, g_3\right) = 0 \;/;\; n - 1 \in \mathbb{N}^+$$

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

09.13.26.0001.01

$$\wp(z; g_2, g_3) = w \;/;\; z = -\frac{1}{\sqrt{w - e_1}} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{e_2 - e_1}{w - e_1}, \frac{e_3 - e_1}{w - e_1}\right) \bigwedge 4 w^3 - g_2 w - g_3 = 4 (w - e_1)(w - e_2)(w - e_3)$$

Representations through equivalent functions

With inverse function

09.13.27.0001.01

$$\wp\left(\wp^{-1}(z; g_2, g_3); g_2, g_3\right) = z$$

09.13.27.0002.01

$$\wp\left(\wp^{-1}(z_1, z_2; g_2, g_3); g_2, g_3\right) = z_1 \;/;\; z_2 = \sqrt{4 z_1^3 - g_2 z_1 - g_3}$$

With related functions

Involving other Weierstrass functions

09.13.27.0003.01

$$\wp(z; g_2, g_3) = -\frac{\partial \zeta(z; g_2, g_3)}{\partial z}$$

09.13.27.0004.01

$$48 \wp(z; g_2, g_3)^4 - 24 g_2 \wp(z; g_2, g_3)^2 - 48 g_3 \wp(z; g_2, g_3) = g_2^2 + \frac{16 \sigma(3 z; g_2, g_3)}{\sigma(z; g_2, g_3)^9}$$

09.13.27.0005.01

$$\wp(z; g_2, g_3) = e_i + \frac{\sigma_i(z; g_2, g_3)^2}{\sigma(z; g_2, g_3)^2} \;/;\; i \in \{1, 2, 3\}$$

09.13.27.0006.01

$$\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3) == -\frac{\sigma(z_1 + z_2; g_2, g_3)\,\sigma(z_1 - z_2; g_2, g_3)}{\sigma(z_1; g_2, g_3)^2\,\sigma(z_2; g_2, g_3)^2}$$

09.13.27.0007.01

$$\wp(z; g_2, g_3) == -\frac{\sigma(z - z_0; g_2, g_3)\,\sigma(z + z_0; g_2, g_3)}{\sigma(z; g_2, g_3)^2\,\sigma(z_0; g_2, g_3)^2} \;/;\; z_0 == \wp^{-1}(0; g_2, g_3)$$

09.13.27.0008.01

$$\frac{\wp'(z_1; g_2, g_3) - \wp'(z_2; g_2, g_3)}{\wp(z_1; g_2, g_3) - \wp(z_2; g_2, g_3)} == 2\,(\zeta(z_1 + z_2; g_2, g_3) - \zeta(z_1; g_2, g_3) - \zeta(z_2; g_2, g_3))$$

Involving Jacobi functions

09.13.27.0009.01

$$\wp(z; g_2, g_3) == (e_1 - e_3)\,\mathrm{ns}\left(\sqrt{e_1 - e_3}\; z \,\middle|\, \frac{e_2 - e_3}{e_1 - e_3}\right)^2 + e_3$$

09.13.27.0010.01

$$\wp(z; g_2, g_3) == e_3 + \frac{e_1 - e_3}{\mathrm{sn}\left(z\sqrt{e_1 - e_3} \,\middle|\, m\right)^2} \;/;\; m == \lambda\left(\frac{\omega_3}{\omega_1}\right)$$

Involving theta functions

09.13.27.0011.01

$$\wp(z; g_2, g_3) == e_i + \frac{\pi^2}{4\,\omega_1^2}\left(\frac{\vartheta_1'(0, q)\,\vartheta_{i+1}\left(\frac{\pi z}{2\,\omega_1}, q\right)}{\vartheta_{i+1}(0, q)\,\vartheta_1\left(\frac{\pi z}{2\,\omega_1}, q\right)}\right)^2$$

09.13.27.0012.01

$$\wp(z; g_2, g_3) == \frac{\pi^2\,\vartheta_1^{(0,2,0)}(1, 0, q)}{12\,\omega_1\,\vartheta_1'(0, q)} - \frac{\partial^2 \log\left(\vartheta_1\left(\frac{\pi z}{2\,\omega_1}, q\right)\right)}{\partial z^2}$$

