Introductions to Factorial

Introduction to the factorials and binomials

General

The factorials and binomials have a very long history connected with their natural appearance in combinatorial problems. Such combinatorial-type problems were known and partially solved even in ancient times. The first mathematical descriptions of binomial coefficients arising from expansions of $(a+b)^n$ for $n == 2, 3, 4, \ldots$ appeared in the works of Chia Hsien (1050), al-Karaji (about 1100), Omar al-Khayyami (1080), Bhaskara Acharya (1150), al-Samaw'al (1175), Yang Hui (1261), Tshu shi Kih (1303), Shih–Chieh Chu (1303), M. Stifel (1544), Cardano (1545), Scheubel (1545), Peletier (1549), Tartaglia (1556), Cardan (1570), Stevin (1585), Faulhaber (1615), Girard (1629), Oughtred (1631), Briggs (1633), Mersenne (1636), Fermat (1636), Wallis (1656), Montmort (1708), and De Moivre (1730). B. Pascal (1653) gave a recursion relation for the binomial, and I. Newton (1676) studied its cases with fractional arguments.

It was known that the factorial $n!$ grows very fast. Its growth speed was estimated by J. Stirling (1730) who found the famous asymptotic formula for the factorial named after him. A special role in the history of the factorial and binomial belongs to L. Euler, who introduced the gamma function $\Gamma(z)$ as the natural extension of factorial $(n! = \Gamma(n+1))$ for noninteger arguments and used notations with parentheses for the binomials (1774, 1781). C. F. Hindenburg (1779) used not only binomials but introduced multinomials as their generalizations. The modern notation $n!$ was suggested by C. Kramp (1808, 1816). C. F. Gauss (1812) also widely used binomials in his mathematical research, but the modern binomial symbol $\binom{n}{k}$ was introduced by A. von Ettinghausen (1826); later Förstemann (1835) gave the combinatorial interpretation of the binomial coefficients.

A. L. Crelle (1831) used a symbol that notates the generalized factorial $a(a+1)(a+2)\ldots(a+n-1)$. Later P. E. Appell (1880) ascribed the name Pochhammer symbol for the notation of this product because it was widely used in the research of L. A. Pochhammer (1890).

While the double factorial $n!!$ was introduced long ago, its extension for complex arguments was suggested only several years ago by J. Keiper and O. I. Marichev (1994) during the implementation of the function `Factorial2` in *Mathematica*.

The classical combinatorial applications of the factorial and binomial functions are the following:

- The factorial $n!$ gives the number of possible placements of n people on n chairs.

- The binomial $\binom{n}{k}$ gives the number of possible selections of k numbers from a larger group of n numbers, for instance on a lotto strip.

- The multinomial $(n; n_1, n_2, \ldots, n_m)$ is the number of ways of putting $n == n_1 + n_2 + \ldots n_m$ different objects into m different boxes with n_k in the k^{th} box, $k == 1, 2, \ldots, m$.

Definitions of factorials and binomials

The factorial $n!$, double factorial $n!!$, Pochhammer symbol $(a)_n$, binomial coefficient $\binom{n}{k}$, and multinomial coefficient $(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)$ are defined by the following formulas. The first formula is a general definition for the complex arguments, and the second one is for positive integer arguments:

$$n! == \Gamma(n+1)$$

$$n! == \prod_{k=1}^{n} k \;/;\; n \in \mathbb{N}^+$$

$$n!! == \left(\frac{2}{\pi}\right)^{\frac{1}{4}(1-\cos(\pi n))} 2^{n/2}\, \Gamma\left(\frac{n}{2}+1\right)$$

$$(a)_n == \frac{\Gamma(a+n)}{\Gamma(a)} \;/;\; (\neg\, (-a \in \mathbb{Z} \land -a \geq 0 \land n \in \mathbb{Z} \land n \leq -a))$$

$$(a)_n == \prod_{k=0}^{n-1} (a+k) \;/;\; n \in \mathbb{N}^+$$

$$\binom{n}{k} == \frac{\Gamma(n+1)}{\Gamma(k+1)\,\Gamma(n-k+1)} == \frac{n!}{k!\,(n-k)!} \;/;\; (\neg\, (n \in \mathbb{Z} \bigwedge k \in \mathbb{Z} \bigwedge k \leq n < 0))$$

$$\binom{n}{k} == 0 \;/;\; n \in \mathbb{Z} \bigwedge k \in \mathbb{Z} \bigwedge k \leq n < 0$$

$$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) == \frac{\Gamma(n+1)}{\prod_{k=1}^{m} \Gamma(n_k+1)} \;/;\; -n \notin \mathbb{N}^+ \bigwedge n == \sum_{k=1}^{m} n_k$$

$$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) == 0 \;/;\; -n \in \mathbb{N}^+ \bigwedge n == \sum_{k=1}^{m} n_k.$$

Remark about values at special points: For $\alpha = a$ and $\nu == n$ integers with $a \leq 0$ and $n \leq -a$, the Pochhammer symbol $(\alpha)_\nu$ cannot be uniquely defined by a limiting procedure based on the previous definition because the two variables α and ν can approach the integers a and n with $a \leq 0$ and $n \leq -a$ at different speeds. For such integers with $a \leq 0$, $n \leq -a$, the following definition is used:

$$(a)_n == \frac{(-1)^n\,(-a)!}{(-a-n)!} \;/;\; -a \in \mathbb{N} \land n \in \mathbb{Z} \land n \leq -a.$$

