Introductions to ArithmeticGeometricMean

Introduction to the Arithmetic-Geometric Mean

General

The arithmetic-geometric mean appeared in the works of J. Landen (1771, 1775) and J.-L. Lagrange (1784-1785) who defined it through the following quite-natural limit procedure:

$$\operatorname{agm}(a, b) == \lim_{n\to\infty} a_n == \lim_{n\to\infty} b_n \;/;\; a_0 == a > b_0 == b > 0 \bigwedge$$

$$a_{n+1} == \frac{1}{2}(a_n + b_n) == \operatorname{agm}(a_0, b_0)\, \vartheta_3\left(0, z^{2^{n+1}}\right)^2 \bigwedge b_{n+1} == \sqrt{a_n\, b_n} == \operatorname{agm}(a_0, b_0)\, \vartheta_4\left(0, z^{2^{n+1}}\right)^2 \bigwedge z == q\left(1 - \left(\frac{b_0}{a_0}\right)^2\right).$$

C. F. Gauss (1791–1799, 1800, 1876) continued to research this limit and in 1800 derived its representation through the hypergeometric function ${}_2F_1(a, b; c; z)$.

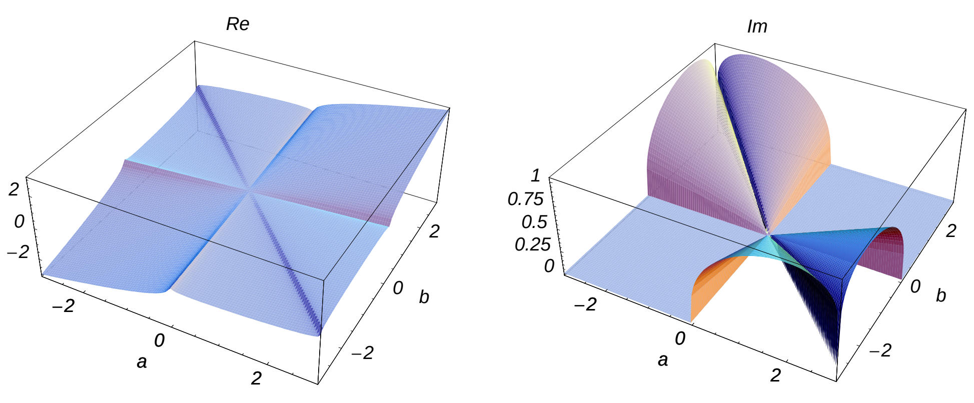
Definition of arithmetic-geometric mean

The arithmetic-geometric mean agm(a, b) is defined through the reciprocal value of the complete elliptic integral $K(z)$ by the formula:

$$\operatorname{agm}(a, b) == \frac{\pi\,(a+b)}{4\, K\left(\left(\frac{a-b}{a+b}\right)^2\right)}.$$

A quick look at the arithmetic-geometric mean

Here is a quick look at the graphic for the arithmetic-geometric mean over the real a–b-plane.



- GraphicsArray -

Connections within the arithmetic-geometric mean group and with other function groups

Representations through more general functions

The arithmetic-geometric mean agm(a, b) can be represented through the reciprocal function of the particular cases of hypergeometric and Meijer G functions:

$$\text{agm}(a, b) = \frac{a+b}{2\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{a-b}{a+b}\right)^2\right)}$$

$$\text{agm}(a, b) = \frac{\pi\,(a+b)}{2} \Big/ G_{2,2}^{1,2}\left(-\left(\frac{a-b}{a+b}\right)^2 \middle| \begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 0, 0 \end{matrix}\right).$$

Representations through related equivalent functions

The definition of the arithmetic-geometric mean agm(a, b) can be interpreted as a representation of agm(a, b) through related equivalent functions—the reciprocal of the complete elliptic integral $K(z)$ with $z = \left(\frac{a-b}{a+b}\right)^2$:

$$\text{agm}(a, b) = \frac{\pi\,(a+b)}{4\,K\left(\left(\frac{a-b}{a+b}\right)^2\right)}.$$

