Introductions to ExpIntegralEi

Introduction to the exponential integrals

General

The exponential-type integrals have a long history. After the early developments of differential calculus, mathematicians tried to evaluate integrals containing simple elementary functions, especially integrals that often appeared during investigations of physical problems. Despite the relatively simple form of the integrands, some of these integrals could not be evaluated through known functions. Examples of integrals that could not be evaluated in known functions are:

$$\int \frac{1}{\log(t)}\, dt$$

$$\int \frac{\sin(t)}{t}\, dt$$

$$\int \frac{\cos(t)}{t}\, dt$$

$$\int \frac{e^t}{t}\, dt.$$

L. Euler (1768) introduced the first integral shown in the preceding list. Later L. Mascheroni (1790, 1819) used it and introduced the second and third integrals, and A. M. Legendre (1811) introduced the last integral shown. T. Caluso (1805) used the first integral in an article and J. von Soldner (1809) introduced its notation through symbol li. F. W. Bessel (1812) used the second and third integrals. C. A. Bretschneider (1843) not only used the second and third integrals, but also introduced similar integrals for the hyperbolic functions:

$$\int \frac{\sinh(t)}{t}\, dt \quad \int \frac{\cosh(t)}{t}\, dt.$$

O. Schlömilch (1846) and F. Arndt (1847) widely used such integrals containing the exponential and trigonometric functions. For the exponential, sine, and cosine integrals, J. W. L. Glaisher (1870) introduced the notations Ei, Si, and Ci. H. Amstein (1895) introduced the branch cut for the logarithmic integral with a complex argument. N. Nielsen (1904) used the notations Si and Ci for corresponding integrals.

Different notations are used for the previous definite integrals by various authors when they are integrated from 0 to z or from z to ∞.

Definitions of exponential integrals

The exponential integral $E_\nu(z)$, exponential integral $\mathrm{Ei}(z)$, logarithmic integral $\mathrm{li}(z)$, sine integral $\mathrm{Si}(z)$, hyperbolic sine integral $\mathrm{Shi}(z)$, cosine integral $\mathrm{Ci}(z)$, and hyperbolic cosine integral $\mathrm{Chi}(z)$ are defined as the following definite integrals, including the Euler gamma constant $\gamma == 0.577216\ldots$:

$$E_\nu(z) = \int_1^\infty \frac{e^{-z t}}{t^\nu}\, dt \;/;\; \mathrm{Re}(z) > 0$$

$$\mathrm{Ei}(z) = \int_0^z \frac{e^t - 1}{t}\, dt + \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right) + \gamma$$

$$\mathrm{li}(z) = \int_0^z \frac{1}{\log(t)}\, dt$$

$$\mathrm{Si}(z) = \int_0^z \frac{\sin(t)}{t}\, dt$$

$$\mathrm{Shi}(z) = \int_0^z \frac{\sinh(t)}{t}\, dt$$

$$\mathrm{Ci}(z) = \int_0^z \frac{\cos(t) - 1}{t}\, dt + \log(z) + \gamma$$

$$\mathrm{Chi}(z) = \int_0^z \frac{\cosh(t) - 1}{t}\, dt + \log(z) + \gamma.$$

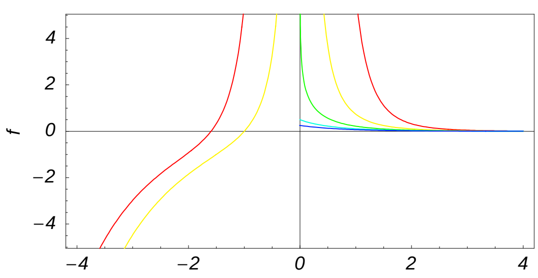
The previous integrals are all interrelated and are called exponential integrals.

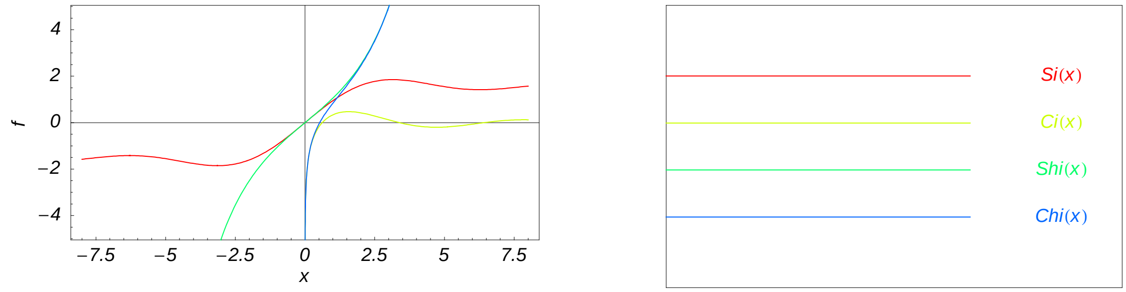
Instead of the above classical definitions through definite integrals, equivalent definitions through infinite series can be used, for example, the exponential integral Ei(z) can be defined by the following formula (see the following sections for the corresponding series for the other integrals):

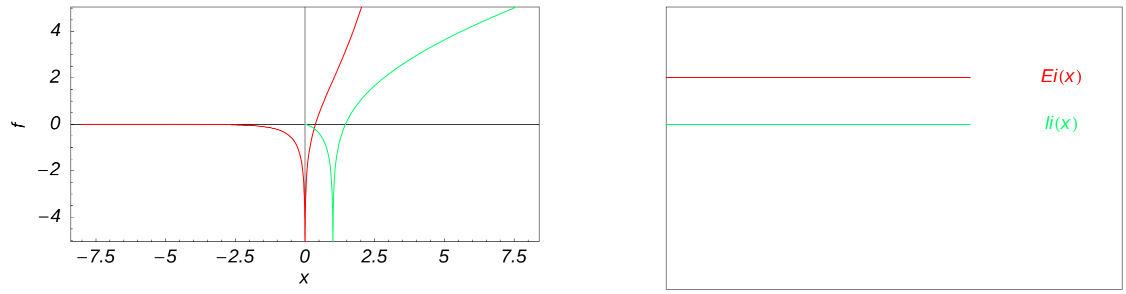
$$\mathrm{Ei}(z) = \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right) + \sum_{k=1}^{\infty} \frac{z^k}{k\, k!} + \gamma.$$

A quick look at the exponential integrals

Here is a quick look at the graphics for the exponential integrals along the real axis.