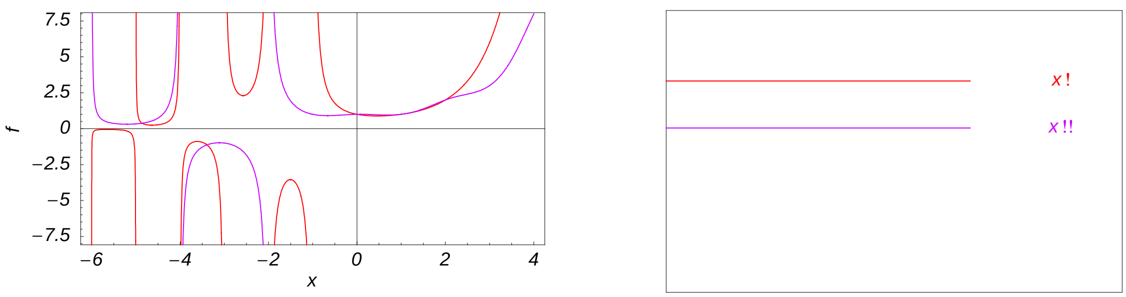
Similarly, for $\nu = n$, $\kappa == k$ negative integers with $k \leq n$, the binomial coefficient $\binom{n}{k}$ cannot be uniquely defined by a limiting procedure based on the previous definition because the two variables ν, κ can approach negative integers n, k with $k \leq n$ at different speeds. For negative integers with $k \leq n$, the following definition is used:

$$\binom{n}{k} == 0 \;/;\; n \in \mathbb{Z} \bigwedge k \in \mathbb{Z} \bigwedge k \leq n < 0.$$

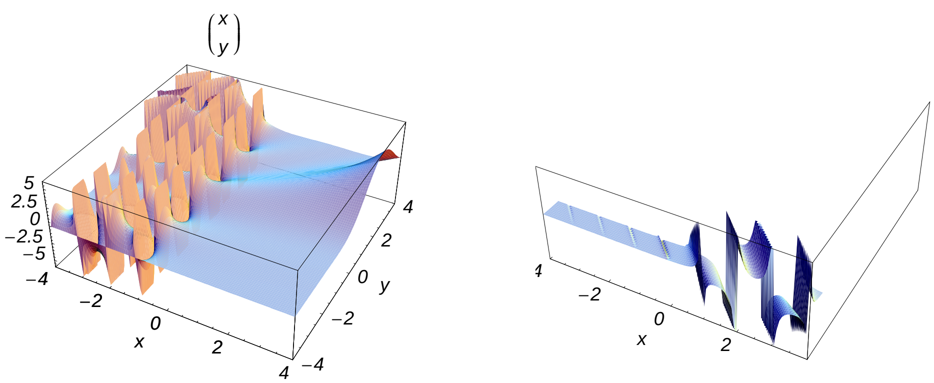
The previous symbols are interconnected and belong to one group that can be called factorials and binomials. These symbols are widely used in the coefficients of series expansions for the majority of mathematical functions.

A quick look at the factorials and binomials

Here is a quick look at the graphics for the factorial the real axis.



And here is a quick view of the bivariate binomial and Pochhammer functions. For positive arguments, both functions are free of singularities. For negative arguments, the functions have a complicated structure with many singularities.



Connections within the group of factorials and binomials and with other function groups

Representations through more general functions

Two factorials $n!$ and $n!!$ are the particular cases of the incomplete gamma function $\Gamma(a, z)$ with the second argument being 0:

$$n! == \Gamma(n+1, 0) \;/;\; \mathrm{Re}(n) > -1$$

$$n!! = \left(\frac{2}{\pi}\right)^{\frac{1}{4}(1-\cos(\pi n))} 2^{n/2}\, \Gamma\!\left(\frac{n}{2}+1,\, 0\right) /;\ \mathrm{Re}(n) > -2.$$

Representations through related equivalent functions

The factorial $n!$, double factorial $n!!$, Pochhammer symbol $(a)_n$, binomial coefficient $\binom{n}{k}$, and multinomial coefficient $(n_1 + n_2 + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_m)$ can be represented through the gamma function by the following formulas:

$$n! = \Gamma(n+1)$$

$$n!! = 2^{n/2} \left(\frac{\pi}{2}\right)^{\frac{1}{4}(\cos(\pi n)-1)} \Gamma\!\left(\frac{n}{2}+1\right)$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$(a)_n = \frac{(-1)^n\, \Gamma(1-a)}{\Gamma(1-a-n)} /;\ n \in \mathbb{Z}$$

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$$(a)_n = \frac{(-1)^n\, \Gamma(1-a)}{\Gamma(1-a-n)} /;\ n \in \mathbb{Z}$$

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\, \Gamma(1-k+n)}$$

$$(n_1 + n_2 + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_m) = \frac{\Gamma(n+1)}{\prod_{k=1}^{m} \Gamma(n_k+1)} /;\ -n \notin \mathbb{N}^+ \bigwedge n = \sum_{k=1}^{m} n_k.$$

Many of these formulas are used as the main elements of the definitions of many functions.

