Introductions to Arg

Introduction to the complex components

General

The study of complex numbers and their characteristics has a long history. It all started with questions about how to understand and interpret the solution of the simple quadratic equation $z^2 == -1$.

It was clear that $1^2 == (-1)^2 == 1$. But it was not clear how to get –1 from something squared.

This problem was intensively discussed in the 16th, 17th, and 18th centuries. As a result, mathematicians proposed a special symbol—the imaginary unit i, which is represented by $i = \sqrt{-1}$:

$$i^2 == -1.$$

L. Euler (1755) introduced the word "complex" (1777) and first used the letter i for denoting $\sqrt{-1}$. Later, C. F. Gauss (1831) introduced the name "imaginary unit" for i.

Accordingly, $i^2 == -1$ and $(-i)^2 == -1$ and the above quadratic equation has two solutions as is expected for a quadratic polynomial:

$$z^2 == -1 \;/;\; z == z_1 == i \bigwedge z == z_2 == -i.$$

The imaginary unit i was interpreted in a geometrical sense as the point with coordinates $\{0, 1\}$ in the Cartesian (Euclidean) x,y-plane with the vertical y-axis upward and the origin $\{0, 0\}$. This geometric interpretation established the following representations of the complex number z through two real numbers x and y as:

$$z == x + i\, y \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R} \Longleftrightarrow (x, y)$$

$$z == r \cos(\varphi) + i\, r \sin(\varphi) \;/;\; r \in \mathbb{R} \wedge r > 0 \bigwedge \varphi \in \mathbb{R},$$

where $r = \sqrt{x^2 + y^2}$ is the distance between points $\{x, y\}$ and $\{0, 0\}$, and φ is the angle between the line connecting the points $\{0, 0\}$ and $\{x, y\}$ and the positive x-axis direction (the so-called polar representation).

The last formula lead to the basic relations:

$$r == \sqrt{x^2 + y^2}$$

$$x == r \cos(\varphi)$$

$$y == r \sin(\varphi)$$

$$\varphi == \tan^{-1}\left(\frac{y}{x}\right) \;/;\; x > 0,$$

which describe the main characteristics of the complex number $z == x + i\,y$—the so-called modulus (absolute value) r, the real part x, the imaginary part y, and the argument φ.

A new era in the theory of complex numbers and functions of complex arguments (analytic functions) arose from the investigations of L. Euler (1727, 1728). In a letter to Goldbach (1731) L. Euler introduced the notation e for the base of the natural logarithm e==2.71828182…, and he proved that e is irrational. Later on L. Euler (1740–1748) found a series expansion for e^z, which lead to the famous very basic formula, connecting exponential and trigonometric functions:

$$e^{i\varphi} == \cos(\varphi) + i\sin(\varphi).$$

This is known as the Euler formula (although it was already derived by R. Cotes in 1714).

The Euler formula allows presentation of the complex number z, using polar coordinates $(r,\ \varphi)$ in the more compact form:

$$z == r\,e^{i\varphi}\ /;\ r \in \mathbb{R} \wedge r \geq 0 \bigwedge \varphi \in [0,\ 2\pi).$$

It also expressed the logarithm of complex numbers through the formula:

$$\log(z) == \log(r) + i\,\varphi\ /;\ r \in \mathbb{R} \wedge r > 0 \bigwedge \varphi \in \mathbb{R}.$$

Taking into account that the cosine and sine have period 2π, it follows that $e^{i\varphi}$ has period $2\pi i$:

$$e^{i\varphi + 2\pi i} == e^{i(\varphi + 2\pi)} == \cos(\varphi + 2\pi) + i\sin(\varphi + 2\pi) == \cos(\varphi) + i\sin(\varphi) == e^{i\varphi}.$$

Generically, the logarithm function $\log(z)$ is the multivalued function:

$$\log(z) == \log(r) + i(\varphi + 2\pi k)\ /;\ r \in \mathbb{R} \wedge r > 0 \bigwedge \varphi \in \mathbb{R} \bigwedge k \in \mathbb{Z}.$$

For specifying just one value for the logarithm $\log(z)$ and one value of the argument φ for a given complex number z, the restriction $-\pi < \varphi \leq \pi$ for the argument φ is generally used.

During the 18th and 19th centuries many mathematicians worked on building the theory of the functions of complex variables, which was called the theory of analytic functions. Today this is a widely used theory, not only for the above-mentioned four complex components (absolute value, argument, real and imaginary parts), but for complimentary characteristics of a complex number $z == x + i\,y == r\,e^{i\varphi}$ such as the conjugate complex number $\bar{z} == x - i\,y$ and the signum (sign) z/r. J. R. Argand (1806, 1814) introduced the word "module" for the absolute value, and A. L. Cauchy (1821) was the first to use the word "conjugate" for complex numbers in the modern sense. Later K. Weierstrass (1841) introduced the notation $|z|$ for the absolute value.

It was shown that the set of complex numbers and the set of real numbers have basic properties in common—they both are fields because they satisfy so-called field axioms. Complex and real numbers exhibit commutativity under addition and multiplication described by the formulas:

$$a + b == b + a$$

$$a\,b == b\,a.$$

Complex and real numbers also have associativity under addition and multiplication described by the formulas:

$(a+b)+c == a+(b+c)$

$(a\,b)\,c == a\,(b\,c)$,

and distributivity described by the formulas:

$a\,(b+c) == a\,b + a\,c$

$(a+b)\,c == a\,c + b\,c$.

(The set of rational numbers $p/q\ /;\ p \in \mathbb{Z} \bigwedge q \in \mathbb{Z}$ also satisfies all of the previous field axioms and is also a field. This set is countable, which means that each rational number can be numerated and placed in a definite position with a corresponding integer number $n\ /;\ n == 1, 2, 3, \ldots$. But the set of rational numbers does not include so-called irrational numbers like $\sqrt{2}$ or π. The set of irrational numbers is much larger and cannot be numerated. The sets of all real and complex numbers form uncountable sets.)

The great success and achievements of the complex number theory stimulated attempts to introduce not only the imaginary unit $i == \sqrt{-1}$ in the Cartesian (Euclidean) plane (x, y), but a similar special third unit j in Cartesian (Euclidean) three-dimensional space $\{x, y, z\}$, which can be used for building a similar theory of (hyper)complex numbers $w == x + i\,y + j\,z$:

$w == x + i\,y + j\,z\ /;\ x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge z \in \mathbb{R} \Longleftrightarrow \{x, y, z\}$.

Unfortunately, such an attempt fails to fulfill the field axioms. Further generalizations to build the so-called quaternions and octonions are needed to obtain mathematically interesting and rich objects.

Definitions of complex components

The complex components include six basic characteristics describing complex numbers−absolute value (modulus) $|z|$, argument (phase) $\mathrm{Arg}(z)$, real part $\mathrm{Re}(z)$, imaginary part $\mathrm{Im}(z)$, complex conjugate $\bar{z}$, and sign function (signum) $\mathrm{sgn}(z)$. It is impossible to define real and imaginary parts of the complex number z through other functions or complex characteristics. They are too basic, so their symbols can be described by simple sentences, for example, "$\mathrm{Re}(z)$ gives the real part of the number z," and "$\mathrm{Im}(z)$ gives the imaginary part of the number z."

All other complex components are defined by the following formulas:

$|x| == x\ /;\ x \in \mathbb{R} \land x \geq 0$

$|x| == -x\ /;\ x \in \mathbb{R} \land x < 0$

$|z| == \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$

$\mathrm{Arg}(z) == -i \log\left(\frac{z}{|z|}\right)$

$\bar{z} == \mathrm{Re}(z) - i\,\mathrm{Im}(z)$

$\mathrm{sgn}(x) == 1\ /;\ x \in \mathbb{R} \land x > 0$

$\mathrm{sgn}(x) == -1\ /;\ x \in \mathbb{R} \land x < 0$

$\operatorname{sgn}(0) == 0$

$\operatorname{sgn}(z) == \frac{z}{|z|} /; z \neq 0.$

Geometrically, the absolute value (or modulus) of a complex number $z == x + i\, y /; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$ is the Euclidean distance from z to the origin, which can also be described by the formula:

$|x + i\, y| == \sqrt{x^2 + y^2}\,.$

Geometrically, the argument of a complex number z is the phase angle (in radians) that the line from 0 to z makes with the positive real axis. So, the complex number $z == x + i\, y /; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$ can be presented by the formulas:

$z == |z|\, e^{i \operatorname{Arg}(z)}$

$x + i\, y == \sqrt{x^2 + y^2}\; e^{i \operatorname{Arg}(x + i\, y)}$

$x + i\, y == \sqrt{x^2 + y^2}\; e^{i \tan^{-1}(x, y)}.$

Geometrically, the real part of a complex number z is the projection of the complex point z on the real axis. So, the real part of the complex number $z == x + i\, y /; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$ can be presented by the formulas:

$\operatorname{Re}(z) == |z| \cos(\operatorname{Arg}(z))$

$\operatorname{Re}(x + i\, y) == \sqrt{x^2 + y^2}\; \cos\left(\tan^{-1}(x, y)\right).$