Connections within the group of exponential integrals and with other function groups

Representations through more general functions

The exponential integrals $E_\nu(z)$, $\text{Ei}(z)$, $\text{li}(z)$, $\text{Si}(z)$, $\text{Shi}(z)$, $\text{Ci}(z)$, and $\text{Chi}(z)$ are the particular cases of the more general hypergeometric and Meijer G functions.

For example, they can be represented through hypergeometric functions ${}_pF_q$ or the Tricomi confluent hypergeometric function U:

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) - \frac{1}{1-\nu}\,{}_1F_1(1-\nu;\,2-\nu;\,-z)$$

$$E_\nu(z) = z^{\nu-1}\,e^{-z}\,U(\nu,\,\nu,\,z)$$

$$\text{Ei}(z) = z\,{}_2F_2(1,\,1;\,2,\,2;\,z) + \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right) + \gamma$$

$$\text{Ei}(z) = -e^z\,U(1,\,1,\,-z) - \log(-z) + \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right)$$

$$\text{li}(z) = \log(z)\,{}_2F_2(1,\,1;\,2,\,2;\,\log(z)) + \frac{1}{2}\left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) + \gamma$$

$$\text{Si}(z) = z\,{}_1F_2\left(\frac{1}{2};\,\frac{3}{2},\,\frac{3}{2};\,-\frac{z^2}{4}\right)$$

$$\text{Shi}(z) = z\,{}_1F_2\left(\frac{1}{2};\,\frac{3}{2},\,\frac{3}{2};\,\frac{z^2}{4}\right)$$

$$\text{Ci}(z) = -\frac{z^2}{4}\,{}_2F_3\left(1,\,1;\,2,\,2,\,\frac{3}{2};\,-\frac{z^2}{4}\right) + \log(z) + \gamma$$

$$\text{Chi}(z) = \frac{z^2}{4} \, {}_2F_3\left(1, 1; 2, 2, \frac{3}{2}; \frac{z^2}{4}\right) + \log(z) + \gamma.$$

Representations of the exponential integrals $E_\nu(z)$ and Ei (z), the sine and cosine integrals Si (z) and Ci (z), and the hyperbolic sine and cosine integrals Shi(z) and Chi (z) through classical Meijer G functions are rather simple:

$$E_\nu(z) = G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \nu \\ \nu-1, 0 \end{matrix}\right)$$

$$\text{Ei}(z) = -G_{2,3}^{1,2}\left(-z \,\middle|\, \begin{matrix} 1, 1 \\ 1, 0, 0 \end{matrix}\right) + \gamma - \frac{1}{2}\left(\log\left(\frac{1}{z}\right) - \log(z)\right)$$

$$\text{Si}(z) = \frac{\sqrt{\pi}}{4} \, z \, G_{1,3}^{1,1}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{2}, -\frac{1}{2} \end{matrix}\right)$$

$$\text{Ci}(z) = -\frac{\sqrt{\pi}}{2} \, G_{1,3}^{2,0}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} 1 \\ 0, 0, \frac{1}{2} \end{matrix}\right) - \frac{1}{2}\left(\log\left(z^2\right) - 2\log(z)\right)$$

$$\text{Shi}(z) = \frac{\sqrt{\pi}}{4} \, z \, G_{1,3}^{1,1}\left(-\frac{z^2}{4} \,\middle|\, \begin{matrix} \frac{1}{2} \\ 0, -\frac{1}{2}, -\frac{1}{2} \end{matrix}\right)$$

$$\text{Chi}(z) = -\frac{\sqrt{\pi}}{2} \, G_{1,3}^{2,0}\left(-\frac{z^2}{4} \,\middle|\, \begin{matrix} 1 \\ 0, 0, \frac{1}{2} \end{matrix}\right) - \frac{1}{2}\left(\log\left(-z^2\right) - 2\log(z)\right).$$

Here γ is the Euler gamma constant $\gamma = 0.577216\ldots$ and the complicated-looking expression containing the two logarithm simplifies piecewise:

$$-\frac{1}{2}\left(\log\left(\frac{1}{z}\right) - \log(z)\right) = \log(z) \;/;\; z \notin \{-\infty, 0\}$$

$$-\frac{1}{2}\left(\log\left(\frac{1}{z}\right) - \log(z)\right) = \log(z) - \pi i \;/;\; z \in \{-\infty, 0\}.$$

But the last four formulas that contain the Meijer G function can be simplified further by changing the classical Meijer G functions to the generalized one. These formulas do not include factors z and terms $\frac{1}{2}\left(\log\left(z^2\right) - 2\log(z)\right)$:

$$\text{Si}(z) = \frac{\sqrt{\pi}}{2} \, G_{1,3}^{1,1}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0, 0 \end{matrix}\right)$$

$$\text{Ci}(z) = -\frac{\sqrt{\pi}}{2} \, G_{1,3}^{2,0}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} 1 \\ 0, 0, \frac{1}{2} \end{matrix}\right)$$

$$\text{Shi}(z) = -\frac{i\sqrt{\pi}}{2} \, G_{1,3}^{1,1}\left(\frac{i z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} 1 \\ \frac{1}{2}, 0, 0 \end{matrix}\right)$$

$$\text{Chi}(z) = -\frac{1}{2} \pi^{3/2} \, G_{2,4}^{2,0}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1}{2}, 1 \\ 0, 0, \frac{1}{2}, \frac{1}{2} \end{matrix}\right).$$