Representations through other factorials and binomials

The factorials and binomials $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1 + n_2 + \ldots + n_m;\, n_1,\, n_2,\, \ldots,\, n_m)$ are interconnected by the following formulas:

$$n! = 2^{\frac{1}{4}(\cos(2 n \pi)-1)-n}\, \pi^{\frac{1}{2}\sin^2(n \pi)}\, (2n)!!$$

$$n! = (n-1)!!\, n!!$$

$$n! = (1)_n$$

$$n!! = 2^{n/2} \left(\frac{\pi}{2}\right)^{\frac{1}{4}(\cos(\pi n)-1)} \left(\frac{n}{2}\right)!$$

$$n!! = 2^{n/2} \left(\frac{\pi}{2}\right)^{\frac{1}{4}(\cos(\pi n)-1)} (1)_{\frac{n}{2}}$$

$$(m)_n = \frac{(m+n-1)!}{(m-1)!} \;/;\; \neg(-m \in \mathbb{N} \land -m-n \in \mathbb{N})$$

$$(-m)_n = \frac{(-1)^n\, m!}{(m-n)!} \;/;\; m \in \mathbb{N} \land n \in \mathbb{N}$$

$$(a)_k = k! \binom{a+k-1}{k} = k! \binom{a+k-1}{a-1}$$

$$(a)_n = n!\,(a-1+n;\, a-1,\, n)$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$\binom{n}{k} = \frac{(1-k+n)_k}{k!}$$

$$\binom{n}{k} = \frac{(k+1)_{n-k}}{(n-k)!}$$

$$\binom{n}{k} = \frac{(-1)^k\,(-n)_k}{k!} \;/;\; k \in \mathbb{Z}$$

$$\binom{n}{k} = (n;\, n-k,\, k)$$

$$(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m) = \frac{n!}{\prod_{k=1}^{m} n_k!} \;/;\; n = \sum_{k=1}^{m} n_k$$

$$(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m) = \frac{(n_m+1)_{n-n_m}}{\prod_{k=1}^{m-1} n_k!} \;/;\; n = \sum_{k=1}^{m} n_k.$$

The best-known properties and formulas for factorials and binomials

Real values for real arguments

For real values of arguments, the values of the factorials and binomials $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m)$ are real (or infinity).

Simple values at zero

The factorials and binomials $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1+n_2+\ldots+n_m;\, n_1,\, n_2,\, \ldots,\, n_m)$ have simple values for zero arguments:

$$0! = 1$$

$$0!! = 1$$

$$(0)_0 = 1$$

$$\binom{0}{0} == 1$$

$$(0; 0) == 1$$

$$(0; 0, 0, \ldots, 0) == 1$$

$$(a)_0 == 1$$

$$(0)_{-n} == \frac{(-1)^n}{n!} /; n \in \mathbb{N}^+$$

$$(0)_n == 0 /; n \in \mathbb{N}^+$$

$$\binom{n}{0} == 1$$

$$\binom{0}{k} == \frac{\sin(k\pi)}{k\pi}.$$

Values at fixed points

Students usually learn the following basic table of values of the factorials $n!$ and $n!!$ in special integer points:

$$(-1)! == \tilde{\infty}$$

$$0! == 1$$

$$1! == 1$$

$$2! == 2$$

$$3! == 6$$

$$(-2)!! == \tilde{\infty}$$

$$(-1)!! == 1$$

$$0!! == 1$$

$$1!! == 1$$

$$2!! == 2$$

$$3!! == 3$$

$$4!! == 8.$$

Specific values for specialized variables

If variable n is a rational or integer number, the factorials $n!$ and $n!!$ can be represented by the following general formulas:

$$(-n)! == \tilde{\infty} /; n \in \mathbb{N}^+$$

$$\left(\frac{p}{q}+n\right)! == \frac{1}{q^n}\,\frac{p}{q}!\prod_{k=1}^{n}(p+k\,q)\;/;\;n\in\mathbb{N}\bigwedge p\in\mathbb{N}^+\bigwedge q\in\mathbb{N}^+\bigwedge p<q$$

$$\left(\frac{p}{q}-n\right)! == \frac{(-1)^n\,q^n}{\prod_{k=1}^{n}(k\,q-q-p)}\,\frac{p}{q}!\;/;\;n\in\mathbb{N}\bigwedge p\in\mathbb{N}^+\bigwedge q\in\mathbb{N}^+\bigwedge p<q$$

$$(-2\,k)!! == \tilde{\infty}\;/;\;k\in\mathbb{N}^+$$

$$(2\,k)!! == \prod_{j=1}^{k}2\,j\;/;\;k\in\mathbb{N}$$

$$(2\,k)!! == 2^k\,k!\;/;\;k\in\mathbb{N}$$

$$(2\,k-1)!! == \prod_{j=1}^{k}(2\,j-1)\;/;\;k\in\mathbb{N}.$$

For some particular values of the variables, the Pochhammer symbol $(a)_n$ has the following meanings:

$$(a)_1 == a$$

$$(a)_2 == a\,(a+1)$$

$$(a)_{-n} == \prod_{k=1}^{n}\frac{1}{a-k}\;/;\;n\in\mathbb{N}^+$$

$$(a)_{-2} == \frac{1}{(a-1)\,(a-2)}$$

$$(a)_{-1} == \frac{1}{a-1}$$

$$\left(-\frac{1}{2}\right)_n == -\frac{(2\,n-2)!}{2^{2\,n-1}\,(n-1)!}$$

$$\left(\frac{1}{2}\right)_n == \frac{(2\,n-1)!}{2^{2\,n-1}\,(n-1)!}$$

$$(1)_n == n!.$$

Some well-known formulas for binomial and multinomial functions are:

$$\binom{n}{k} == 0\;/;\;-k\in\mathbb{N}^+\bigvee k-n\in\mathbb{N}^+$$

$$\binom{n}{1} == n$$

$$\binom{n}{2} == \frac{(n-1)\,n}{2}$$

$$\binom{n}{n} == 1$$

$(n; n) = 1$

$(n_1 + n_2; n_1, n_2) = \binom{n_1 + n_2}{n_2}$.

Analyticity

The factorials and binomials $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)$ are defined for all complex values of their variables. The factorials, binomials, and multinomials are analytical functions of their variables and do not have branch cuts and branch points. The functions $n!$ and $n!!$ do not have zeros: $n! \neq 0$; $n!! \neq 0$. Therefore, the functions $1/n!$ and $1/n!!$ are entire functions with an essential singular point at $z = \tilde{\infty}$.

Poles and essential singularities

The factorials and binomials $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)$ have an essential singularity for infinite values of any argument. This singular point is also the point of convergence of the poles (except $k = \tilde{\infty}$ for $\binom{n}{k}$).

The function $n!$ has an infinite set of singular points: $n = -k /; k \in \mathbb{N}^+$ are the simple poles with residues $(-1)^{k-1} / ((k-1)!)$.

The function $n!!$ has an infinite set of singular points: $n = -2k /; k - 1 \in \mathbb{N}^+$ are the simple poles with residues $(-1)^{k-1} / ((2k-2)!!)$.

For fixed a, the function $(a)_n$ has an infinite set of singular points: $n = -a - k /; k \in \mathbb{N}$ are the simple poles with residues $(-1)^k / (k!\, \Gamma(a))$.

For fixed n, the function $(a)_n$ has an infinite set of singular points: $a = -k - n /; k \in \mathbb{N}$ are the simple poles with residues $(-1)^k / (k!\, \Gamma(-n-k)) /; k + n \notin \mathbb{N}$.

For fixed k, the function $\binom{n}{k}$ has an infinite set of singular points: $n = -j /; j \in \mathbb{N}^+$ are the simple poles with residues $(-1)^j / (j!\, k!\, (-j-k)!) /; k \notin \mathbb{Z}$.

By variable n_k, $1 \leq k \leq m$, (with the other variables fixed) the function $(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)$ has an infinite set of singular points: $n_k = -\tilde{N}_k - j /; j \in \mathbb{N}^+$ are the simple poles with residues $(-1)^{j-1} / \left(\Gamma\left(1 - j - \tilde{N}_k\right) \prod_{r=1}^{k-1} \Gamma(n_r + 1) \prod_{r=k+1}^{m} \Gamma(n_r + 1)\, (j-1)!\right) /; \tilde{N}_k = \sum_{r=1}^{k-1} n_r + \sum_{r=k+1}^{m} n_r \bigwedge j \in \mathbb{N}^+$.

Periodicity

The factorials and binomials $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)$ do not have periodicity.

Parity and symmetry

The factorials and binomials $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1+n_2+\ldots+n_m;\, n_1, n_2, \ldots, n_m)$ have mirror symmetry:

$$\overline{n}! = \overline{n!}$$

$$\overline{n}!! = \overline{n!!}$$

$$(\overline{a})_{\overline{n}} = \overline{(a)_n}$$

$$\binom{\overline{n}}{\overline{k}} = \overline{\binom{n}{k}}$$

$$(\overline{n_1}+\overline{n_2}+\ldots+\overline{n_m};\, \overline{n_1}, \overline{n_2}, \ldots, \overline{n_m}) = \overline{(n_1+n_2+\ldots+n_m;\, n_1, n_2, \ldots, n_m)}.$$

The multinomial $(n_1+n_2+\ldots+n_m;\, n_1, n_2, \ldots, n_m)$ has permutation symmetry:

$$(n_1+n_2;\, n_1, n_2) = (n_1+n_2;\, n_2, n_1)$$

$$(n_1+n_2+\ldots+n_k+\ldots+n_j+\ldots+n_m;\, n_1, n_2, \ldots, n_k, \ldots, n_j, \ldots, n_m) = (n_1+n_2+\ldots+n_j+\ldots+n_k+\ldots+n_m;\, n_1, n_2, \ldots, n_j, \ldots, n_k, \ldots, n_m) \;/;\; n_k \neq n_j \bigwedge k \neq j.$$