Geometrically, the imaginary part of a complex number z is the projection of complex point z on the imaginary axis. So, the imaginary part of the complex number $z == x + i\, y /; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$ can be presented by the formulas:

$\operatorname{Im}(z) == |z| \sin(\operatorname{Arg}(z))$

$\operatorname{Im}(x + i\, y) == \sqrt{x^2 + y^2}\; \sin\left(\tan^{-1}(x, y)\right).$

Geometrically, the complex conjugate of a complex number z is the complex point $\bar{z}$, which is symmetrical to z with respect to the real axis. So, the conjugate value $\bar{z}$ of the complex number $z == x + i\, y /; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$ can be presented by the formulas:

$\bar{z} == |z|\, e^{-i \operatorname{Arg}(z)}$

$\overline{x - i\, y} == x - i\, y.$

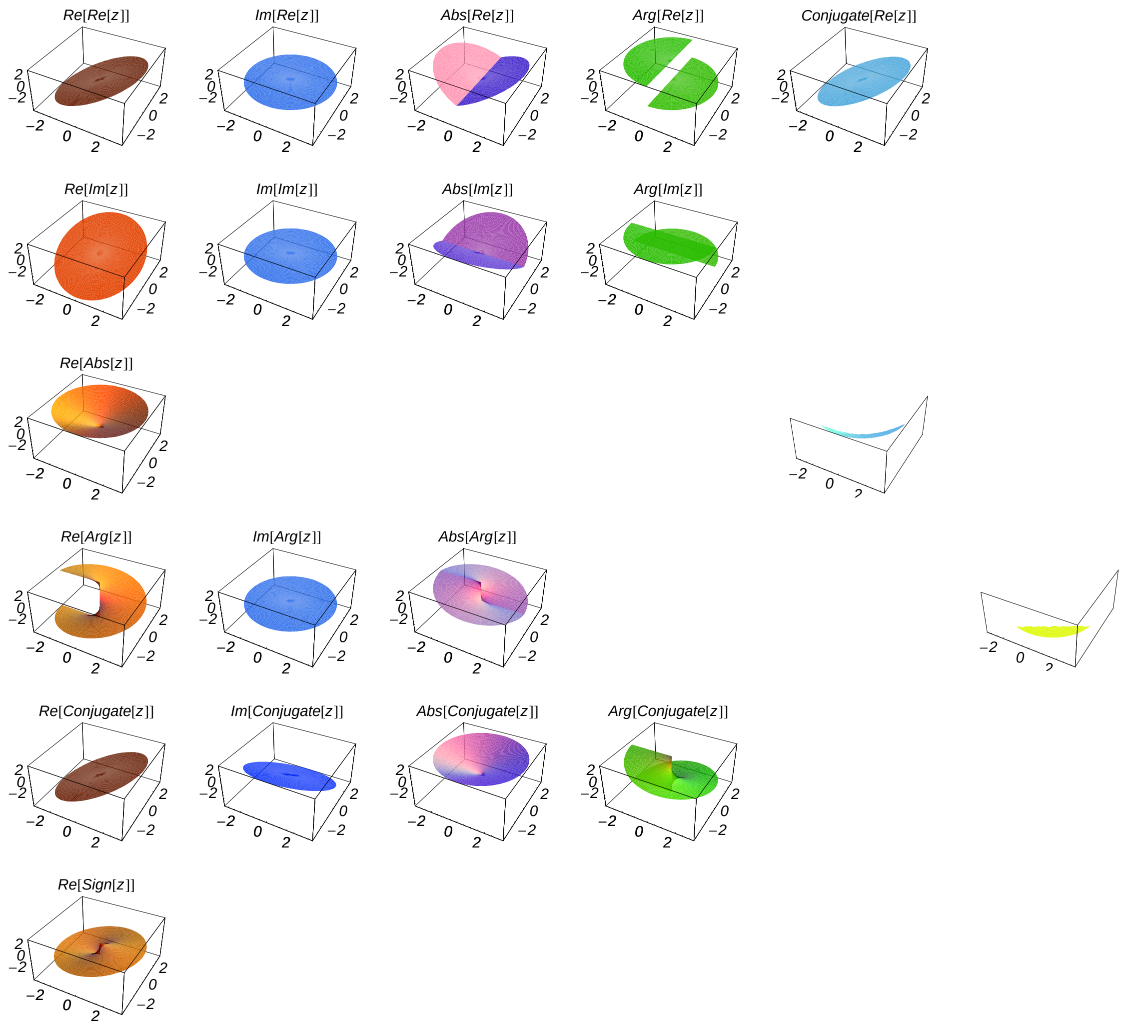
Geometrically, the sign function (signum) is the complex point $\operatorname{sgn}(z)$ that lays on the intersection of the unit circle $|z| == 1$ and the line from 0 to z (if $z \neq 0$). So, the conjugate value $\operatorname{sgn}(z)$ of the complex number $z == x + i\, y /; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$ can be presented by the formulas:

$\operatorname{sgn}(z) == \frac{z}{\sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}} /; z \neq 0$

$$\operatorname{sgn}(x + i\, y) = \frac{x + i\, y}{\sqrt{x^2 + y^2}} \;/;\; \{x,\, y\} \neq \{0,\, 0\}.$$

A quick look at the complex components

Here is a quick look at the graphics for the complex components of the complex components over the complex z-plane. The empty graphic indicates that the function value is not real.



Connections within the group of complex components and with other function groups

Representations through more general functions

All six complex component functions $z \to |z|$, $z \to \operatorname{Arg}(z)$, $z \to \operatorname{Re}(z)$, $z \to \operatorname{Im}(z)$, $z \to \bar{z}$, and sz → gn(z) cannot be easily represented by more generalized functions because most of them are analytic functions of their arguments. But sometimes such representations can be found through Meijer G functions, for example:

$$|x|^a = \frac{\pi}{\Gamma(-a)} \sec\left(\frac{a\,\pi}{2}\right) G_{2,2}^{1,1}\left(1 - x \,\middle|\, \begin{matrix} a + 1,\; \frac{a+1}{2} \\ 0,\; \frac{a+1}{2} \end{matrix}\right) /;\; x < 1.$$

Representations through other functions

All six complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z) satisfy numerous internal relations of the type $f(z) = g(f_1(z), z)$, where $f(z)$ and $f_1(z)$ are different complex components and g is a basic arithmetic operation or (composition of) elementary functions. The most important of these relations are represented in the following table:

z	**|z|**	**Arg(z)**	**Re(z)**	**Im(z)**	$\bar{z}$
|z|		$\lvert z\rvert = z\, e^{-i \operatorname{Arg}(z)}$	$\lvert z\rvert = \sqrt{2 z \operatorname{Re}(z) - z^2}$	$\lvert z\rvert = \sqrt{z^2 - 2 i z \operatorname{Im}(z)}$	$\lvert z\rvert = \sqrt{z \bar{z}}$
Arg (z)	$\operatorname{Arg}(z) = -i \log\left(\frac{z}{\lvert z\rvert}\right)$		$\operatorname{Arg}(z) = \tan^{-1}(\operatorname{Re}(z), -i (z - \operatorname{Re}(z)))$	$\operatorname{Arg}(z) = \tan^{-1}(z - i \operatorname{Im}(z), \operatorname{Im}(z))$	$\operatorname{Arg}(z) = \frac{1}{2} i (\log($ $- 2\log(z))$
Re (z)	$\operatorname{Re}(z) = \frac{z^2 + \lvert z\rvert^2}{2 z}$	$\operatorname{Re}(z) = \frac{z}{2} e^{-2 i \operatorname{Arg}(z)} \left(1 + e^{2 i \operatorname{Arg}(z)}\right)$		$\operatorname{Re}(z) = z - i \operatorname{Im}(z)$	$\operatorname{Re}(z) = \frac{z + \bar{z}}{2}$
Im (z)	$\operatorname{Im}(z) = \frac{i \left(\lvert z\rvert^2 - z^2\right)}{2 z}$	$\operatorname{Im}(z) = \frac{z}{2} i\, e^{-2 i \operatorname{Arg}(z)} \left(1 - e^{2 i \operatorname{Arg}(z)}\right)$	$\operatorname{Im}(z) = i (\operatorname{Re}(z) - z)$		$\operatorname{Im}(z) = \frac{z - \bar{z}}{2 i}$
$\bar{z}$	$\bar{z} = \frac{\lvert z\rvert^2}{z}$	$\bar{z} = e^{-2 i \operatorname{Arg}(z)} z$	$\bar{z} = 2 \operatorname{Re}(z) - z$	$\bar{z} = z - 2 i \operatorname{Im}(z)$	
sgn (z)	$\operatorname{sgn}(z) = \frac{z}{\lvert z\rvert} /; z \neq 0$	$\operatorname{sgn}(z) = e^{i \operatorname{Arg}(z)}$	$\operatorname{sgn}(z) = \frac{z}{\sqrt{2 z \operatorname{Re}(z) - z^2}}$	$\operatorname{sgn}(z) = \frac{z}{\sqrt{z^2 - 2 i z \operatorname{Im}(z)}}$	$\operatorname{sgn}(z) = \frac{z}{\sqrt{z \bar{z}}} /;$