The corresponding representations of the logarithmic integral $\mathrm{li}(z)$ through the classical Meijer G function is more complicated and includes composition of the G function and a logarithmic function:

$$\mathrm{li}(z) = \gamma - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - G_{2,3}^{1,2}\left(-\log(z) \,\middle|\, \begin{matrix} 1, 1 \\ 1, 0, 0 \end{matrix}\right).$$

All six exponential integrals of one variable are the particular cases of the incomplete gamma function:

$$\mathrm{Ei}(z) = -\Gamma(0, -z) + \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right) - \log(-z)$$

$$\mathrm{Si}(z) = \frac{i}{2}\left(\Gamma(0, -i z) - \Gamma(0, i z) + \log(-i z) - \log(i z)\right)$$

$$\mathrm{Shi}(z) = \frac{1}{2}\left(\Gamma(0, z) - \Gamma(0, -z) - \log(-z) + \log(z)\right)$$

$$\mathrm{Ci}(z) = \log(z) - \frac{1}{2}\left(\Gamma(0, -i z) + \Gamma(0, i z) + \log(-i z) + \log(i z)\right)$$

$$\mathrm{Chi}(z) = -\frac{1}{2}\left(\Gamma(0, -z) + \Gamma(0, z) + \log(-z) - \log(z)\right)$$

$$\mathrm{li}(z) = -\Gamma(0, -\log(z)) + \frac{1}{2}\left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) - \log(-\log(z)).$$

Representations through related equivalent functions

The exponential integral $E_\nu(z)$ can be represented through the incomplete gamma function or the regularized incomplete gamma function:

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu, z)$$

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu)\,Q(1-\nu, z).$$

Relations to inverse functions

The exponential integral $E_\nu(z)$ is connected with the inverse of the regularized incomplete gamma function $Q^{-1}(a, z)$ by the following formula:

$$E_\nu\left(Q^{-1}(1-\nu, z)\right) = Q^{-1}(1-\nu, z)^{\nu-1}\,\Gamma(1-\nu)\,z.$$

Representations through other exponential integrals

The exponential integrals $E_\nu(z)$, $\mathrm{Ei}(z)$, $\mathrm{li}(z)$, $\mathrm{Si}(z)$, $\mathrm{Shi}(z)$, $\mathrm{Ci}(z)$, and $\mathrm{Chi}(z)$ are interconnected through the following formulas:

$$\mathrm{Ei}(z) = -E_1(-z) + \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right) - \log(-z)$$

$$\mathrm{Ei}(z) = \mathrm{li}(e^z) \;/;\; -\pi < \mathrm{Im}(z) \le \pi$$

$$\mathrm{Ei}(\log(z)) = \mathrm{li}(z)$$

$$\mathrm{Ei}(z) = \mathrm{Ci}(i z) - i \,\mathrm{Si}(i z) - \frac{1}{2}\left(\log\left(\frac{1}{z}\right) - \log(z)\right) - \log(i z)$$

$$\mathrm{Ei}(z) = \mathrm{Chi}(z) + \mathrm{Shi}(z) - \frac{1}{2}\left(\log\left(\frac{1}{z}\right) + \log(z)\right)$$

$$\mathrm{li}(z) = -E_1(-\log(z)) + \frac{1}{2}\left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) - \log(-\log(z))$$

$$\mathrm{li}(z) = \mathrm{Ei}(\log(z))$$

$$\mathrm{li}(z) = \mathrm{Ci}(i \log(z)) - i \,\mathrm{Si}(i \log(z)) - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) - \log(\log(z))\right) - \log(i \log(z))$$

$$\mathrm{li}(z) = \mathrm{Chi}(\log(z)) + \mathrm{Shi}(\log(z)) - \frac{1}{2}\left(\log\left(\frac{1}{\log(z)}\right) + \log(\log(z))\right)$$

$$\mathrm{Si}(z) = \frac{i}{2}\left(E_1(-i z) - E_1(i z) + \log(-i z) - \log(i z)\right)$$

$$\mathrm{Si}(z) = \frac{i}{4}\left(2\left(\mathrm{Ei}(-i z) - \mathrm{Ei}(i z)\right) + \log\left(\frac{i}{z}\right) - \log\left(-\frac{i}{z}\right) - \log(-i z) + \log(i z)\right)$$

$$\mathrm{Si}(z) = \frac{1}{2 i}\left(\mathrm{li}\left(e^{i z}\right) - \mathrm{li}\left(e^{-i z}\right)\right) - \frac{\pi}{2}\,\mathrm{sgn}(\mathrm{Re}(z)) \;/;\; |\mathrm{Re}(z)| < \pi$$

$$\mathrm{Si}(z) = -i \,\mathrm{Shi}(i z)$$

$$\mathrm{Shi}(z) = \frac{1}{2}\left(E_1(z) - E_1(-z) - \log(-z) + \log(z)\right)$$

$$\mathrm{Shi}(z) = \frac{1}{4}\left(2\left(\mathrm{Ei}(z) - \mathrm{Ei}(-z)\right) + \log\left(\frac{1}{z}\right) - \log\left(-\frac{1}{z}\right) + \log(-z) - \log(z)\right)$$

$$\mathrm{Shi}(z) = \frac{1}{2}\left(\mathrm{li}(e^{z}) - \mathrm{li}(e^{-z})\right) - \frac{i \pi}{2}\,\mathrm{sgn}(\mathrm{Im}(z)) \;/;\; |\mathrm{Im}(z)| < \pi$$

$$\mathrm{Shi}(z) = -i \,\mathrm{Si}(i z)$$

$$\mathrm{Ci}(z) = -\frac{1}{2}\left(E_1(-i z) + E_1(i z) + \log(-i z) + \log(i z)\right) + \log(z)$$

$$\mathrm{Ci}(z) = \frac{1}{4}\left(2\left(\mathrm{Ei}(-i z) + \mathrm{Ei}(i z)\right) + \log\left(\frac{i}{z}\right) + \log\left(-\frac{i}{z}\right) - \log(-i z) - \log(i z)\right) + \log(z)$$