Series representations

The factorials $n!$, $n!!$, and $(a)_n$ have the following series expansions in the regular points:

$$n! \propto n_0!\left(1+\psi(n_0+1)(n-n_0)+\frac{1}{2}\left(\psi(n_0+1)^2+\psi^{(1)}(n_0+1)\right)(n-n_0)^2+\ldots\right) /;\; (n\to n_0) \bigwedge -n_0 \notin \mathbb{N}^+$$

$$n!! \propto n_0!!\left(1+\frac{1}{4}\left(\log(4)+2\,\psi\left(\frac{n_0}{2}+1\right)+\pi\log\left(\frac{2}{\pi}\right)\sin(n_0\,\pi)\right)(n-n_0)+\ldots\right) /;\; (n\to n_0) \bigwedge -\frac{n_0}{2} \notin \mathbb{N}^+$$

$$(a)_n \propto \Gamma(n)\,a+\Gamma(n)\,(\psi(n)+\gamma)\,a^2+\ldots /;\; (a\to 0)$$

$$(a)_n = \sum_{k=0}^{n}(-1)^{k+n}\,S_n^{(k)}\,a^k /;\; n \in \mathbb{N}$$

$$(a)_n \propto \frac{\Gamma(b+n)}{\Gamma(b)}\left(1+(\psi(b+n)-\psi(b))(a-b)+\frac{1}{2}\left(\psi(b)^2-2\,\psi(b+n)\,\psi(b)+\psi(b+n)^2-\psi^{(1)}(b)+\psi^{(1)}(b+n)\right)(a-b)^2+\ldots\right) /;\; \{a\to b\}$$

$$(a)_n = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k+n}\,S_n^{(k)}\binom{k}{j}b^j\,(a-b)^{k-j} /;\; n\in\mathbb{N}.$$

The series expansions of $n!$ and $n!!$ near singular points are given by the following formulas:

$$n! \propto \frac{(-1)^{m-1}}{(m-1)!}\left(\frac{1}{n+m}+\psi(m)+\frac{1}{6}\left(3\,\psi(m)^2+\pi^2-3\,\psi^{(1)}(m)\right)(n+m)+\ldots\right) /;\; (n\to -m) \bigwedge m\in\mathbb{N}^+$$

$$n!! \propto \frac{(-1)^{m-1} 2^{1-m}}{(m-1)!} \left(\frac{1}{n+2m} + \frac{1}{2} (\log(2) + \psi(m)) + \frac{1}{24} \left(3 \log^2(2) + \pi^2 + 3 \psi(m)^2 + \pi^2 \log(8) - 3 \pi^2 \log(\pi) + \log(64) \psi(m) - 3 \psi^{(1)}(m)\right) (n+2m) + \ldots \right) /; (n \to -2m) \bigwedge m \in \mathbb{N}^+.$$

Asymptotic series expansions

The asymptotic behavior of the factorials and binomials $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m)$ can be described by the following formulas (only the main terms of asymptotic expansion are given). The first is the famous Stirling's formula:

$$n! \propto \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \left(1 + O\left(\frac{1}{n}\right)\right) /; |\operatorname{Arg}(n)| < \pi \bigwedge (|n| \to \infty)$$

$$n!! \propto \left(\frac{2}{\pi}\right)^{\frac{1}{4}(1-\cos(\pi n))} \sqrt{\pi}\, n^{\frac{n+1}{2}} e^{-\frac{n}{2}} \left(1 + O\left(\frac{1}{n}\right)\right) /; |\operatorname{Arg}(n)| < \pi \bigwedge (|n| \to \infty)$$

$$(a)_n \propto z^n \left(1 + \frac{(n-1)n}{2a} + O\left(\frac{1}{a^2}\right)\right) /; (|a| \to \infty) \bigwedge |\operatorname{Arg}(a+n)| < \pi$$

$$(a)_n \propto \frac{\sqrt{2\pi}}{\Gamma(a)} e^{-n} n^{a+n-\frac{1}{2}} \left(1 + O\left(\frac{1}{n}\right)\right) /; (|n| \to \infty) \bigwedge |\operatorname{Arg}(a+n)| < \pi$$

$$\binom{n}{k} \propto \frac{n^k}{\Gamma(k+1)} \left(1 + O\left(\frac{1}{n}\right)\right) /; (|n| \to \infty) \bigwedge |\operatorname{Arg}(n+1)| < \pi$$

$$\binom{n}{k} \propto \frac{\Gamma(n+1)\sin(\pi(k-n))\, k^{-n-1}}{\pi} \left(1 + O\left(\frac{1}{k}\right)\right) /; (|k| \to \infty) \bigwedge |\operatorname{Arg}(k-n)| < \pi$$

$$(n_1 + n_2 + \ldots + n_m; n_1, n_2, \ldots, n_m) \propto \frac{n_1^{a-1}}{\prod_{k=2}^m \Gamma(n_k+1)} \left(1 + O\left(\frac{1}{n_1}\right)\right) /; (|n_1| \to \infty) \bigwedge a == \sum_{k=2}^m n_k + 1 \bigwedge |\operatorname{Arg}(a+n_1)| < \pi.$$

Integral representations

The factorial $n!$ and binomial $\binom{n}{k}$ can also be represented through the following integrals:

$$n! == \int_0^\infty t^n e^{-t} \, dt /; n \in \mathbb{N}$$

$$n! == \int_0^1 \log^n\left(\frac{1}{t}\right) dt /; \operatorname{Re}(n) > -1$$

$$n! == \int_0^\infty \left(e^{-t} - \sum_{k=0}^m \frac{(-t)^k}{k!}\right) t^n \, dt /; m \in \mathbb{N}^+ \bigwedge -m-1 < \operatorname{Re}(n) < -m$$

$$n! == \int_1^{\infty} t^n e^{-t} \, dt + \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \, (k+n+1)}$$

$$\binom{n}{k} == \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i k t} \left(1 + e^{i t}\right)^n dt \,/;\, k \in \mathbb{R} \, k > -1 \bigwedge n \in \mathbb{R}.$$