Other internal relations between complex components of the type $f(z) = g(f_1(z), f_2(z), z)$, where $f(z)$ and $f_j(z)$ are different complex components that also exist. Some of them are shown here:

z	**|z|**	**Arg(z)**	**Re(z)**	**Im(z)**	$\bar{z}$
|z|		$z (\cos(\operatorname{Arg}(z)) - i \sin(\operatorname{Arg}(z)))$	$\lvert z\rvert = \frac{\operatorname{Re}(z)}{\cos(\operatorname{Arg}(z))}$	$\lvert z\rvert = \frac{\operatorname{Im}(z)}{\sin(\operatorname{Arg}(z))}$	
Arg (z)	$\operatorname{Arg}(z) = i (\log(\lvert z\rvert) - \log(z))$		$\cos(\operatorname{Arg}(z)) = \frac{\operatorname{Re}(z)}{\lvert z\rvert}$	$\sin(\operatorname{Arg}(z)) = \frac{\operatorname{Im}(z)}{\lvert z\rvert}$	$\operatorname{Arg}(z) = \frac{1}{2} i (\log(z \bar{z}$ $2 \log(z))$
Re (z)		$\operatorname{Re}(z) = \lvert z\rvert \cos(\operatorname{Arg}(z))$		$\operatorname{Re}(z) = \operatorname{Im}(i z)$	$\operatorname{Re}(z) = \bar{z} + i \operatorname{Im}(z)$
Im (z)		$\operatorname{Im}(z) = \lvert z\rvert \sin(\operatorname{Arg}(z))$	$\operatorname{Im}(z) = -\operatorname{Re}(i z)$		$\operatorname{Im}(z) = i (\bar{z} - \operatorname{Re}(z)$
$\bar{z}$	$\bar{z} = \frac{\lvert z\rvert^2}{z}$	$\bar{z} = \lvert z\rvert\, e^{-i \operatorname{Arg}(z)}$		$\bar{z} = \operatorname{Re}(z) - i \operatorname{Im}(z)$	
sgn (z)				$\operatorname{sgn}(z) = \frac{z}{\sqrt{\operatorname{Im}(z)^2 + \operatorname{Re}(z)^2}} /; z \neq 0$	

Here are some more formulas of the last type:

$$\lvert z\rvert = \sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}$$

$$\operatorname{Arg}(z) = \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))$$

$$\operatorname{Arg}(z) = \tan^{-1}\left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)}\right) /; \operatorname{Re}(z) > 0$$

$$\bar{z} = \lvert z\rvert \cos(\operatorname{Arg}(z)) - i \lvert z\rvert \sin(\operatorname{Arg}(z))$$

$$\operatorname{sgn}(z) = \frac{\operatorname{Re}(z) + i \operatorname{Im}(z)}{\sqrt{\operatorname{Re}(z)^2 + \operatorname{Im}(z)^2}}$$

$$\operatorname{sgn}(x) = \theta(x) - \theta(-x) \;/;\; x \in \mathbb{R}$$

$$\operatorname{sgn}(x) = 2\,\theta(x) - 1 \;/;\; x \in \mathbb{R} \land x \neq 0,$$

(here $\theta(x)$ is the Heaviside theta function, also called the unit step function).

The first table can be rewritten using the notations $z = x + i y \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$:

$x + i y$ /; $x \in \mathbb{R} \bigwedge y \in \mathbb{R}$	**$\lvert x + i y \rvert$**	**$\operatorname{Arg}(x + i y)$**	**$\operatorname{Re}(x + i y)$**	**$\operatorname{Im}(x + i y)$**
$\lvert x + i y \rvert$		$\lvert x + i y \rvert = (x + i y)\, e^{-i \operatorname{Arg}(x+i y)}$	$\lvert x + i y \rvert = \sqrt{2 (x + i y) x - (x + i y)^2}$	$\lvert x + i y \rvert = \sqrt{(x + i y)^2 - 2 i (}$
$\operatorname{Arg}(x + i y)$	$\operatorname{Arg}(x + i y) = -i \log\left(\frac{x+i y}{\sqrt{x^2+y^2}}\right)$		$\operatorname{Arg}(x + i y) = \tan^{-1}(x, -i (x + i y - x))$	$\operatorname{Arg}(x + i y) = \tan^{-1}$ $i y, y)$
$\operatorname{Re}(x + i y)$	$\operatorname{Re}(x + i y) = \frac{(x+i y)^2+x^2+y^2}{2 (x+i y)}$	$\operatorname{Re}(x + i y) = \frac{x+i y}{2} e^{-2 i \operatorname{Arg}(x+i y)} \left(1 + e^{2 i \operatorname{Arg}(x+i y)}\right)$		$\operatorname{Re}(x + i y) = (x + i$
$\operatorname{Im}(x + i y)$	$\operatorname{Im}(x + i y) = \frac{i \left(x^2+y^2-(x+i y)^2\right)}{2 (x+i y)}$	$\operatorname{Im}(x + i y) = \frac{x+i y}{2} i\, e^{-2 i \operatorname{Arg}(x+i y)} \left(1 - e^{2 i \operatorname{Arg}(x+i y)}\right)$	$\operatorname{Im}(x + i y) = i (x - (x + i y))$	
$\overline{x + i y}$	$\overline{x + i y} = \frac{x^2+y^2}{x+i y}$	$\overline{x + i y} = e^{-2 i \operatorname{Arg}(x+i y)} (x + i y)$	$\overline{x + i y} = 2 x - (x + i y)$	$\overline{x + i y} = x + i y - 2$
$\operatorname{sgn}(x + i y)$	$\operatorname{sgn}(x + i y) = \frac{x+i y}{\sqrt{x^2+y^2}}$ /; $\{x, y\} \neq \{0, 0\}$	$\operatorname{sgn}(x + i y) = e^{i \operatorname{Arg}(x+i y)}$	$\operatorname{sgn}(x + i y) = \frac{x+i y}{\sqrt{2 (x+i y) x-\left(x^2+y^2\right)}}$	$\operatorname{sgn}(x + i y) = \frac{x+i y}{\sqrt{(x+i y)^2-2 i (x+i y) y}}$

The best-known properties and formulas for complex components

Real values for real arguments

For real values of argument z, the values of all six complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and $\operatorname{sgn}(z)$ are real.

Simple values at zero

The six complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and $\operatorname{sgn}(z)$ have the following values for the argument $z = 0$:

$$|0| = 0$$

$$\operatorname{Arg}(0) = (-\pi, \pi]$$

$$\operatorname{Re}(0) = 0$$

$\operatorname{Im}(0) = 0$

$\overline{0} = 0$

$\operatorname{sgn}(0) = 0.$

Arg(0) is not a uniquely defined number. Depending on the argument of z, the limit $\lim_{|z| \to 0} \operatorname{Arg}(z)$ can take any value in the interval $(-\pi, \ \pi]$.

Specific values for specialized variable

The six complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and $\operatorname{sgn}(z)$ have the following values for some concrete numeric arguments:

z	$\lvert z\rvert$	$\operatorname{Arg}(z)$	$\operatorname{Re}(z)$	$\operatorname{Im}(z)$	$\bar{z}$	$\operatorname{sgn}(z)$
0	0	$\in(-\pi,\pi]$	0	0	0	0
1	1	0	1	0	1	1
-1	1	π	-1	0	-1	-1
i	1	$\frac{\pi}{2}$	0	1	$-i$	i
$-i$	1	$-\frac{\pi}{2}$	0	-1	i	$-i$
$1+i$	$\sqrt{2}$	$\frac{\pi}{4}$	1	1	$1-i$	$\frac{1+i}{\sqrt{2}}$
$-1+i$	$\sqrt{2}$	$\frac{3\pi}{4}$	-1	1	$-1-i$	$\frac{-1+i}{\sqrt{2}}$
$-1-i$	$\sqrt{2}$	$-\frac{3\pi}{4}$	-1	-1	$-1+i$	$-\frac{1+i}{\sqrt{2}}$
$1-i$	$\sqrt{2}$	$-\frac{\pi}{4}$	1	-1	$1+i$	$\frac{1-i}{\sqrt{2}}$
$\sqrt{3}+i$	2	$\frac{\pi}{6}$	$\sqrt{3}$	1	$\sqrt{3}-i$	$e^{\frac{i\pi}{6}}$
$1+i\sqrt{3}$	2	$\frac{\pi}{3}$	1	$\sqrt{3}$	$1-i\sqrt{3}$	$e^{\frac{i\pi}{3}}$
$-1+i\sqrt{3}$	2	$\frac{2\pi}{3}$	-1	$\sqrt{3}$	$-1-i\sqrt{3}$	$e^{\frac{2i\pi}{3}}$
$-\sqrt{3}+i$	2	$\frac{5\pi}{6}$	$-\sqrt{3}$	1	$-\sqrt{3}-i$	$e^{\frac{5i\pi}{6}}$
$-\sqrt{3}-i$	2	$-\frac{5\pi}{6}$	$-\sqrt{3}$	-1	$-\sqrt{3}+i$	$e^{-\frac{5i\pi}{6}}$
$-1-i\sqrt{3}$	2	$-\frac{2\pi}{3}$	-1	$-\sqrt{3}$	$-1+i\sqrt{3}$	$e^{-\frac{2i\pi}{3}}$
$1-i\sqrt{3}$	2	$-\frac{\pi}{3}$	1	$-\sqrt{3}$	$1+i\sqrt{3}$	$e^{-\frac{i\pi}{3}}$
$\sqrt{3}-i$	2	$-\frac{\pi}{6}$	$\sqrt{3}$	-1	$\sqrt{3}+i$	$e^{-\frac{i\pi}{6}}$
2	2	0	2	0	2	1
-2	2	π	-2	0	-2	-1
π	π	0	π	0	π	1
$3i$	3	$\frac{\pi}{2}$	0	3	$-3i$	i
$-2i$	2	$-\frac{\pi}{2}$	0	-2	$2i$	$-i$
$2+i$	$\sqrt{5}$	$\operatorname{ArcTan}\left[\frac{1}{2}\right]$	2	1	$2-i$	$\frac{2+i}{\sqrt{5}}$