Involving other related functions

09.13.27.0013.01

$$\wp(z; g_2, g_3) == \frac{1}{\sqrt[3]{4}}\left(\frac{a}{3} + x\right) \;/;\; \{x, y\} == \mathrm{eexp}\left(\frac{1}{\sqrt[3]{2}}\, z; a, b\right) \bigwedge$$

$$a == \left(i\sqrt[6]{3}\left(\sqrt[6]{3}\,\left(i + \sqrt{3}\right)\left(9\,g_3 + \sqrt{81\,g_3^2 - 3\,g_2^3}\right)^{2/3} + \left(3\,i + \sqrt{3}\right) i\,g_2\right)\right) \Big/ \left(2\sqrt[3]{2}\,\sqrt[3]{9\,g_3 + \sqrt{81\,g_3^2 - 3\,g_2^3}}\right) \bigwedge$$

$$b == \left(-\sqrt[6]{3}\left(3\left(3\,i + \sqrt{3}\right) g_3 + \sqrt{27\,g_3^2 - g_2^3} + i\sqrt{81\,g_3^2 - 3\,g_2^3}\right)\sqrt[3]{9\,g_3 + \sqrt{81\,g_3^2 - 3\,g_2^3}} + \right.$$

$$\left. 2\,g_2\left(9\,g_3 + \sqrt{81\,g_3^2 - 3\,g_2^3}\right)^{2/3} + \sqrt[3]{3}\,\left(i + \sqrt{3}\right) i\,g_2^2\right) \Big/ \left(2\,2^{2/3}\left(9\,g_3 + \sqrt{81\,g_3^2 - 3\,g_2^3}\right)^{2/3}\right)$$

Theorems

Rational representation of double periodic functions

Any elliptic function $\mathcal{E}(z)$ with periods ρ_1 and ρ_2 can be expressed as a rational function of the Weierstrassian elliptic functions $\wp(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$ and their derivative $\wp'(z; g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3))$ with the same periods $\rho_1 == 2\,\omega_1,\ \rho_2 == 2\,\omega_3$.

Solution of Korteweg-de Vries equation

The elongation of a water wave in a shallow channel is described by the Korteweg-de Vries equation

$$\frac{\partial w(z,t)}{\partial t} == 3\,w(z,t)\,\frac{\partial w(z,t)}{\partial z} - \frac{1}{2}\,\frac{\partial^3 w(z,t)}{\partial z^3}.$$

A solution to this nonlinear partial differential equation is given by

$$w(z,t) = 2\,c_1\,\wp(c_1(z - c_2\,t) + \omega'(g_2, g_3); g_2, g_3) - \frac{c_2}{3}\ /;\ \omega(g_2, g_3) > 0,\ i\,\omega'(g_2, g_3) > 0.$$

Algebraic independence of some Weierstrass functions

If the algebraic numbers z_1 , z_2, z_3 are linearly independent over $\mathbb{Q}$ and g_2, g_3 are algebraic numbers, then two of the three numbers $\wp(z_1; g_2, g_3)$, $\wp(z_2; g_2, g_3)$, $\wp(z_3; g_2, g_3)$ are algebraically independent over $\mathbb{Q}$.

The Picard solution of a special case of the Painlevé VI- equation

The Picard solution of a special case of the Painlevé VI–equation,

$$w''(z) == \frac{1}{2}\left(\frac{1}{w(z)-z} + \frac{1}{w(z)} + \frac{1}{w(z)-1}\right) w'(z)^2 - \left(\frac{1}{w(z)-z} + \frac{1}{z} + \frac{1}{z-1}\right) w'(z) + \frac{w(z)\,(w(z)-1)\,(w(z)-z)}{2\left(z^2\,(z-1)^2\right)}\,\frac{(z-1)\,z}{(w(z)-z)^2},$$

is given by

$$w(z) == \wp(c_1\,K(z) + c_2\,K(1-z), \{\omega_1(z), \omega_2(z)\}) + \frac{z+1}{3}\ /;\ \{\omega_1(z), \omega_2(z)\} = \{\omega(K(z),\ K(1-z)),\ \omega'(K(z),\ K(1-z))\}.$$

Uniformization of elliptic curves

The general elliptic curve $y^2 == f(x) == a_0\,x^4 + 4\,a_1\,x^3 + 6\,a_2\,x^2 + 4\,a_3\,x + a_4$ can be uniformized (meaning x and y can both be parameterized as a single-valued analytic function) through

$$x(z) == x_0 + \frac{f'(x_0)}{4\left(\wp(z; g_2, g_3) - \frac{1}{24} f''(x_0)\right)}$$

$$y(z) == -\frac{f'(x_0)\, \wp'(z; g_2, g_3)}{4\left(\wp(z; g_2, g_3) - \frac{1}{24} f''(x_0)\right)^2}.$$

Here x_0 is any zero of $f(x)$, and $g_2 = a_0\, a_4 - 4\, a_1\, a_3 + 3\, a_2^2$, $g_3 = a_0\, a_2\, a_4 + 2\, a_1\, a_2\, a_3 - a_2^3 - a_0\, a_3^2 - a_1^2\, a_4$.

History

– N. H. Abel (1827)
– K. Weierstrass (1855, 1862)
– C. Hermite (1849) first used the notation $\wp$

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