Transformations

The following formulas describe some of the main types of transformations between and among factorials and binomials:

$$(-n)! == \frac{\pi \csc(\pi n)}{(n-1)!}$$

$$(n+1)! == (n+1) \, n!$$

$$(n+m)! == (n+1)_m \, n!$$

$$(n-1)! == \frac{n!}{n}$$

$$(n-m)! == \frac{(-1)^m \, n!}{(-n)_m} \,/;\, m \in \mathbb{Z}$$

$$(-n)!! == \frac{1}{(n-2)!!} \left(\frac{\pi}{2}\right)^{\cos^2\left(\frac{n\pi}{2}\right)} \csc\left(\frac{n\pi}{2}\right)$$

$$(n+2)!! == (n+2) \, n!!$$

$$(n+2m)!! == 2^m \left(\frac{n}{2}+1\right)_m n!! \,/;\, m \in \mathbb{Z}$$

$$(n-2)!! == \frac{n!!}{n}$$

$$(n-2m)!! == \frac{(-1)^m \, 2^{-m} \, n!!}{\left(-\frac{n}{2}\right)_m} \,/;\, m \in \mathbb{Z}$$

$$(a)_{k+m n} == (a)_k \, m^{m n} \prod_{j=0}^{m-1} \left(\frac{a+j+k}{m}\right)_n \,/;\, m \in \mathbb{N}$$

$$(a m + b)_n == m^n \prod_{k=0}^{m-1} \left(a + \frac{b+k}{m}\right)_{\frac{n}{m}} \,/;\, m \in \mathbb{N}^+.$$

Some of these transformations can be called addition formulas, for example:

$$(a+b)_n == \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a+k)_{n-k} \, (-b)_k \,/;\, n \in \mathbb{N}$$

$$(a+b)_n = n! \sum_{k=0}^{n} \frac{(a)_k\,(b)_{n-k}}{k!\,(n-k)!} \;/;\; n \in \mathbb{N}$$

$$(a)_{m+n} = (a)_m\,(a+m)_n.$$

Multiple argument transformations are, for example:

$$(2n)! = \frac{2^{2n}\,n}{\sqrt{\pi}}\,(n-1)!\left(n-\frac{1}{2}\right)!$$

$$(m\,n)! = n\,m^{m\,n+\frac{1}{2}}\,(2\pi)^{\frac{1-m}{2}} \prod_{k=0}^{m-1}\left(\frac{k}{m}+n-1\right)! \;/;\; m \in \mathbb{N}^+$$

$$(2n)!! = 2^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}\sin^2(n\pi)} (n-1)!!\,n!!$$

$$(m\,n)!! = n\,m^{\frac{1}{2}(m\,n+1)}\;2^{\frac{1-m}{2}}\left(\frac{\pi}{2}\right)^{\frac{1}{4}(1-m+\cos(m\,n\,\pi))} \prod_{k=0}^{m-1}\left(\frac{2k}{m}+n-2\right)!! \;/;\; m \in \mathbb{N}^+$$

$$(2a)_{2n} = 2^{2n}\,(a)_n\left(a+\frac{1}{2}\right)_n.$$

The following transformations are for products of the functions:

$$n!\,(-n)! = n\,\pi\,\csc(n\,\pi)$$

$$n!\,m! = \frac{(m+n)!}{\binom{m+n}{n}}$$

$$\frac{n!}{m!} = (m+1)_{n-m}$$

$$\frac{m!\,n!}{(m+n+1)!} = \mathrm{B}(m+1,\,n+1)$$

$$(-n)!!\,n!! = n\left(\frac{\pi}{2}\right)^{\cos^2\left(\frac{n\pi}{2}\right)} \csc\left(\frac{n\,\pi}{2}\right)$$

$$n!!\,m!! = \frac{1}{\binom{\frac{m+n}{2}}{\frac{n}{2}}}\,2^{\frac{m+n}{2}}\left(\frac{2}{\pi}\right)^{\frac{1}{4}(2-\cos(m\,\pi)-\cos(n\,\pi))} \frac{m+n}{2}!$$

$$\frac{n!!}{m!!} = 2^{\frac{n-m}{2}}\left(\frac{2}{\pi}\right)^{\frac{1}{4}(\cos(m\,\pi)-\cos(n\,\pi))} \left(\frac{m}{2}+1\right)_{\frac{n-m}{2}}$$

$$\frac{m!!\,n!!}{(m+n+2)!!} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{4}(\cos(m\,\pi)+\cos(n\,\pi)-\cos(\pi\,(m+n))-1)} \mathrm{B}\left(\frac{m}{2}+1,\,\frac{n}{2}+1\right).$$

Identities

The factorials $n!$ and $n!!$ can be defined as the solutions of the following corresponding functional equations:

$$f(n) == n\, f(n-1) \;/;\; f(n) == n!\, g(n) \bigwedge g(n) == g(n-1) \bigwedge f(1) == 1$$

$$f(n) == n\, f(n-2) \;/;\; f(n) == n!!\, g(n) \bigwedge g(n) == g(n-2) \bigwedge f(1) == 1.$$

The factorial $n!$ is the unique nonzero solution of the functional equation $f(n) == n\, f(n-1)$ that is logarithmically convex for all real $n > 0$; that is, for which $\log(f(n))$ is a convex function for $n > 0$.