$$\mathrm{Ci}(z) = \frac{1}{2}\left(\mathrm{li}\left(e^{-i z}\right) + \mathrm{li}\left(e^{i z}\right)\right) + \frac{\pi i}{2}\,\mathrm{sgn}(\mathrm{Im}(z))\left(1 - \mathrm{sgn}(\mathrm{Re}(z))\right) \;/;\; |\mathrm{Re}(z)| < \pi$$

$$\mathrm{Ci}(z) = \mathrm{Chi}(i z) - \log(i z) + \log(z)$$

$$\mathrm{Chi}(z) = -\frac{1}{2}\left(E_1(-z) + E_1(z) + \log(-z) - \log(z)\right)$$

$$\mathrm{Chi}(z) = \frac{1}{4}\left(2\left(\mathrm{Ei}(-z) + \mathrm{Ei}(z)\right) + \log\left(\frac{1}{z}\right) + \log\left(-\frac{1}{z}\right) - \log(-z) + 3\log(z)\right)$$

$$\text{Chi}(z) = \frac{1}{2}\left(\text{li}(e^{-z}) + \text{li}(e^{z})\right) + \frac{\pi i}{2}\,\text{sgn}(\text{Im}(z)) + \frac{1}{2}\left(\log\left(\frac{1}{z}\right) + \log(z)\right) /;\ |\text{Im}(z)| < \pi$$

$$\text{Chi}(z) = \text{Ci}(i z) + \log(z) - \log(i z).$$

The best-known properties and formulas for exponential integrals

Real values for real arguments

For real values of parameter ν and positive argument z, the values of the exponential integral $E_\nu(z)$ are real (or infinity). For real values of argument z, the values of the exponential integral Ei(z), the sine integral Si (z), and the hyperbolic sine integral Shi (z) are real. For real positive values of argument z, the values of the logarithmic integral li (z), the cosine integral Ci(z), and the hyperbolic cosine integral Chi(z) are real.

Simple values at zero

The exponential integrals have rather simple values for argument $z = 0$:

$$E_0(0) = \tilde{\infty}$$

$$\text{Ei}(0) = -\infty$$

$$\text{li}(0) = 0$$

$$\text{Si}(0) = 0$$

$$\text{Shi}(0) = 0$$

$$\text{Ci}(0) = -\infty$$

$$\text{Chi}(0) = -\infty$$

$$E_\nu(0) = \frac{1}{\nu - 1} /;\ \text{Re}(\nu) > 1.$$

Specific values for specialized parameter

If the parameter ν equals $0, -1 - 2, \ldots$, the exponential integral $E_\nu(z)$ can be expressed through an exponential function multiplied by a simple rational function. If the parameter ν equals $1, 2, 3, \ldots$, the exponential integral $E_\nu(z)$ can be expressed through the exponential integral Ei (z), and the exponential and logarithmic functions:

$$E_0(z) = \frac{e^{-z}}{z}$$

$$E_{-1}(z) = \frac{e^{-z}\,(z + 1)}{z^2}$$

$$E_1(z) = -\text{Ei}(-z) + \frac{1}{2}\left(\log(-z) - \log\left(-\frac{1}{z}\right)\right) - \log(z).$$

The previous formulas are the particular cases of the following general formula:

$$E_n(z) = z^{n-1}\left(\frac{(-1)^n}{(n-1)!}\left(\text{Ei}(-z) + \frac{1}{2}\left(\log\left(-\frac{1}{z}\right) - \log(-z)\right) + \log(z)\right) + e^{-z}\sum_{k=0}^{-n}\frac{z^k}{(1-n)_{k+n}} - e^{-z}\sum_{k=1-n}^{-1}\frac{z^k}{(1-n)_{k+n}}\right) /; n \in \mathbb{Z}.$$

If the parameter ν equals $\pm\frac{1}{2}$, $\pm\frac{3}{2}$ $\pm\frac{5}{2}$, …, the exponential integral $E_\nu(z)$ can be expressed through the probability integral erf (z), and the exponential and power functions, for example:

$$E_{-\frac{1}{2}}(z) = \frac{\sqrt{\pi}}{2\,z^{3/2}}\,\text{erfc}\left(\sqrt{z}\right) + \frac{e^{-z}}{z}$$

$$E_{\frac{1}{2}}(z) = \frac{\sqrt{\pi}}{\sqrt{z}}\,\text{erfc}\left(\sqrt{z}\right).$$

The previous formulas can be generalized by the following general representation of this class of particular cases:

$$E_{n+\frac{1}{2}}(z) = z^{n-\frac{1}{2}}\left(\text{erfc}\left(\sqrt{z}\right)\Gamma\left(\frac{1}{2}-n\right) + e^{-z}\sum_{k=0}^{-n-1}\frac{z^{k+\frac{1}{2}}}{\left(\frac{1}{2}-n\right)_{k+n+1}} - e^{-z}\sum_{k=-n}^{-1}\frac{z^{k+\frac{1}{2}}}{\left(\frac{1}{2}-n\right)_{k+n+1}}\right) /; n \in \mathbb{Z}.$$

Analyticity

The exponential integrals $E_\nu(z)$, Ei(z), li (z), Si (z), Shi (z), Ci(z), and Chi(z) are defined for all complex values of the parameter ν and the variable z. The function $E_\nu(z)$ is an analytical functions of ν and z over the whole complex ν- and z-planes excluding the branch cut on the z-plane. For fixed z, the exponential integral $E_\nu(z)$ is an entire function of ν. The sine integral Si(z) and the hyperbolic sine integral Shi(z) are entire functions of z.

Poles and essential singularities

For fixed ν, the function $E_\nu(z)$ has an essential singularity at $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is a branch point for generic ν. For fixed z, the function $E_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

The exponential integral Ei(z), the cosine integral Ci(z), and the hyperbolic cosine integral Chi(z) have an essential singularity at $z = \tilde{\infty}$.