Restricted arguments have the following formulas for the six complex components $\lvert z\rvert$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z):

z	**|z|**	**Arg(z)**	**Re(z)**	**Im(z)**	$\bar{z}$	**sgn(z)**
x /; $x \in \mathbb{R} \land x > 0$	x	0	x	0	x	1
x /; $x \in \mathbb{R} \land x < 0$	$-x$	π	x	0	x	-1
x /; $x \in \mathbb{R}$	$\sqrt{x^2}$	$\tan^{-1}(x, 0)$	x	0	x	$\frac{x}{\sqrt{x^2}}$ /; $x \neq 0$
$i\,y$ /; $y \in \mathbb{R}$	$\sqrt{y^2}$	$\tan^{-1}(0, y)$	0	y	$-i\,y$	i /; $y > 0$ 0 /; $y == 0$ $-i$ /; $y < 0$
$x + i\,y$ /; $x \in \mathbb{R} \land y \in \mathbb{R}$	$\sqrt{x^2 + y^2}$	$\tan^{-1}(x, y)$	x	y	$x - i\,y$	$\frac{x+i\,y}{\sqrt{x^2+y^2}}$ /; $\{x, y\} \neq \{0, 0\}$
$r\,e^{i\,\varphi}$ /; $r \in \mathbb{R} \land r > 0 \land \varphi \in \mathbb{R}$	r	$\varphi + 2\pi \left\lfloor \frac{\pi-\varphi}{2\pi} \right\rfloor$	$r\cos(\varphi)$	$r\sin(\varphi)$	$r\,e^{-i\,\varphi}$	$e^{i\,\varphi}$

The values of complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z) at any infinity can be described through the following:

z	**|z|**	**Arg(z)**	**Re (z)**	**Im(z)**	$\bar{z}$	**sgn (z)**
∞	∞	0	∞	0	∞	1
$-\infty$	∞	π	$-\infty$	0	$-\infty$	-1
$i\,\infty$	∞	$\frac{\pi}{2}$	0	∞	$-i\,\infty$	i
$-i\,\infty$	∞	$-\frac{\pi}{2}$	0	$-\infty$	$i\,\infty$	$-i$
$\tilde{\infty}$	∞	$\in (-\pi, \pi]$	¿	¿	$\tilde{\infty}$	¿
¿	¿	¿	¿	¿	¿	¿

Analyticity

All six complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z) are not analytical functions. None of them fulfills the Cauchy–Riemann conditions and as such the value of the derivative depends on the direction. The functions $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, and $\operatorname{Im}(z)$ are real-analytic functions of the variable z (except, maybe, $z \neq 0$). The real and the imaginary parts of $\bar{z}$ and sgn (z) are real-analytic functions of the variable z.

Sets of discontinuity

The four complex components $|z|$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, and $\bar{z}$ are continuous functions in $\mathbb{C}$.
The function sgn(z) has discontinuity at point $z == 0$.

The function $\operatorname{Arg}(z)$ is a single-valued, continuous function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above. Its behavior can be described by the following formulas:

$$\lim_{\epsilon \to +0} \operatorname{Arg}(x + i\,\epsilon) == \operatorname{Arg}(x) == \pi \text{ /; } x < 0$$

$$\lim_{\epsilon \to +0} \operatorname{Arg}(x - i\,\epsilon) == -\pi \text{ /; } x < 0.$$

Periodicity

All six complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z) do not have any periodicity.

Parity and symmetry

All six complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z) have mirror symmetry:

$$|\bar{z}| = |z| \quad \operatorname{Arg}(\bar{z}) = -\operatorname{Arg}(z) \;/;\; z \notin (-\infty, 0)$$

$$\operatorname{Re}(\bar{z}) = \overline{\operatorname{Re}(z)} \quad \operatorname{Im}(\bar{z}) = -\overline{\operatorname{Im}(z)}$$

$$\bar{\bar{z}} = z \quad \operatorname{sgn}(\bar{z}) = \overline{\operatorname{sgn}(z)}\ .$$

The absolute value $|z|$ is an even function. The four complex components $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z) are odd functions. The argument $\operatorname{Arg}(z)$ is an odd function for almost all z:

$$|-z| = |z|$$

$$\operatorname{Arg}(-z) = -\operatorname{Arg}(z) \;/;\; z \notin (-\infty, 0)$$

$$\operatorname{Arg}(-z) = \operatorname{Arg}(z) - \frac{\sqrt{-z}}{\sqrt{z}}\, i\, \pi$$

$$\operatorname{Re}(-z) = -\operatorname{Re}(z) \quad \operatorname{Im}(-z) = -\operatorname{Im}(z)$$

$$\overline{-z} = -\bar{z} \quad \operatorname{sgn}(-z) = -\operatorname{sgn}(z).$$

Homogeneity

The six complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z) have the following homogeneity properties:

$$|\lambda\, z| = |\lambda|\, |z|$$

$$\operatorname{Arg}(a\, z) = \operatorname{Arg}(z) \;/;\; a \in \mathbb{R} \land a > 0$$

$$\operatorname{Re}(a\, z) = a\operatorname{Re}(z) \;/;\; a \in \mathbb{R}$$

$$\operatorname{Im}(a\, z) = a\operatorname{Im}(z) \;/;\; a \in \mathbb{R}$$

$$\overline{a\, z} = \bar{a}\, \bar{z}$$

$$\operatorname{sgn}(a\, z) = \operatorname{sgn}(a)\operatorname{sgn}(z).$$

Scale symmetry

Some complex components have scale symmetry:

$$|z^a| = |z|^a \;/;\; a \in \mathbb{R}$$

$$\operatorname{sgn}(z^a) = \operatorname{sgn}(z)^a \;/;\; a \in \mathbb{R}.$$

Series representations

The functions $|x|$ and sgn(x) with real x have the following series expansions near point $x = 0$:

$$|x| = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4k^2 - 1} T_{2k}(x) + \frac{2}{\pi} \;/;\; x \in \mathbb{R} \land -1 < x < 1$$

$$|x| = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \left(2k + \frac{1}{2}\right) \left(-\frac{1}{2}\right)_k P_{2k}(x) \;/;\; x \in \mathbb{R} \land -1 < x < 1$$

$$|x| = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(-\frac{1}{2}\right)_k H_{2k}(x) \;/;\; x \in \mathbb{R} \land -1 < x < 1$$

$$\operatorname{sgn}(x) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} (2k+1) k!} H_{2k+1}(x) \;/;\; x \in \mathbb{R} \land -1 < x < 1$$

$$\operatorname{sgn}(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k - 1} T_{2k-1}(x) \;/;\; x \in \mathbb{R} \land -1 < x < 1$$

$$\operatorname{sgn}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (4k+3)(2k)!}{2^{2k+1} (k+1)! \, k!} P_{2k+1}(x) \;/;\; x \in \mathbb{R} \land -1 < x < 1.$$

Integral representations

The function sgn(x) with real x has the following contour integral representation:

$$\operatorname{sgn}(x) = \frac{1}{\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\Gamma(-s)(x+1)^{-s}}{\Gamma(1-s)} \, ds \;/;\; 0 < \gamma \bigwedge x > -2.$$

Limit representations

The functions $|x|$ and sgn (x) with real x have the following limit representations:

$$|x| = \lim_{n \to \infty} x \frac{p_n(x) - p_n(-x)}{p_n(x) + p_n(-x)} \;/;\; n \in \mathbb{N} \bigwedge -1 < x < 1 \bigwedge p_n(x) = \prod_{k=0}^{n-1} \left(x + e^{-\frac{k}{\sqrt{n}}}\right).$$

$$\operatorname{sgn}(x) = \lim_{m+n \to \infty} \frac{4\, n! \, \Gamma\left(m + \frac{3}{2}\right) \Gamma(m+n+2)}{\sqrt{\pi}\; m! \, \Gamma\left(n + \frac{1}{2}\right) \Gamma\left(m + n + \frac{3}{2}\right)} x \frac{{}_3F_2\left(-m, \frac{1}{2} - n, m+n+2; \frac{3}{2}, \frac{3}{2}; x^2\right)}{{}_3F_2\left(-n, -m - \frac{1}{2}, m+n+\frac{3}{2}; \frac{1}{2}, 1; x^2\right)} \;/;\; -1 < x < 1 \land n \in \mathbb{N} \land m \in \mathbb{N}$$

$$\operatorname{sgn}(x) = \lim_{m+n \to \infty} \frac{4(m+1)(2n+1)}{\pi} x \frac{{}_4F_3\left(-m, m+2, \frac{1}{2} - n, n + \frac{3}{2}; \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; x^2\right)}{{}_4F_3\left(-n, n+1, -m - \frac{1}{2}, m + \frac{3}{2}; \frac{1}{2}, 1, 1; x^2\right)} \;/;\; -1 < x < 1 \land n \in \mathbb{N} \land m \in \mathbb{N}.$$

The last two representations are sometimes called generalized Padé approximations.