The factorials and binomials $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1 + n_2 + \ldots + n_m;\, n_1, n_2, \ldots, n_m)$ satisfy the following recurrence identities:

$$n! == \frac{1}{n+1} (n+1)!$$

$$n! == n\, (n-1)!$$

$$n!! == \frac{1}{n+2} (n+2)!!$$

$$n!! == n\, (n-2)!$$

$$(a)_n == \frac{1}{a-1} (a-1)_{n+1}$$

$$(a)_n == a\, (a+1)_{n-1}$$

$$(a)_n == \frac{a}{a+n} (a+1)_n$$

$$(a)_n == \frac{a+n-1}{a-1} (a-1)_n$$

$$(a)_n == \frac{1}{a+n} (a)_{n+1}$$

$$(a)_n == (a+n-1)\, (a)_{n-1}$$

$$\binom{n}{k} == \frac{n-k+1}{n+1} \binom{n+1}{k}$$

$$\binom{n}{k} == \frac{n}{n-k} \binom{n-1}{k}$$

$$\binom{n}{k} == \frac{k+1}{n-k} \binom{n}{k+1}$$

$$\binom{n}{k} == \frac{n-k+1}{k} \binom{n}{k-1}$$

$$(n_1 + n_2 + \ldots + n_m;\, n_1, n_2, \ldots, n_m) = \frac{n_l + 1}{\sum_{j=1}^{m} n_j + 1} (n_1 + n_2 + \ldots + n_m + 1;\, n_1, n_2, \ldots, n_{l-1}, n_l + 1, n_{l+1}, \ldots, n_m)$$

$$(n_1 + n_2 + \ldots + n_m;\, n_1, n_2, \ldots, n_m) = \frac{\sum_{j=1}^{m} n_j}{n_l} (n_1 + n_2 + \ldots + n_m - 1;\, n_1, n_2, \ldots, n_{l-1}, n_l - 1, n_{l+1}, \ldots, n_m).$$

The previous formulas can be generalized to the following recurrence identities with a jump of length n:

$$n! = \frac{(n+m)!}{(n+1)_m}$$

$$n! = (-1)^m (-n)_m (n-m)! \,/;\, m \in \mathbb{Z}$$

$$n!! = \frac{2^{-m} (n + 2m)!!}{\left(\frac{n}{2} + 1\right)_m} \,/;\, m \in \mathbb{Z}$$

$$n!! = (-1)^m 2^m \left(-\frac{n}{2}\right)_m (n - 2m)!! \,/;\, m \in \mathbb{Z}$$

$$(a)_n = \frac{\Gamma(a+m)\,\Gamma(a+n)}{\Gamma(a)\,\Gamma(a+m+n)} (a+m)_n$$

$$(a)_n = \frac{\Gamma(a-m)\,\Gamma(a+n)}{\Gamma(a)\,\Gamma(a-m+n)} (a-m)_n$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a+m+n)} (a)_{n+m}$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a-m+n)} (a)_{n-m}$$

$$\binom{n}{k} = \frac{(n-k+1)_m}{(n+1)_m} \binom{n+m}{k}$$

$$\binom{n}{k} = \frac{(n-m+1)_m}{(n-m+1)_m} \binom{n-m}{k}$$

$$\binom{n}{k} = \frac{(k+1)_m}{(n-m-k+1)_m} \binom{n}{k+m}$$

$$\binom{n}{k} = \frac{(n-k+1)_m}{(k-m+1)_m} \binom{n}{k-m}$$

$$(n_1 + n_2 + \ldots + n_m;\, n_1, n_2, \ldots, n_m) = \frac{(n_l + 1)_p}{\left(\sum_{j=1}^{m} n_j + 1\right)_p} (n_1 + n_2 + \ldots + n_m + p;\, n_1, n_2, \ldots, n_{l-1}, n_l + p, n_{l+1}, \ldots, n_m)$$

$$(n_1 + n_2 + \ldots + n_m;\, n_1, n_2, \ldots, n_m) = \frac{1}{(n_l - p + 1)_p} \left(\sum_{j=1}^{m} n_j - p + 1\right)_m (n_1 + n_2 + \ldots + n_m - p;\, n_1, n_2, \ldots, n_{l-1}, n_l - p\,, n_{l+1}, \ldots, n_m).$$

The Pochhammer symbol $(a)_n$ and binomial $\binom{n}{k}$ satisfy the following functional identities:

$$(a)_n = \frac{(-1)^n}{(1-a)_{-n}} \,/;\, n \in \mathbb{Z}$$

$$(a)_n = \frac{1}{(a-m)_m} (a-m)_{n+m}$$

$$(a)_n = (a)_m (a+m)_{n-m}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k} \,/;\, k \in \mathbb{Z}.$$