The best-known properties and formulas for the arithmetic-geometric mean

Values in points

The arithmetic-geometric mean agm(a, b) can be exactly evaluated in some points, for example:

$$\text{agm}(a, a) = a$$

$$\text{agm}(0, b) = 0$$

$$\text{agm}(1, b) = \frac{\pi}{2\,K(1-b^2)}$$

$$\text{agm}\left(a, \sqrt{2}\,a\right) = a\sqrt{\frac{2}{\pi}}\,\Gamma\left(\frac{3}{4}\right)^2$$

$$\text{agm}\left(1, \frac{\vartheta_4(0, z)^2}{\vartheta_3(0, z)^2}\right) = \frac{1}{\vartheta_3(0, z)^2} \;/;\; -1 < z < 1$$

$$\text{agm}(1, \infty) = \infty.$$

Real values for real arguments

For real values of arguments a, b (with $a\,b > 0$), the values of the arithmetic-geometric mean agm(a, b) are real.

Analyticity

The arithmetic-geometric mean agm(a, b) is an analytical function of a and b that is defined over $\mathbb{C}^2$.

Poles and essential singularities

The arithmetic-geometric mean agm(a, b) does not have poles and essential singularities.

Branch points and branch cuts

The arithmetic-geometric mean agm(a, b) on the $\frac{a}{b}$-plane has two branch points: $\frac{a}{b} = 0$ and $\frac{a}{b} = \tilde{\infty}$. It is a single-valued function on the $\frac{a}{b}$-plane cut along the interval $(-\infty, 0)$, where it is continuous from above:

$$\lim_{\epsilon \to +0} \text{agm}(a + i\,\epsilon, 1) == \text{agm}(a, 1) \;/;\; a < 0$$

$$\lim_{\epsilon \to +0} \text{agm}(a - i\,\epsilon, 1) == a\,\text{agm}\left(1, \frac{1}{a}\right) \;/;\; a < 0.$$

Periodicity

The arithmetic-geometric mean agm(a, b) does not have periodicity.

Parity and symmetries

The arithmetic-geometric mean agm(a, b) is an odd function and has mirror and permutation symmetry:

$$\text{agm}(-a, -b) == -\text{agm}(a, b) \;/;\; a \notin \mathbb{R} \bigwedge b \notin \mathbb{R}$$

$$\text{agm}\left(\overline{a}, \overline{b}\right) == \overline{\text{agm}(a, b)} \;/;\; \frac{a}{b} \notin (-\infty, 0)$$

$$\text{agm}(b, a) == \text{agm}(a, b).$$

The arithmetic-geometric mean agm(a, b) is the homogenous function:

$$\text{agm}(c\,a, c\,b) == c\,\text{agm}(a, b) \;/;\; c > 0.$$

Series representations

The arithmetic-geometric mean agm(a, b) has the following series representations at the points $\frac{b}{a} \to 0$, $\frac{b}{a} \to 1$, and $\left|\frac{b}{a}\right| \to \infty$:

$$\text{agm}(a, b) \propto \frac{\pi\,a}{2\left(\log(4) - \log\left(\frac{b}{a}\right)\right)} + \frac{\pi\left(\log\left(\frac{b}{4\,a}\right) + 1\right) b^2}{8\,a\left(\log(4) - \log\left(\frac{b}{a}\right)\right)^2} + \ldots \;/;\; \left(\frac{b}{a} \to 0\right)$$

$$\text{agm}(a, b) \propto a + \frac{b - a}{2} - \frac{(b - a)^2}{16\,a} + \ldots \;/;\; \left(\frac{b}{a} \to 1\right)$$

$$\text{agm}(a, b) \propto \frac{\pi\,b}{2\log\left(\frac{4\,b}{a}\right)} + \frac{\pi\,a^2\left(1 - \log\left(\frac{4\,b}{a}\right)\right)}{8\,b\log^2\left(\frac{4\,b}{a}\right)} + \ldots \;/;\; \left(\left|\frac{b}{a}\right| \to \infty\right).$$