Transformations

The values of all complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z) at the points $-z$, $i\,z$, $-i\,z$, $a\,z \;/;\; a \in \mathbb{R}$, $\frac{1}{z}$, z^a, e^z, and $e^{i\,z}$ are given by the following identities:

z	$-z$	$i\,z$	$-i\,z$	$a\,z$ /; $a \in$
$\lvert z\rvert$	$\lvert -z\rvert == \lvert z\rvert$	$\lvert i\,z\rvert == \lvert z\rvert$	$\lvert -i\,z\rvert == \lvert z\rvert$	$\lvert a\,z\rvert == a\,\lvert$ $\lvert a\,z\rvert == -c$
$\operatorname{Arg}(z)$	$\operatorname{Arg}(-z) == \operatorname{Arg}(z) - \frac{\sqrt{-z}}{\sqrt{z}}\,i\,\pi$	$\operatorname{Arg}(i\,z) == \operatorname{Arg}(z) - \frac{\pi}{2} - \frac{(-1)^{3/4}\,\pi\,\sqrt{i\,z}}{\sqrt{z}}$	$\operatorname{Arg}(-i\,z) == \operatorname{Arg}(z) + \frac{\pi}{2} - \frac{\sqrt[4]{-1}\,\pi\,\sqrt{-i\,z}}{\sqrt{z}}$	$\operatorname{Arg}(a\,z) =$
$\operatorname{Re}(z)$	$\operatorname{Re}(-z) == -\operatorname{Re}(z)$	$\operatorname{Re}(i\,z) == -\operatorname{Im}(z)$	$\operatorname{Re}(-i\,z) == \operatorname{Im}(z)$	$\operatorname{Re}(a\,z) ==$
$\operatorname{Im}(z)$	$\operatorname{Im}(-z) == -\operatorname{Im}(z)$	$\operatorname{Im}(i\,z) == \operatorname{Re}(z)$	$\operatorname{Im}(-i\,z) == -\operatorname{Re}(z)$	$\operatorname{Im}(a\,z) ==$
$\bar{z}$	$\overline{-z} == -\bar{z}$	$\overline{i\,z} == -i\,\bar{z}$	$\overline{-i\,z} == i\,\bar{z}$	$\overline{a\,z} == a\,\bar{z}$
$\operatorname{sgn}(z)$	$\operatorname{sgn}(-z) == -\operatorname{sgn}(z)$	$\operatorname{sgn}(i\,z) == i\operatorname{sgn}(z)$	$\operatorname{sgn}(-i\,z) == -i\operatorname{sgn}(z)$	$\operatorname{sgn}(a\,z) =$ $\operatorname{sgn}(a\,z) =$

z	$\frac{1}{z}$	z^a	e^z	$e^{i\,z}$
$\lvert z\rvert$	$\left\lvert\frac{1}{z}\right\rvert == \frac{1}{\lvert z\rvert}$	$\lvert z^a\rvert == \lvert z\rvert^{\operatorname{Re}(a)}\exp(-\operatorname{Im}(a)\operatorname{Arg}(z))$	$\lvert e^z\rvert = e^{\operatorname{Re}(z)}$	$\left\lvert e^{i\,z}\right\rvert == e^{-}$
$\operatorname{Arg}(z)$	$\operatorname{Arg}\left(\frac{1}{z}\right) == -\sqrt{z}\,\sqrt{z^{-1}}\,\operatorname{Arg}(z)$ $\operatorname{Arg}\left(\frac{1}{z}\right) == -\operatorname{Arg}(z)$ /; $\operatorname{Arg}(z) \neq \pi$	$\operatorname{Arg}(z^a) == 2\,\pi\left\lfloor\frac{\pi-\operatorname{Im}(a\log(z))}{2\,\pi}\right\rfloor + \operatorname{Im}(a\log(z))$	$\operatorname{Arg}(e^z) == \operatorname{Im}(z) + 2\,\pi\left\lfloor\frac{\pi-\operatorname{Im}(z)}{2\,\pi}\right\rfloor$	$\operatorname{Arg}\left(e^{i\,z}\right)$
$\operatorname{Re}(z)$	$\operatorname{Re}\left(\frac{1}{z}\right) == \frac{\operatorname{Re}(z)}{\lvert z\rvert^2}$	$\operatorname{Re}(z^a) == \lvert z\rvert^{\operatorname{Re}(a)}\,e^{-\operatorname{Im}(a)\operatorname{Arg}(z)}$ $\cos(\operatorname{Im}(a)\log(\lvert z\rvert) + \operatorname{Arg}(z)\operatorname{Re}(a))$	$\operatorname{Re}(e^z) == e^{\operatorname{Re}(z)}\cos(\operatorname{Im}(z))$	$\operatorname{Re}\left(e^{i\,z}\right) =$
$\operatorname{Im}(z)$	$\operatorname{Im}\left(\frac{1}{z}\right) == -\frac{\operatorname{Im}(z)}{\lvert z\rvert^2}$	$\operatorname{Im}(z^a) == \lvert z\rvert^{\operatorname{Re}(a)}\,e^{-\operatorname{Im}(a)\operatorname{Arg}(z)}$ $\sin(\operatorname{Im}(a)\log(\lvert z\rvert) + \operatorname{Arg}(z)\operatorname{Re}(a))$	$\operatorname{Im}(e^z) == e^{\operatorname{Re}(z)}\sin(\operatorname{Im}(z))$	$\operatorname{Im}\left(e^{i\,z}\right) =$
$\bar{z}$	$\overline{\frac{1}{z}} == \frac{1}{\bar{z}}$	$\overline{z^a} == \left(\frac{1}{\bar{z}}\right)^{-\bar{a}}$	$\overline{e^z} == e^{\bar{z}}$	$\overline{e^{i\,z}} == e^{-i}$
$\operatorname{sgn}(z)$	$\operatorname{sgn}\left(\frac{1}{z}\right) == \frac{\lvert z\rvert}{z}$	$\operatorname{sgn}(z^a) == \lvert z\rvert^{i\operatorname{Im}(a)}\exp(i\operatorname{Re}(a)\operatorname{Arg}(z))$	$\operatorname{sgn}(e^z) == e^{i\operatorname{Im}(z)}$	$\operatorname{sgn}\left(e^{i\,z}\right)$

The values of all complex components $\lvert z\rvert$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z) at the points $z_1 + z_2$, $z_1\,z_2$, and $\frac{z_1}{z_2}$ are described by the following table:

z	$z_1 + z_2$	$z_1\,z_2$	$\frac{z_1}{z_2}$
$\lvert z\rvert$	$\lvert z_1 + z_2\rvert == \lvert\lvert z_1\rvert - \lvert z_2\rvert\rvert + \lvert z_1\rvert + \lvert z_2\rvert - \lvert z_1 - z_2\rvert$	$\lvert z_1\,z_2\rvert == \lvert z_1\rvert\,\lvert z_2\rvert$	$\left\lvert\frac{z_1}{z_2}\right\rvert == \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$
$\operatorname{Arg}(z)$	$\operatorname{Arg}(z_1 + z_2) == \tan^{-1}(\operatorname{Re}(z_1) + \operatorname{Re}(z_2),\ \operatorname{Im}(z_1) + \operatorname{Im}(z_2))$	$\operatorname{Arg}(z_1\,z_2) == \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) +$ $2\,\pi\left\lfloor\frac{\pi-\operatorname{Arg}(z_1)-\operatorname{Arg}(z_2)}{2\,\pi}\right\rfloor$	$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) == \operatorname{Arg}(z_1)$ - $2\,\pi\left\lfloor\left(\pi - \operatorname{Arg}(z_1\right.\right.$
$\operatorname{Re}(z)$	$\operatorname{Re}(z_1 + z_2) == \operatorname{Re}(z_1) + \operatorname{Re}(z_2)$	$\operatorname{Re}(z_1\,z_2) == \operatorname{Re}(z_1)\operatorname{Re}(z_2) - \operatorname{Im}(z_2)\operatorname{Im}(z_1)$	$\operatorname{Re}\left(\frac{1}{z}\right) == \frac{\operatorname{Re}(z)}{\lvert z\rvert^2}$
$\operatorname{Im}(z)$	$\operatorname{Im}(z_1 + z_2) == \operatorname{Im}(z_1) + \operatorname{Im}(z_2)$	$\operatorname{Im}(z_1\,z_2) == \operatorname{Im}(z_2)\operatorname{Re}(z_1) + \operatorname{Im}(z_1)\operatorname{Re}(z_2)$	$\operatorname{Im}\left(\frac{1}{z}\right) == -\frac{\operatorname{Im}(z)}{\lvert z\rvert^2}$
$\bar{z}$	$\overline{z_1 + z_2} == \overline{z_1} + \overline{z_2}$	$\overline{z_1\,z_2} == \overline{z_1}\,\overline{z_2}$	$\overline{\frac{1}{z}} == \frac{1}{\bar{z}}$
$\operatorname{sgn}(z)$	$\operatorname{sgn}(z_1 + z_2) == \frac{z_1+z_2}{\lvert z_1+z_2\rvert}$	$\operatorname{sgn}(z_1\,z_2) == \operatorname{sgn}(z_1)\operatorname{sgn}(z_2)$	$\operatorname{sgn}\left(\frac{1}{z}\right) == \frac{\lvert z\rvert}{z}$

Some complex components can be easily evaluated in more general cases of the points including symbolic sums and products of z_k, $k = 1, \ldots, n$, for example:

$$\left|\prod_{k=1}^{n} z_k\right| = \prod_{k=1}^{n} |z_k|$$

$$\operatorname{Arg}\left(\prod_{k=1}^{n} z_k\right) = \sum_{k=1}^{n} \operatorname{Arg}(z_k) + 2\pi \left\lfloor \frac{\pi - \sum_{k=1}^{n} \operatorname{Arg}(z_k)}{2\pi} \right\rfloor /; n \in \mathbb{N}^+$$

$$\operatorname{Arg}(z^n) = n \operatorname{Arg}(z) + 2\pi \left\lfloor \frac{\pi - n \operatorname{Arg}(z)}{2\pi} \right\rfloor /; n \in \mathbb{N}^+$$

$$\operatorname{Re}\left(\sum_{k=1}^{n} z_k\right) = \sum_{k=1}^{n} \operatorname{Re}(z_k)$$

$$\operatorname{Re}(z^n) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^j \binom{n}{2j} \operatorname{Im}(z)^{2j} \operatorname{Re}(z)^{n-2j} /; n \in \mathbb{N}^+$$

$$\operatorname{Im}\left(\sum_{k=1}^{n} z_k\right) = \sum_{k=1}^{n} \operatorname{Im}(z_k)$$

$$\operatorname{Im}(z^n) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^j \binom{n}{2j+1} \operatorname{Im}(z)^{2j+1} \operatorname{Re}(z)^{n-2j-1} /; n \in \mathbb{N}^+$$

$$\overline{\sum_{k=1}^{n} z_k} = \sum_{k=1}^{n} \overline{z_k}$$

$$\overline{\prod_{k=1}^{n} z_k} = \prod_{k=1}^{n} \overline{z_k}$$

$$\operatorname{sgn}\left(\prod_{k=1}^{n} z_k\right) = \prod_{k=1}^{n} \operatorname{sgn}(z_k).$$

The previous tables and formulas can be modified or simplified for particular cases when some variables become real or satisfy special restrictions, for example:

z	x^a /; $x \in \mathbb{R} \land x > 0$	z^a /; $a \in \mathbb{R}$	e^{x+iy} /; $x \in \mathbb{R} \land y \in \mathbb{R}$
$\lvert z\rvert$	$\lvert x^a\rvert = x^{\operatorname{Re}(a)}$	$\lvert z^a\rvert = \lvert z\rvert^a$	$\lvert e^{x+iy}\rvert = e^x$
$\operatorname{Arg}(z)$	$\operatorname{Arg}(x^a) = \tan^{-1}(\cos(\operatorname{Im}(a)\log(x)), \sin(\operatorname{Im}(a)\log(x)))$	$\operatorname{Arg}(z^a) = a\operatorname{Arg}(z) + 2\pi\left\lfloor \frac{\pi - a\operatorname{Arg}(z)}{2\pi}\right\rfloor$	$\operatorname{Arg}(e^{x+iy}) = \tan^{-1}(\cos(y), \sin(y))$
$\operatorname{Re}(z)$	$\operatorname{Re}(x^a) = x^{\operatorname{Re}(a)}\cos(\operatorname{Im}(a)\log(x))$	$\operatorname{Re}(z^a) = \lvert z\rvert^a \cos(a\tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z)))$	$\operatorname{Re}(e^{x+iy}) = e^x\cos(y)$
$\operatorname{Im}(z)$	$\operatorname{Im}(x^a) = x^{\operatorname{Re}(a)}\sin(\operatorname{Im}(a)\log(x))$	$\operatorname{Im}(z^a) = \lvert z\rvert^a \sin(a\tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z)))$	$\operatorname{Im}(e^{x+iy}) = e^x\sin(y)$
$\bar{z}$	$\overline{x^a} = x^{\operatorname{Re}(a)}(\cos(\operatorname{Im}(a)\log(x)) - i\sin(\operatorname{Im}(a)\log(x)))$	$\overline{z^a} = \lvert z\rvert^a(\cos(a\tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))) - i\sin(a\tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))))$	$\overline{e^{x+iy}} = e^{x-iy}$
$\operatorname{sgn}(z)$	$\operatorname{sgn}(x^a) = x^{i\operatorname{Im}(a)}$	$\operatorname{sgn}(z^a) = \operatorname{sgn}(z)^a$	$\operatorname{sgn}(e^{x+iy}) = e^{iy}$

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) /; -\pi < \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) \le \pi$$

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) == \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) \;/;\; -\pi < \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) \leq \pi.$$

Taking into account that complex components have numerous representations through other complex components and elementary functions such as the logarithm, exponential function, or the inverse tangent function, all of the previous formulas can be transformed into different equivalent forms. Here are some of the resulting formulas for the power function $z \to z^\alpha$:

$$|z^a| == \exp(i\, a \operatorname{Im}(\log(z))) \;/;\; i\, a \in \mathbb{R}$$

$$|z^a| == \exp(i\, a \operatorname{Arg}(z)) \;/;\; i\, a \in \mathbb{R}$$

$$|z^a| == \exp(\operatorname{Re}(a \log(z)))$$

$$|z^a| == \exp(\operatorname{Re}(a) \log(|z|) - \operatorname{Im}(a) \operatorname{Arg}(z))$$

$$|z^a| == |z|^{\operatorname{Re}(a)} \exp\left(-\operatorname{Im}(a) \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\right)$$

$$\operatorname{Arg}(z^a) == a \operatorname{Arg}(z) \;/;\; a \in \mathbb{R} \bigwedge -\pi < a \operatorname{Arg}(z) \leq \pi$$

$$\operatorname{Arg}(z^a) == \operatorname{Arg}\left(e^{i\, a \operatorname{Arg}(z)}\right) \;/;\; a \in \mathbb{R}$$

$$\operatorname{Arg}(z^a) == \operatorname{Im}(a \log(z)) \;/;\; -\frac{\pi}{\log(z)} < \operatorname{Im}(a) \leq \frac{\pi}{\log(z)} \bigwedge z \in \mathbb{R} \bigwedge z > 0$$

$$\operatorname{Arg}(z^a) == \operatorname{Im}(a) \log(|z|) + \operatorname{Arg}(z) \operatorname{Re}(a) \;/;\; -\frac{\pi}{\log(z)} < \operatorname{Im}(a) \leq \frac{\pi}{\log(z)} \bigwedge z \in \mathbb{R} \bigwedge z > 0$$

$$\operatorname{Arg}(z^a) == \tan^{-1}\left(\cos\left(a \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\right), \sin\left(a \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\right)\right)$$

$$\operatorname{Arg}(z^a) == 2\pi \left\lfloor \frac{\pi - \operatorname{Im}(a \log(z))}{2\pi} \right\rfloor + \operatorname{Im}(a \log(z))$$

$$\operatorname{Arg}(z^a) == 2\pi \left\lfloor \frac{\pi - \operatorname{Im}(a) \log(|z|) - \operatorname{Arg}(z) \operatorname{Re}(a)}{2\pi} \right\rfloor + \operatorname{Im}(a) \log(|z|) + \operatorname{Arg}(z) \operatorname{Re}(a)$$

$$\operatorname{Arg}(z^a) == \tan^{-1}\left(\cos\left(\operatorname{Im}(a) \log(|z|) + \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z)) \operatorname{Re}(a)\right), \sin\left(\operatorname{Im}(a) \log(|z|) + \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z)) \operatorname{Re}(a)\right)\right)$$