The function li(z) does not have poles and essential singularities.

The sine integral Si(z) and the hyperbolic sine integral Shi(z) have an essential singularity at $z = \tilde{\infty}$.

Branch points and branch cuts

For fixed z, the function $E_\nu(z)$ does not have branch points and branch cuts.

For fixed ν, not being a nonpositive integer, the function $E_\nu(z)$ has two branch points $z = 0$ and $z = \tilde{\infty}$, and branch cuts along the interval $(-\infty, 0)$. At the same time, the point $z = \tilde{\infty}$ is an essential singularity for this function.

The exponential integral Ei (z), the cosine integral Ci(z), and the hyperbolic cosine integral Chi(z) have two branch points $z = 0$ and $z = \tilde{\infty}$.

The function li(z) has three branch points $z = 0$, $z = 1$, and $z = \tilde{\infty}$.

The sine integral Si(z) and hyperbolic sine integral Shi(z) do not have branch points or branch cuts.

For fixed ν, not being a nonpositive integer, the function $E_\nu(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above:

$$\lim_{\epsilon\to+0} E_\nu(x+i\,\epsilon) = E_\nu(x) \;/;\; x<0$$

$$\lim_{\epsilon\to+0} E_\nu(x-i\,\epsilon) = E_\nu(x) - \frac{2\,\pi\,i\,e^{-\pi\,i\,\nu}\,x^{\nu-1}}{\Gamma(\nu)} \;/;\; x<0.$$

The function Ei(z) is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it has discontinuities from both sides:

$$\lim_{\epsilon\to+0} \mathrm{Ei}(x+i\,\epsilon) = \mathrm{Ei}(x) + \pi\,i \;/;\; x<0$$

$$\lim_{\epsilon\to+0} \mathrm{Ei}(x-i\,\epsilon) = \mathrm{Ei}(x) - \pi\,i \;/;\; x<0.$$

The function li(z) is a single-valued function on the z-plane cut along the interval $(-\infty, 1)$. It is continuous from above along the interval $(-\infty, 0)$ and it has discontinuities from both sides along the interval $(0, 1)$:

$$\lim_{\epsilon\to+0} \mathrm{li}(x+i\,\epsilon) = \mathrm{li}(x) \;/;\; x<0$$

$$\lim_{\epsilon\to+0} \mathrm{li}(x-i\,\epsilon) = \mathrm{Ei}(\log(-x) - i\,\pi) \;/;\; x<0$$

$$\lim_{\epsilon\to+0} \mathrm{li}(x+i\,\epsilon) = \mathrm{li}(x) + \pi\,i \;/;\; 0<x<1$$

$$\lim_{\epsilon\to+0} \mathrm{li}(x-i\,\epsilon) = \mathrm{li}(x) - \pi\,i \;/;\; 0<x<1.$$

The cosine integral Ci(z) and hyperbolic cosine integral Chi(z) are single-valued functions on the z-plane cut along the interval $(-\infty, 0)$ where they are continuous from above:

$$\lim_{\epsilon\to+0} \mathrm{Ci}(x+i\,\epsilon) = \mathrm{Ci}(x) \;/;\; x<0$$

$$\lim_{\epsilon\to+0} \mathrm{Chi}(x+i\,\epsilon) = \mathrm{Chi}(x) \;/;\; x<0.$$

From below, these functions have discontinuity that are described by the formulas:

$$\lim_{\epsilon\to+0} \mathrm{Ci}(x-i\,\epsilon) = \mathrm{Ci}(x) - 2\,\pi\,i \;/;\; x<0$$

$$\lim_{\epsilon\to+0} \mathrm{Chi}(x-i\,\epsilon) = \mathrm{Chi}(x) - 2\,\pi\,i \;/;\; x<0.$$

Periodicity

The exponential integrals $E_\nu(z)$, Ei(z), li (z), Si (z), Shi (z), Ci(z), and Chi(z) do not have periodicity.

Parity and symmetry

The exponential integral Ei(z) has mirror symmetry:

$\text{Ei}(\bar{z}) == \overline{\text{Ei}(z)}$.

The sine integral Si(z) and the hyperbolic sine integral Shi(z) are odd functions and have mirror symmetry:

$$\text{Si}(-z) == -\text{Si}(z) \qquad \text{Si}(\bar{z}) == \overline{\text{Si}(z)}$$
$$\text{Shi}(-z) == -\text{Shi}(z) \qquad \text{Shi}(\bar{z}) == \overline{\text{Shi}(z)}.$$

The exponential integral $E_\nu(z)$, logarithmic integral li(z), cosine integral Ci(z), and hyperbolic cosine integral Chi(z) have mirror symmetry (except on the branch cut interval (-∞, 0)):

$$E_{\bar{\nu}}(\bar{z}) == \overline{E_\nu(z)} \;/;\; z \notin (-\infty, 0)$$

$$\text{li}(\bar{z}) == \overline{\text{li}(z)} \;/;\; z \notin (-\infty, 0)$$

$$\text{Ci}(\bar{z}) == \overline{\text{Ci}(z)} \;/;\; z \notin (-\infty, 0)$$

$$\text{Chi}(\bar{z}) == \overline{\text{Chi}(z)} \;/;\; z \notin (-\infty, 0).$$