Representations of derivatives

The derivatives of the functions $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1+n_2+\ldots+n_m; n_1, n_2, \ldots, n_m)$ have rather simple representations that include the corresponding functions as factors:

$$\frac{\partial n!}{\partial n} = n!\, \psi(n+1)$$

$$\frac{\partial n!!}{\partial n} = \frac{1}{2} n!! \left(\log(2) + \psi\left(\frac{n}{2}+1\right) + \frac{\pi}{2} \log\left(\frac{2}{\pi}\right) \sin(n\pi)\right)$$

$$\frac{\partial (a)_n}{\partial a} = (a)_n (\psi(a+n) - \psi(a))$$

$$\frac{\partial (a)_n}{\partial n} = (a)_n\, \psi(a+n)$$

$$\frac{\partial \binom{n}{k}}{\partial n} = \binom{n}{k} (\psi(n+1) - \psi(n-k+1))$$

$$\frac{\partial \binom{n}{k}}{\partial k} = \binom{n}{k} (\psi(1-k+n) - \psi(k+1))$$

$$\frac{\partial (n_1+n_2+\ldots+n_m; n_1, n_2, \ldots, n_m)}{\partial n_m} = (n_1+n_2+\ldots+n_m; n_1, n_2, \ldots, n_m) (\psi(n+1) - \psi(n_m+1)) \,/;\, n = \sum_{k=1}^{m} n_k.$$

The symbolic derivatives of the n^{th} order form factorials and binomials $n!$, $n!!$, $(a)_n$, $\binom{n}{k}$, and $(n_1+n_2+\ldots+n_m; n_1, n_2, \ldots, n_m)$ have much more complicated representations, which can include recursive function calls, regularized generalized hypergeometric functions ${}_{m+2}\tilde{F}_{m+1}$, or Stirling numbers $S_n^{(k)}$:

$$\frac{\partial^m n!}{\partial z^m} == \Gamma(n+1)\, R(m, n+1) \;/;\; R(m, z) == \psi(z)\, R(m-1, z) + R^{(0,1)}(m-1, z) \bigwedge R(0, z) == 1 \bigwedge m \in \mathbb{N}^+$$

$$\frac{\partial^m (a)_n}{\partial a^m} == \frac{(-1)^m\, m!\, \Gamma(a+n)^{m+1}}{\Gamma(-n)}\, {}_{m+2}\tilde{F}_{m+1}(a_1, a_2, \ldots, a_{m+1}, n+1; a_1+1, a_2+1, \ldots, a_{m+1}+1; 1) \;/;$$
$$a_1 == a_2 == \ldots == a_{m+1} == a+n \bigwedge m \in \mathbb{N}^+ \bigwedge n \notin \mathbb{N}$$

$$\frac{\partial^m (a)_n}{\partial a^m} == \sum_{k=1}^{n} (-1)^{k+n}\, S_n^{(k)}\, (k-m+1)_m\, a^{k-m} \;/;\; m \in \mathbb{N} \land n \in \mathbb{N}$$

$$\frac{\partial^m \binom{n}{k}}{\partial n^m} == \frac{(-1)^{m-1} \sin(\pi k)\, m!}{\pi}\, \Gamma(n+1)^{m+1}\, {}_{m+2}\tilde{F}_{m+1}(a_1, a_2, \ldots, a_{m+1}, k+1; a_1+1, a_2+1, \ldots, a_{m+1}+1; 1) \;/;$$
$$a_1 == a_2 == \ldots == a_{m+1} == n+1 \bigwedge m \in \mathbb{N}^+ \bigwedge k \notin \mathbb{N}^+$$

$$\frac{\partial^m \binom{n}{k}}{\partial n^m} == \frac{1}{k!} \sum_{j=1}^{k} (-1)^{j+k}\, S_k^{(j)}\, (j-m+1)_m\, (1-k+n)^{j-m} \;/;\; m \in \mathbb{N}^+ \bigwedge k \in \mathbb{N}^+$$

$$\frac{\partial^m \binom{n}{k}}{\partial k^m} == \pi^{m-1}\, \Gamma(k-n)$$
$$\sum_{j=0}^{m} \binom{m}{j} \sin\left(\pi\left(\frac{m-j}{2} + k - n\right)\right) j!\, {}_{m+2}\tilde{F}_{m+1}(a_1, a_2, \ldots, a_{m+1}, -n; a_1+1, a_2+1, \ldots, a_{m+1}+1; 1) \left(-\frac{\Gamma(k-n)}{\pi}\right)^j \;/;$$
$$a_1 == a_2 == \ldots == a_{m+1} == k-n \bigwedge m \in \mathbb{N}^+$$

$$\frac{\partial^u (n_1+n_2+\ldots+n_m; n_1, n_2, \ldots, n_m)}{\partial n_m^u} ==$$
$$\frac{(-1)^u\, u!\, \Gamma(s+1)^{u+1}}{\prod_{k=1}^{m-1} \Gamma(n_k+1)\, \Gamma(n_m-s)}\, {}_{u+2}\tilde{F}_{u+1}(a_1, a_2, \ldots, a_{u+1}, s-n_m+1; a_1+1, a_2+1, \ldots, a_{u+1}+1; 1) \;/;$$
$$a_1 == a_2 == \ldots == a_{u+1} == s+1 \bigwedge s == \sum_{k=1}^{m} n_k \bigwedge u \in \mathbb{N}^+ \bigwedge s - n_m \notin \mathbb{N}.$$

Applications of factorials and binomials

Applications of factorials and binomials include combinatorics, number theory, discrete mathematics, and calculus.

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