Product representation

The arithmetic-geometric mean agm$(1, b)$ has the following infinite product representation:

$$\text{agm}(1, b) = \prod_{k=0}^{\infty} \frac{1}{2}(q_k + 1) /; q_0 = b \bigwedge q_{k+1} = \frac{2\sqrt{q_k}}{q_k + 1}.$$

Integral representation

The arithmetic-geometric mean agm(a, b) has the following integral representation:

$$\text{agm}(a, b) = \frac{\pi}{2} \Big/ \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{a^2 \cos^2(t) + b^2 \sin^2(t)}} \, dt /; a > 0 \bigwedge b > 0.$$

Limit representation

The arithmetic-geometric mean agm(a, b) has the following limit representation, which is often used for the definition of agm(a, b):

$$\text{agm}(a, b) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n /; a_0 = a > b_0 = b > 0 \bigwedge$$
$$a_{n+1} = \frac{1}{2}(a_n + b_n) = \text{agm}(a_0, b_0)\, \vartheta_3\left(0, z^{2^{n+1}}\right)^2 \bigwedge b_{n+1} = \sqrt{a_n b_n} = \text{agm}(a_0, b_0)\, \vartheta_4\left(0, z^{2^{n+1}}\right)^2 \bigwedge z = q\left(1 - \left(\frac{b_0}{a_0}\right)^2\right).$$

Transformations

The homogeneity property of the arithmetic-geometric mean agm(a, b) leads to the following transformations:

$$\text{agm}(a, b) = a \, \text{agm}\left(1, \frac{b}{a}\right) /; a > 0$$

$$\text{agm}(c\, a, c\, b) = c \, \text{agm}(a, b) /; c > 0$$

$$\text{agm}(-a, -b) = -\text{agm}(a, b) /; a \notin \mathbb{R} \bigwedge b \notin \mathbb{R}$$

$$\text{agm}(1, z) = \frac{1}{a} \text{agm}(a, a\, z) /; a > 0.$$

Another group of transformations is based on the first of the following properties:

$$\text{agm}\left(\frac{a+b}{2}, \sqrt{a\, b}\right) = \text{agm}(a, b)$$

$$\text{agm}\left(1, \sqrt{1 - z^2}\right) = \text{agm}(z + 1, 1 - z)$$

$$\text{agm}\left(1, \frac{2\sqrt{b}}{b+1}\right) = \frac{2}{b+1} \text{agm}(1, b).$$

Representations of derivatives

The first derivatives of the arithmetic-geometric mean agm(a, b) have rather simple representations:

$$\frac{\partial \operatorname{agm}(a,\, b)}{\partial a} = \frac{\operatorname{agm}(a,\, b)}{a\,(a-b)\,\pi}\left(a\,\pi - 2\operatorname{agm}(a,\, b)\,E\!\left(\frac{(a-b)^2}{(a+b)^2}\right)\right)$$

$$\frac{\partial \operatorname{agm}(a,\, b)}{\partial b} = \frac{\operatorname{agm}(a,\, b)}{(a-b)\,b\,\pi}\left(2\operatorname{agm}(a,\, b)\,E\!\left(\frac{(a-b)^2}{(a+b)^2}\right) - b\,\pi\right)$$

The n^{th}-order symbolic derivatives are much more complicated. Here is an example:

$$\frac{\partial^n \operatorname{agm}(a,\, b)}{\partial a^n} = \operatorname{agm}(a,\, b)\,\delta_n + \frac{\pi}{4\,b^n}\left(\frac{b\,\delta_{n-1}}{K\!\left(\left(\frac{a-b}{a+b}\right)^2\right)} + b\,n\,n!\sum_{q=1}^{n-1}\frac{(-1)^q}{(q+1)!\,(n-q-1)!}\,K\!\left(\left(\frac{a-b}{a+b}\right)^2\right)^{-q-1}\right.$$
$$\sum_{k_1=0}^{n-\sum_{j=1}^{p}k_j-1}\;\sum_{k_2=0}^{n-\sum_{j=1}^{p}k_j-1}\cdots\sum_{k_{q-1}=0}^{n-\sum_{j=1}^{p}k_j-1}\left(\prod_{p=1}^{q-1}\binom{n-\sum_{j=1}^{p-1}k_j-1}{k_p}\right)\left(\prod_{i=1}^{q-1}A(k_i,\, a,\, b)\right)A\!\left(n-\sum_{j=1}^{q-1}k_j-1,\, a,\, b\right)+$$
$$(a+b)\,(n+1)!\sum_{q=1}^{n}\frac{(-1)^q}{(q+1)!\,(n-q)!}\,K\!\left(\left(\frac{a-b}{a+b}\right)^2\right)^{-q-1}\sum_{k_1=0}^{n-\sum_{j=1}^{p}k_j}\;\sum_{k_2=0}^{n-\sum_{j=1}^{p}k_j}\cdots$$
$$\left.\sum_{k_{q-1}=0}^{n-\sum_{j=1}^{p}k_j}\left(\prod_{p=1}^{q-1}\binom{n-\sum_{j=1}^{p-1}k_j}{k_p}\right)\left(\prod_{i=1}^{q-1}A(k_i,\, a,\, b)\right)A\!\left(n-\sum_{j=1}^{q-1}k_j,\, a,\, b\right)\right) /;\; A(r,\, a,\, b) = K\!\left(\left(\frac{a-b}{a+b}\right)^2\right)\delta_r +$$
$$\frac{\pi}{2}\sum_{m=1}^{r}\frac{1}{m!}\sum_{s=0}^{m}\frac{1}{(m-s)!\,2^{m-2s}}\left((2\,s-m+1)_{2(m-s)}\left(\frac{a+b}{a-b}\right)^m {}_2\tilde{F}_1\!\left(\frac{1}{2},\, \frac{1}{2};\, 1-s;\, \left(\frac{a-b}{a+b}\right)^2\right)\sum_{q=0}^{m}(-1)^q\binom{m}{q}\left(\frac{a-b}{a+b}\right)^q\right.$$
$$\left.\sum_{u_1=0}^{r}\sum_{u_2=0}^{r}\cdots\sum_{u_{m-q}=0}^{r}\delta_{r,\sum_{i=1}^{m-q}u_i}\,(u_1+u_2+\ldots+u_{m-q};\, u_1,\, u_2,\, \ldots,\, u_{m-q})\prod_{i=1}^{m-q}\left(\delta_{u_i}-\frac{2\,(-1)^{u_i}\,b^{u_i+1}\,u_i!}{(a+b)^{u_i+1}}\right)\right) /;\; n\in\mathbb{N}.$$

Differential equations

The arithmetic-geometric mean $\operatorname{agm}(a, b)$ satisfies the following second-order ordinary nonlinear differential equation:

$$2\,a\left(b^2-a^2\right)\left(\frac{\partial w(a)}{\partial a}\right)^2 - a\,w(a)^2 + \left(\left(3\,a^2-b^2\right)\frac{\partial w(a)}{\partial a} + a\left(a^2-b^2\right)\frac{\partial^2 w(a)}{\partial a^2}\right)w(a) = 0 /;\; w(a) = \operatorname{agm}(a,\, b).$$

It can also be represented as partial solutions of the following partial differential equation:

$$\operatorname{agm}(a,\, b) - a\,\frac{\partial \operatorname{agm}(a,\, b)}{\partial a} - b\,\frac{\partial \operatorname{agm}(a,\, b)}{\partial b} = 0.$$

Inequalities

The arithmetic-geometric mean $\operatorname{agm}(a, b)$ lies between the middle geometric mean and middle arithmetic mean, which is shown in the following famous inequality:

$$\sqrt{a\,b} \le \operatorname{agm}(a,\, b) \le \frac{a+b}{2}.$$

Applications of the arithmetic-geometric mean

Applications of the arithmetic-geometric mean include fast high-precision computation of π, $\log(z)$, e^z, $\sin(z)$, $\cos(z)$, and so on.

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