Series representations

The exponential integrals $E_\nu(z)$, Ei(z), li(z), Si(z), Shi(z), Ci(z), and Chi(z) have the following series expansions through series that converge on the whole z-plane:

$$E_\nu(z) \propto \Gamma(1-\nu)\, z^{\nu-1} - \frac{1}{1-\nu} + \frac{z}{2-\nu} - \frac{z^2}{2\,(3-\nu)} + \ldots \;/;\; (z \to 0) \bigwedge \nu \notin \mathbb{N}^+$$

$$E_\nu(z) == \Gamma(1-\nu)\, z^{\nu-1} - \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(k-\nu+1)\, k!} \;/;\; \nu \notin \mathbb{N}^+$$

$$E_n(z) == \frac{(-z)^{n-1}}{(n-1)!}\, (\psi(n) - \log(z)) - \sum_{\substack{k=0 \\ k\neq n-1}}^{\infty} \frac{(-1)^k z^k}{(k-n+1)\, k!} \;/;\; n \in \mathbb{N}^+$$

$$E_{-n}(z) == n!\, z^{-n-1}\, e^{-z} \sum_{k=0}^{n} \frac{z^k}{k!} \;/;\; n \in \mathbb{N}$$

$$\text{Ei}(z) \propto \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right) + \gamma + z + \frac{z^2}{4} + \frac{z^3}{18} + \ldots \;/;\; (z \to 0)$$

$$\text{Ei}(z) == \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right) + \sum_{k=1}^{\infty} \frac{z^k}{k\, k!} + \gamma$$

$$\text{li}(z) \propto \frac{1}{2}\left(\log(z-1) - \log\left(\frac{1}{z-1}\right)\right) + \gamma + \frac{z-1}{2}\left(1 - \frac{z-1}{12} + \frac{(z-1)^2}{36} + \ldots\right) \;/;\; (z \to 1)$$

$$\text{li}(z) == \frac{1}{2}\left(\log(z-1) - \log\left(\frac{1}{z-1}\right)\right) + \gamma + \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \sum_{j=1}^{k+1} \frac{B_j\, S_k^{(j-1)}}{j}\, (1-z)^{k+1}$$

$$\text{Si}(z) \propto z\left(1 - \frac{z^2}{18} + \frac{z^4}{600} - \ldots\right) /; (z \to 0)$$

$$\text{Si}(z) = z\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(1+2k)^2 (2k)!}$$

$$\text{Shi}(z) \propto z\left(1 + \frac{z^2}{18} + \frac{z^4}{600} + \ldots\right) /; (z \to 0)$$

$$\text{Shi}(z) = z\sum_{k=0}^{\infty} \frac{z^{2k}}{(1+2k)^2 (2k)!}$$

$$\text{Ci}(z) \propto \log(z) + \gamma - \frac{1}{4} z^2 \left(1 - \frac{z^2}{24} + \frac{z^4}{1080} - \ldots\right) /; (z \to 0)$$

$$\text{Ci}(z) = \log(z) + \gamma + \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k z^{2k}}{k (2k)!}$$

$$\text{Chi}(z) \propto \log(z) + \gamma + \frac{z^2}{4}\left(1 + \frac{z^2}{24} + \frac{z^4}{1080} + \ldots\right) /; (z \to 0)$$

$$\text{Chi}(z) = \log(z) + \gamma + \frac{1}{2}\sum_{k=1}^{\infty} \frac{z^{2k}}{k (2k)!}.$$

Interestingly, closed-form expressions for the truncated version of the Taylor series at the origin can be expressed through the generalized hypergeometric function ${}_pF_q$, for example:

$$E_\nu(z) = F_\infty(z, \nu) /; \left(\left(F_n(z, \nu) = \Gamma(1-\nu) z^{\nu-1} - \sum_{k=0}^{n} \frac{(-1)^k z^k}{(k-\nu+1) k!} = \Gamma(1-\nu, z) z^{\nu-1} + \frac{(-1)^n z^{n+1}}{(-n+\nu-2)(n+1)!} {}_2F_2(1, n-\nu+2; n+2, n-\nu+3; -z)\right) \bigwedge n \in \mathbb{N}\right)$$

$$\text{Ei}(z) = F_\infty(z) /; \left(\left(F_n(z) = \frac{1}{2}\left(\log(z) - \log\left(\frac{1}{z}\right)\right) + \gamma + z\sum_{k=0}^{n} \frac{z^k}{(k+1)^2 k!} = \text{Ei}(z) - \frac{z^{n+2}}{(n+2)(n+2)!} {}_2F_2(1, n+2; n+3, n+3; z)\right) \bigwedge n \in \mathbb{N}\right)$$

$$\text{Si}(z) = F_\infty(z) /; \left(\left(F_n(z) = z\sum_{k=0}^{n} \frac{(-1)^k z^{2k}}{(2k+1)^2 (2k)!} = \text{Si}(z) + \frac{(-1)^n z^{2n+3}}{(2n+3)^2 (2n+2)!} {}_2F_3\left(1, n+\frac{3}{2}; n+2, n+\frac{5}{2}, n+\frac{5}{2}; -\frac{z^2}{4}\right)\right) \bigwedge n \in \mathbb{N}\right)$$

$$\text{Ci}(z) = F_\infty(z) /; \left(\left(F_n(z) = \log(z) + \gamma + \frac{1}{2}\sum_{k=1}^{n} \frac{(-1)^k z^{2k}}{k (2k)!} = \text{Ci}(z) + \frac{(-1)^n z^{2n+2}}{4 (n+1)^2 (2n+1)!} {}_2F_3\left(1, n+1; n+\frac{3}{2}, n+2, n+2; -\frac{z^2}{4}\right)\right) \bigwedge n \in \mathbb{N}\right).$$