$$\operatorname{Re}(z^a) == |z|^a \cos(a \operatorname{Arg}(z)) \;/;\; a \in \mathbb{R}$$

$$\operatorname{Re}(z^a) == \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-a)_{j+l}}{(l-j)!\, j!\, \left(\frac{1}{2}\right)_j} (1 - \operatorname{Re}(z))^l \left(\frac{\operatorname{Im}(z)^2}{4(\operatorname{Re}(z) - 1)}\right)^j \;/;\; a \in \mathbb{R}$$

$$\operatorname{Re}(z^a) == |z|^{\operatorname{Re}(a)} e^{-\operatorname{Im}(a) \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))} \cos\left(\operatorname{Im}(a) \log(|z|) + \operatorname{Re}(a) \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\right)$$

$$\operatorname{Im}(z^a) == |z|^a \sin(a \operatorname{Arg}(z)) \;/;\; a \in \mathbb{R}$$

$$\operatorname{Im}(z^a) == |z|^{\operatorname{Re}(a)} e^{-\operatorname{Im}(a) \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))} \sin\left(\operatorname{Im}(a) \log(|z|) + \operatorname{Re}(a) \tan^{-1}(\operatorname{Re}(z), \operatorname{Im}(z))\right)$$

$$\overline{z^a} == \bar{z}^a \;/;\; a \in \mathbb{R}$$

$$\overline{x^a} == x^{\bar{a}} \;/;\; x \in \mathbb{R} \land x > 0$$

$\overline{z^a} = \bar{z}^{\bar{a}}$ /; $\mathrm{Arg}(z) \neq \pi$

$\overline{z^a} = |z|^{\mathrm{Re}(a)} \exp(-\mathrm{Arg}(z)\, \mathrm{Im}(a) - i\, (\mathrm{Im}(a) \log(|z|) + \mathrm{Arg}(z)\, \mathrm{Re}(a)))$

$\overline{z^a} = |z|^{\mathrm{Re}(a)}\, e^{-\mathrm{Im}(a) \tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z))} \left(\cos\left(\mathrm{Im}(a) \log(|z|) + \mathrm{Re}(a) \tan^{-1}(\mathrm{Re}(z),\, \mathrm{Im}(z))\right) - i \sin\left(\mathrm{Im}(a) \log(|z|) + \mathrm{Re}(a) \tan^{-1}(\mathrm{Re}(z),\, \mathrm{Im}(z))\right)\right)$

$\mathrm{sgn}(z^a) = \exp(a\, \mathrm{Re}(\log(z)))$ /; $i\, a \in \mathbb{R}$

$\mathrm{sgn}(z^a) = |z|^a$ /; $i\, a \in \mathbb{R}$

$\mathrm{sgn}(z^a) = z^a \exp(-\mathrm{Re}(a \log(z)))$

$\mathrm{sgn}(z^a) = |z|^{i\, \mathrm{Im}(a)} \exp\left(i\, \mathrm{Re}(a) \tan^{-1}(\mathrm{Re}(z),\, \mathrm{Im}(z))\right)$

$\mathrm{sgn}(z^a) = \exp(i\, (\mathrm{Im}(a) \log(|z|) + \mathrm{Arg}(z)\, \mathrm{Re}(a)))$.

Similar identities can be derived for the exponent functions, such as:

$\mathrm{Arg}(e^z) = \mathrm{Im}(z)$ /; $-\pi < \mathrm{Im}(z) \leq \pi$

$\mathrm{Arg}(e^z) = \pi - (\pi - \mathrm{Im}(z)) \bmod (2\pi)$

$\mathrm{Arg}\left(e^{i z}\right) = \pi - (\pi - \mathrm{Re}(z)) \bmod (2\pi)$.

Some arithmetical operations involving complex components or elementary functions of complex components are:

$|z_1|\, |z_2| = |z_1\, z_2|$

$|z|^a = |z^a|$ /; $a \in \mathbb{R}$

$|z_1|^2 + |z_2|^2 = \frac{1}{2} \left(|z_1 - z_2|^2 + |z_1 + z_2|^2\right)$

$\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) = \mathrm{Arg}(z_1\, z_2) - 2\pi \left\lfloor \frac{\pi - \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)}{2\pi} \right\rfloor$

$\mathrm{Re}(z_1) + \mathrm{Re}(z_2) = \mathrm{Re}(z_1 + z_2)$

$\mathrm{Im}(z_1) + \mathrm{Im}(z_2) = \mathrm{Im}(z_1 + z_2)$

$\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$

$\overline{z_1}\, \overline{z_2} = \overline{z_1\, z_2}$

$\bar{z}^a = \overline{z^a}$ /; $a \in \mathbb{R}$

$e^{i\, \mathrm{Arg}(z)} = \frac{z}{|z|}$

$e^{i\, \mathrm{Arg}(z)} = \cos(\mathrm{Arg}(z)) + i \sin(\mathrm{Arg}(z))$

$e^{i\, \mathrm{Arg}(z)} = \cos\left(\tan^{-1}\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)\right) + i \sin\left(\tan^{-1}\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right)\right)$ /; $-\frac{\pi}{2} < \mathrm{Arg}(\mathrm{Re}(z)) \leq \frac{\pi}{2}$.

Complex characteristics

The next two tables describe all the complex components applied to all complex components $\lvert z\rvert$, $\mathrm{Arg}(z)$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\bar{z}$, and sgn(z) at the points z and $z == x + i\, y\, /;\, x \in \mathbb{R} \bigwedge y \in \mathbb{R}$:

z	**Abs**	**Arg**	**Re**	**Im**	**Conjugate**
Abs	$\lvert\lvert z\rvert\rvert == \lvert z\rvert$	$\lvert \mathrm{Arg}(z)\rvert == \sqrt{\tan^{-1}(\mathrm{Re}(z), \mathrm{Im}(z))^2}$	$\lvert \mathrm{Re}(z)\rvert == \lvert z\rvert\, \lvert\cos(\mathrm{Arg}(z))\rvert$	$\lvert \mathrm{Im}(z)\rvert == \sqrt{\mathrm{Im}(z)^2}$ $\lvert \mathrm{Im}(z)\rvert == \lvert z\rvert\, \lvert\sin(\mathrm{Arg}(z))\rvert$	$\lvert\bar{z}\rvert == \lvert z\rvert$
Arg	$\mathrm{Arg}(\lvert z\rvert) == 0$	$\mathrm{Arg}(\mathrm{Arg}(z)) == \tan^{-1}(\mathrm{Arg}(z), 0)$	$\mathrm{Arg}(\mathrm{Re}(z)) == \tan^{-1}(\mathrm{Re}(z), 0)$	$\mathrm{Arg}(\mathrm{Im}(z)) == \tan^{-1}(0, \mathrm{Im}(z))$ $\mathrm{Arg}(\mathrm{Im}(z)) == (1 - \theta(\mathrm{Im}(z)))\, \pi$	$\mathrm{Arg}(\bar{z}) == \tan^{-1}(\mathrm{Re}(x), -\mathrm{Im}(z))$
Re	$\mathrm{Re}(\lvert z\rvert) == \lvert z\rvert$	$\mathrm{Re}(\mathrm{Arg}(z)) == \mathrm{Arg}(z)$	$\mathrm{Re}(\mathrm{Re}(z)) == \mathrm{Re}(z)$	$\mathrm{Re}(\mathrm{Im}(z)) == \mathrm{Im}(z)$	$\mathrm{Re}(\bar{z}) == \mathrm{Re}(z)$
Im	$\mathrm{Im}(\lvert z\rvert) == 0$	$\mathrm{Im}(\mathrm{Arg}(z)) == 0$	$\mathrm{Im}(\mathrm{Re}(z)) == 0$	$\mathrm{Im}(\mathrm{Im}(z)) == 0$	$\mathrm{Im}(\bar{z}) == -\mathrm{Im}(z)$
Conjugate	$\overline{\lvert z\rvert} == \lvert z\rvert$	$\overline{\mathrm{Arg}(z)} == \mathrm{Arg}(z)$	$\overline{\mathrm{Re}(z)} == \mathrm{Re}(z)$	$\overline{\mathrm{Im}(z)} == \mathrm{Im}(z)$	$\bar{\bar{z}} == z$
Sign	$\mathrm{sgn}(\lvert z\rvert) == 1\, /;\, z \neq 0$	$\mathrm{sgn}(\mathrm{Arg}(z)) == \frac{\mathrm{Arg}(z)}{\sqrt{\mathrm{Arg}(z)^2}}$	$\mathrm{sgn}(\mathrm{Re}(z)) == \frac{\mathrm{Re}(z)}{\sqrt{\mathrm{Re}(z)^2}}$	$\mathrm{sgn}(\mathrm{Im}(z)) == \frac{\mathrm{Im}(z)}{\sqrt{\mathrm{Im}(z)^2}}$	$\mathrm{sgn}(\bar{z}) == \frac{\bar{z}}{\lvert z\rvert}\, /;\, z \neq 0$

$\boldsymbol{x + i\, y\, /;\, x \in \mathbb{R} \bigwedge y \in \mathbb{R}}$	**Abs**	**Arg**	**Re**	**Im**	**Conjugate**
Abs	$\lvert\lvert x + i\, y\rvert\rvert == \sqrt{x^2 + y^2}$	$\lvert \mathrm{Arg}(x + i\, y)\rvert == \sqrt{\tan^{-1}(x, y)^2}$	$\lvert \mathrm{Re}(x + i\, y)\rvert == \sqrt{x^2}$	$\lvert \mathrm{Im}(x + i\, y)\rvert == \sqrt{y^2}$	$\lvert\overline{x + i\, y}\rvert == \sqrt{x^2}$
Arg	$\mathrm{Arg}(\lvert x + i\, y\rvert) == 0$	$\mathrm{Arg}(\mathrm{Arg}(x + i\, y)) == \tan^{-1}(\tan^{-1}(x, y), 0)$	$\mathrm{Arg}(\mathrm{Re}(x + i\, y)) == \tan^{-1}(x, 0)$ $\mathrm{Arg}[\mathrm{Re}[x + i\, y]] == (1 - \mathrm{UnitStep}[x])\, \pi$	$\mathrm{Arg}(\mathrm{Im}(x + i\, y)) == \tan^{-1}(y, 0)$ $\mathrm{Arg}(\mathrm{Im}(x + i\, y)) == (1 - \theta(y))\, \pi$	$\mathrm{Arg}(\bar{z}) == -\mathrm{Arg}($ $\mathrm{Arg}(z) \neq \pi$
Re	$\mathrm{Re}(\lvert x + i\, y\rvert) == \sqrt{x^2 + y^2}$	$\mathrm{Re}(\mathrm{Arg}(x + i\, y)) == \tan^{-1}(x, y)$	$\mathrm{Re}(\mathrm{Re}(x + i\, y)) == x$	$\mathrm{Re}(\mathrm{Im}(x + i\, y)) == y$	$\mathrm{Re}(\overline{x + i\, y}) == x$
Im	$\mathrm{Im}(\lvert x + i\, y\rvert) == 0$	$\mathrm{Im}(\mathrm{Arg}(x + i\, y)) == 0$	$\mathrm{Im}(\mathrm{Re}(x + i\, y)) == 0$	$\mathrm{Im}(\mathrm{Im}(x + i\, y)) == 0$	$\mathrm{Im}(\overline{x + i\, y}) == -$
Conjugate	$\overline{\lvert x + i\, y\rvert} == \sqrt{x^2 + y^2}$	$\overline{\mathrm{Arg}(x + i\, y)} == \tan^{-1}(x, y)$	$\overline{\mathrm{Re}(x + i\, y)} == x$	$\overline{\mathrm{Im}(x + i\, y)} == y$	$\overline{\overline{x + i\, y}} == x + i\, y$
Sign	$\mathrm{sgn}(\lvert x + i\, y\rvert) == 1\, /;\, x + i\, y \neq 0$	$\mathrm{sgn}(\mathrm{Arg}(x + i\, y)) == \frac{\tan^{-1}(x,y)}{\sqrt{\tan^{-1}(x,y)^2}}$	$\mathrm{sgn}(\mathrm{Re}(x + i\, y)) == \mathrm{sgn}(x) == \frac{x}{\sqrt{x^2}}$	$\mathrm{sgn}(\mathrm{Im}(x + i\, y)) == \mathrm{sgn}(y) == \frac{y}{\sqrt{y^2}}$	$\mathrm{sgn}(\overline{x + i\, y}) == \frac{x - i\, y}{\sqrt{x^2+y^2}}$

Differentiation

The derivatives of five complex components $\lvert x\rvert$, $\mathrm{Re}(x)$, $\mathrm{Im}(x)$, $\bar{x}$, and sgn(x) at the real point $x \in \mathbb{R}$ can be interpreted in a real-analytic or distributional sense and are given by the following formulas:

$$\frac{\partial |x|}{\partial x} == \operatorname{sgn}(x)$$

$$\frac{\partial \operatorname{Re}(x)}{\partial x} == 1$$

$$\frac{\partial \operatorname{Im}(x)}{\partial x} == 0$$

$$\frac{\partial \overline{x}}{\partial x} == 1$$

$$\frac{\partial \operatorname{sgn}(x)}{\partial x} == 2\,\delta(x),$$

where $\delta(x)$ is the Dirac delta function.

It is impossible to make a classical, direction-independent interpretation of these derivatives for complex values of variable x because the complex components do not fulfill the Cauchy-Riemann conditions.

Indefinite integration

The indefinite integrals of some complex components at the real point $x \in \mathbb{R}$ can be represented by the following formulas:

$$\int |x|\, dx == \frac{x\,|x|}{2}$$

$$\int \operatorname{sgn}(x)\, dx == |x|.$$

Definite integration

The definite integrals of some complex components in the complex plane can also be represented through complex components, for example:

$$\int_{-1}^{1} |t|\, dt == 1$$

$$\int_{-z}^{z} |t|\, dt == z\sqrt{\operatorname{Im}(z)^2 + \operatorname{Re}(z)^2}$$

$$\int_{0}^{z} |t|\, dt == \frac{|z|\,z}{2}$$

$$\int_{-z}^{z} \operatorname{sgn}(t)\, dt == 0$$

$$\int_{0}^{z} \operatorname{sgn}(t)\, dt == \sqrt{z^2}\ .$$

Some definite integrals including absolute values can be easily evaluated, for example (in the Hadamard sense of integration, the next identity is correct for all complex values of n):

$$\int_0^1 \int_0^1 |x-y|^n \, dx \, dy = \frac{2}{(n+1)(n+2)} \;/;\; \operatorname{Re}(n) > -2.$$

Integral transforms

Fourier integral transforms of the absolute value and signum functions $|t|$ and sgn(t) can be evaluated through generalized functions:

$$\mathcal{F}_t[|t|](z) = -\sqrt{\frac{2}{\pi}} \frac{1}{z^2}$$

$$\mathcal{F}_t^{-1}[|t|](z) = -\sqrt{\frac{2}{\pi}} \frac{1}{z^2}$$

$$\mathcal{F}c_t[|t|](z) = -\sqrt{\frac{2}{\pi}} \frac{1}{z^2}$$

$$\mathcal{F}s_t[|t|](z) = -\sqrt{\frac{\pi}{2}} \delta'(z)$$

$$\mathcal{F}c_t[\operatorname{sgn}(t)](z) = \sqrt{\frac{\pi}{2}} \delta(z)$$

$$\mathcal{F}s_t[\operatorname{sgn}(t)](z) = \sqrt{\frac{2}{\pi}} \frac{1}{z}.$$

Laplace integral transforms of these functions can be evaluated in a classical sense and have the following values:

$$\mathcal{L}_t[|t|](z) = \frac{1}{z^2}$$

$$\mathcal{L}_t[\operatorname{sgn}(t)](z) = \frac{1}{z}.$$

Differential equations

The absolute value function $|x|$ for real $x \in \mathbb{R}$ satisfies the following simple first-order differential equation understandable in a distributional sense:

$$w'(x) = \operatorname{sign}(w(x)) \;/;\; w(x) = |x|.$$

In a similar manner:

$$w'(x) = 1 \;/;\; w(x) = \operatorname{Re}(x)$$

$$w'(x) = 0 \;/;\; w(x) = \operatorname{Im}(x)$$

$$w'(x) = 2\pi\,\delta(x) \;/;\; w(x) = \operatorname{Arg}(x).$$

Inequalities

All six complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$ and sgn(z) satisfy numerous inequalities. The best known are so-called triangle inequalities for absolute values:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|.$$

Some other inequalities can be described by the following formulas:

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

$$|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$|\operatorname{Arg}(z)| \leq \pi$$

$$|\operatorname{Re}(z)| \leq |z|$$

$$|\operatorname{Im}(z)| \leq |z|$$

$$\operatorname{Re}(\operatorname{sgn}(z)) \leq 1$$

$$\operatorname{Im}(\operatorname{sgn}(z)) \leq 1$$

$$|\operatorname{sgn}(z)| \leq 1.$$

Zeros

The six complex components $|z|$, $\operatorname{Arg}(z)$, $\operatorname{Re}(z)$, $\operatorname{Im}(z)$, $\bar{z}$, and sgn(z) have the set of zeros described by the following formulas:

$$|z| == 0 \;/;\; z == 0$$

$$\operatorname{Arg}(z) == 0 \;/;\; z \in \mathbb{R} \land z > 0$$

$$\operatorname{Re}(z) == 0 \;/;\; i\, z \in \mathbb{R}$$

$$\operatorname{Im}(z) == 0 \;/;\; z \in \mathbb{R}$$

$$\bar{z} == 0 \;/;\; z == 0$$

$$\operatorname{sgn}(z) == 0 \;/;\; z == 0.$$

Applications of complex components

All six complex components are used throughout mathematics, the exact sciences, and engineering.

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