Asymptotic series expansions

The asymptotic behavior of the exponential integrals $E_\nu(z)$, $\text{Ei}(z)$, $\text{li}(z)$, $\text{Si}(z)$, $\text{Shi}(z)$, $\text{Ci}(z)$, and $\text{Chi}(z)$ can be described by the following formulas (only the main terms of the asymptotic expansions are given):

$$E_\nu(z) \propto \frac{1}{z} e^{-z} \left(1 + O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty)$$

$$\text{Ei}(z) \propto \frac{1}{2} \left(\log(z) - \log\left(\frac{1}{z}\right)\right) - \log(-z) + \frac{1}{z} e^{z} \left(1 + O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty)$$

$$\text{li}(z) \propto \frac{1}{2} \left(\log(\log(z)) - \log\left(\frac{1}{\log(z)}\right)\right) - \log(-\log(z)) + \frac{z}{\log(z)} \left(1 + O\left(\frac{1}{\log(z)}\right)\right) /; (|z| \to \infty \lor z \to 0)$$

$$\text{Si}(z) \propto \frac{\pi \sqrt{z^2}}{2 z} - \frac{\cos(z)}{z} \left(1 + O\left(\frac{1}{z^2}\right)\right) - \frac{\sin(z)}{z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

$$\text{Shi}(z) \propto -\frac{\pi \sqrt{-z^2}}{2 z} + \frac{\cosh(z)}{z} \left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{\sinh(z)}{z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

$$\text{Ci}(z) \propto \log(z) - \frac{\log(z^2)}{2} + \frac{\sin(z)}{z} \left(1 + O\left(\frac{1}{z^2}\right)\right) - \frac{\cos(z)}{z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty)$$

$$\text{Chi}(z) \propto \log(z) - \frac{\log(-z^2)}{2} + \frac{\sinh(z)}{z} \left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{\cosh(z)}{z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty).$$

The previous formulas are valid in any direction of approaching point z to infinity ($|z| \to \infty$). In particular cases, these formulas can be simplified to the following relations:

$$E_\nu(z) \propto \frac{1}{z} e^{-z} \left(1 + O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty)$$

$$\text{Ei}(z) \propto \frac{1}{z} e^{z} \left(1 + O\left(\frac{1}{z}\right)\right) /; (|z| \to \infty) \land \text{Re}(z) > 0$$

$$\text{li}(z) \propto \frac{z}{\log(z)} \left(1 + O\left(\frac{1}{\log(z)}\right)\right) /; (|z| \to \infty \lor z \to 0) \land \text{Re}(z) > 0$$

$$\text{Si}(z) \propto \frac{\pi}{2} - \frac{\cos(z)}{z} \left(1 + O\left(\frac{1}{z^2}\right)\right) - \frac{\sin(z)}{z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (z \to \infty)$$

$$\text{Shi}(z) \propto \frac{\pi i}{2} + \frac{\cosh(z)}{z} \left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{\sinh(z)}{z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty) \bigwedge \text{Arg}(z) \neq -\frac{\pi}{2}$$

$$\text{Ci}(z) \propto \frac{\sin(z)}{z} \left(1 + O\left(\frac{1}{z^2}\right)\right) - \frac{\cos(z)}{z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty) \bigwedge |\text{Arg}(z)| < \pi$$

$$\text{Chi}(z) \propto \frac{\sinh(z)}{z} \left(1 + O\left(\frac{1}{z^2}\right)\right) + \frac{\cosh(z)}{z^2} \left(1 + O\left(\frac{1}{z^2}\right)\right) /; (|z| \to \infty) \land \text{Re}(z) \neq 0.$$

Integral representations

The exponential integrals $E_\nu(z)$, Ei(z), Si (z), and Ci(z) can also be represented through the following equivalent integrals:

$$E_\nu(z) = z^{\nu-1} \int_z^\infty t^{-\nu} e^{-t} \, dt \;/;\; |\text{Arg}(z)| < \pi$$

$$\text{Ei}(x) = -\mathcal{P} \int_{-x}^\infty \frac{e^{-t}}{t} \, dt \;/;\; x \in \mathbb{R}$$

$$\text{Ei}(x) = \mathcal{P} \int_{-\infty}^x \frac{e^t}{t} \, dt \;/;\; x \in \mathbb{R}$$

$$\text{Si}(z) = \frac{\pi}{2} - \int_z^\infty \frac{\sin(t)}{t} \, dt$$

$$\text{Ci}(z) = -\int_z^\infty \frac{\cos(t)}{t} \, dt \;/;\; |\text{Arg}(z)| < \pi.$$

The symbol $\mathcal{P}$ in the second and third integrals means that these integrals evaluate as the Cauchy principal value of the singular integral:

$$\mathcal{P} \int_a^b \frac{f(t)}{t-x} \, dt = \lim_{\epsilon \to 0} \left(\int_a^{x-\epsilon} \frac{f(t)}{t-x} \, dt + \int_{x+\epsilon}^b \frac{f(t)}{t-x} \, dt \right) /;\; a < x < b.$$

Transformations

The arguments of the exponential integrals Ei(z), Si (z), Shi (z), Ci(z), and Chi(z) that contain square roots can sometimes be simplified:

$$\text{Ei}\left(\sqrt{z^2}\right) = \text{Ei}(z) + \frac{1}{2}\left(\log\left(\frac{1}{z}\right) - \log(z) + \log(-i\, z) + \log(i\, z)\right) + \left(\frac{\sqrt{z^2}}{z} - 1\right) \text{Shi}(z)$$

$$\text{Si}\left(\sqrt{z^2}\right) = \frac{\sqrt{z^2}}{z} \text{Si}(z)$$

$$\text{Shi}\left(\sqrt{z^2}\right) = \frac{\sqrt{z^2}}{z} \text{Shi}(z)$$

$$\text{Ci}\left(\sqrt{z^2}\right) = \text{Ci}(z) - \log(z) + \log\left(\sqrt{z^2}\right)$$

$$\text{Chi}\left(\sqrt{z^2}\right) = \text{Chi}(z) - \log(z) + \log\left(\sqrt{z^2}\right).$$

Identities

The exponential integral $E_\nu(z)$ satisfies the following recurrence identities:

$$E_\nu(z) = \frac{1}{z}\left(e^{-z} - \nu\, E_{\nu+1}(z)\right)$$

$$E_\nu(z) = \frac{1}{\nu - 1} \left(e^{-z} - z\, E_{\nu-1}(z)\right).$$

All of the preceding formulas can be generalized to the following recurrence identities with a jump of length n:

$$E_\nu(z) = (-1)^n (\nu)_n\, z^{-n}\, E_{n+\nu}(z) - e^{-z} \sum_{k=0}^{n-1} (\nu)_k\, (-z)^{-k-1} \;/;\; n \in \mathbb{N}$$

$$E_\nu(z) = \frac{z^n}{(1-\nu)_n}\, E_{\nu-n}(z) - e^{-z} \sum_{k=0}^{n-1} \frac{z^k}{(1-\nu)_{k+1}} \;/;\; n \in \mathbb{N}.$$

Simple representations of derivatives

The derivative of the exponential integral $E_\nu(z)$ with respect to the variable z has a simple representation through itself, but with a different parameter:

$$\frac{\partial E_\nu(z)}{\partial z} = -E_{\nu-1}(z).$$

The derivative of the exponential integral $E_\nu(z)$ by its parameter ν can be represented through the regularized hypergeometric function ${}_2\tilde{F}_2$:

$$\frac{\partial E_\nu(z)}{\partial \nu} = z^{\nu-1}\, \Gamma(1-\nu) \left(\log(z) - \psi(1-\nu)\right) - \Gamma(1-\nu)^2\, {}_2\tilde{F}_2(1-\nu, 1-\nu; 2-\nu, 2-\nu; -z).$$

The derivatives of the other exponential integrals Ei(z), li (z), Si (z), Shi (z), Ci(z), and Chi(z) have simple representations through simple elementary functions:

$$\frac{\partial \mathrm{Ei}(z)}{\partial z} = \frac{e^z}{z}$$

$$\frac{\partial \mathrm{li}(z)}{\partial z} = \frac{1}{\log(z)}$$

$$\frac{\partial \mathrm{Si}(z)}{\partial z} = \frac{\sin(z)}{z}$$

$$\frac{\partial \mathrm{Shi}(z)}{\partial z} = \frac{\sinh(z)}{z}$$

$$\frac{\partial \mathrm{Ci}(z)}{\partial z} = \frac{\cos(z)}{z}$$

$$\frac{\partial \mathrm{Chi}(z)}{\partial z} = \frac{\cosh(z)}{z}.$$

The symbolic n^{th}-order derivatives with respect to the variable z of all exponential integrals $E_\nu(z)$, Ei(z), li (z), Si (z), Shi (z), Ci(z), and Chi(z) have the following representations:

$$\frac{\partial^n E_\nu(z)}{\partial z^n} = (-1)^n E_{\nu-n}(z) \,/;\, n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{li}(z)}{\partial z^n} = z^{1-n} \sum_{k=0}^{n-1} (-1)^k k!\, S_{n-1}^{(k)} \log^{-k-1}(z) \,/;\, n \in \mathbb{N}^+$$

$$\frac{\partial^n \mathrm{Ei}(z)}{\partial z^n} = {}_2\tilde{F}_2(1, 1; 2, 2-n; z)\, z^{1-n} + (-1)^{n-1} (n-1)!\, z^{-n} \,/;\, n \in \mathbb{N}^+$$

$$\frac{\partial^n \mathrm{Si}(z)}{\partial z^n} = 2^{n-2} \pi z^{1-n} {}_2\tilde{F}_3\left(\frac{1}{2}, 1; \frac{3}{2}, 1-\frac{n}{2}, \frac{3-n}{2}; -\frac{z^2}{4}\right) /;\, n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{Shi}(z)}{\partial z^n} = 2^{n-2} \pi z^{1-n} {}_2\tilde{F}_3\left(\frac{1}{2}, 1; \frac{3}{2}, 1-\frac{n}{2}, \frac{3-n}{2}; \frac{z^2}{4}\right) /;\, n \in \mathbb{N}$$

$$\frac{\partial^n \mathrm{Ci}(z)}{\partial z^n} = (-1)^{n-1} z^{-n} (n-1)! - 2^{n-3} \sqrt{\pi}\, z^{2-n} {}_2\tilde{F}_3\left(1, 1; 2, \frac{3-n}{2}, 2-\frac{n}{2}; -\frac{z^2}{4}\right) /;\, n \in \mathbb{N}^+$$

$$\frac{\partial^n \mathrm{Chi}(z)}{\partial z^n} = (-1)^{n-1} z^{-n} (n-1)! + 2^{n-3} \sqrt{\pi}\, z^{2-n} {}_2\tilde{F}_3\left(1, 1; 2, \frac{3-n}{2}, 2-\frac{n}{2}; \frac{z^2}{4}\right) /;\, n \in \mathbb{N}^+.$$

Differential equations

The exponential integrals $E_\nu(z)$, $\mathrm{Ei}(z)$, $\mathrm{Si}(z)$, $\mathrm{Shi}(z)$, $\mathrm{Ci}(z)$, and $\mathrm{Chi}(z)$ satisfy the following linear differential equations of second or third orders:

$$z\, w''(z) + (z - \nu + 2)\, w'(z) + (1 - \nu)\, w(z) = 0 \,/;\, w(z) = c_1 E_\nu(z) + c_2$$

$$z\, w^{(3)}(z) + 2\, w''(z) - z\, w'(z) = 0 \,/;\, w(z) = c_1 \mathrm{Ei}(z) + c_2 \mathrm{Ei}(-z) + c_3$$

$$z\, w^{(3)}(z) + 2\, w''(z) + z\, w'(z) = 0 \,/;\, w(z) = c_1 \mathrm{Si}(z) + c_2 \mathrm{Ci}(z) + c_3$$

$$z\, w^{(3)}(z) + 2\, w''(z) - z\, w'(z) = 0 \,/;\, w(z) = c_1 \mathrm{Shi}(z) + c_2 \mathrm{Chi}(z) + c_3,$$

where c_1, c_2, and c_3 are arbitrary constants.

The logarithmic integral $\mathrm{li}(z)$ satisfies the following ordinary second-order nonlinear differential equation:

$$z\, w''(z) + w'(z)^2 = 0 \,/;\, w(z) = \mathrm{li}(z).$$

Applications of exponential integrals

Applications of exponential integrals include number theory, quantum field theory, Gibbs phenomena, and solutions of Laplace equations in semiconductor physics.

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