Introductions to Gamma2

Introduction to the gamma functions

General

The gamma function $\Gamma(z)$ is applied in exact sciences almost as often as the well-known factorial symbol $n!$. It was introduced by the famous mathematician L. Euler (1729) as a natural extension of the factorial operation $n!$ from positive integers n to real and even complex values of this argument. This relation is described by the formula:

$$\Gamma(n) = (n-1)!.$$

Euler derived some basic properties and formulas for the gamma function. He started investigations of $n!$ from the infinite product:

$$\frac{1}{\Gamma(z)} = z\, e^{z\gamma} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}.$$

The gamma function $\Gamma(z)$ has a long history of development and numerous applications since 1729 when Euler derived his famous integral representation of the factorial function. In modern notation it can be rewritten as the following:

$$n! = \Gamma(n+1) = \int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^n dt = \int_0^\infty \tau^n e^{-\tau}\, d\tau \;/;\; \operatorname{Re}(n) > -1.$$

The history of the gamma function is described in the subsection "General" of the section "Gamma function." Since the famous work of J. Stirling (1730) who first used series for $\log(n!)$ to derive the asymptotic formula for $n!$, mathematicians have used the logarithm of the gamma function $\log(\Gamma(z))$ for their investigations of the gamma function $\Gamma(z)$. Investigators of mention include: C. Siegel, A. M. Legendre, K. F. Gauss, C. J. Malmstén, O. Schlömilch, J. P. M. Binet (1843), E. E. Kummer (1847), and G. Plana (1847). M. A. Stern (1847) proved convergence of the Stirling's series for the derivative of $\log(\Gamma(z))$. C. Hermite (1900) proved convergence of the Stirling's series for $\log(\Gamma(z+1))$ if z is a complex number.

During the twentieth century, the function $\log(\Gamma(z))$ was used in many works where the gamma function was applied or investigated. The appearance of computer systems at the end of the twentieth century demanded more careful attention to the structure of branch cuts for basic mathematical functions to support the validity of the mathematical relations everywhere in the complex plane. This lead to the appearance of a special log-gamma function $\log\Gamma(z)$, which is equivalent to the logarithm of the gamma function $\log(\Gamma(z))$ as a multivalued analytic function, except that it is conventionally defined with a different branch cut structure and principal sheet. The log-gamma function $\log\Gamma(z)$ was introduced by J. Keiper (1990) for *Mathematica*. It allows a concise formulation of many identities related to the Riemann zeta function $\zeta(z)$.

The importance of the gamma function and its Euler integral stimulated some mathematicians to study the incomplete Euler integrals, which are actually equal to the indefinite integral of the expression $\tau^n e^{-\tau}$. They were introduced in an article by A. M. Legendre (1811). Later, P. Schlömilch (1871) introduced the name "incomplete gamma function" for such an integral. These functions were investigated by J. Tannery (1882), F. E. Prym (1877), and M. Lerch (1905) (who gave a series representation for the incomplete gamma function). N. Nielsen (1906) and other mathematicians also had special interests in these functions, which were included in the main handbooks of special functions and current computer systems like *Mathematica*.

The needs of computer systems lead to the implementation of slightly more general incomplete gamma functions and their regularized and inverse versions. In addition to the classical gamma function $\Gamma(z)$, *Mathematica* includes the following related set of gamma functions: incomplete gamma function $\Gamma(a, z)$, generalized incomplete gamma function $\Gamma(a, z_1, z_2)$, regularized incomplete gamma function $Q(a, z)$, generalized regularized incomplete gamma function $Q(a, z_1, z_2)$, log-gamma function $\log\Gamma(z)$, inverse of the regularized incomplete gamma function $Q^{-1}(a, z)$, and inverse of the generalized regularized incomplete gamma function $Q^{-1}(a, z_1, z_2)$.

Definitions of gamma functions

The gamma function $\Gamma(z)$, the incomplete gamma function $\Gamma(a, z)$, the generalized incomplete gamma function $\Gamma(a, z_1, z_2)$, the regularized incomplete gamma function $Q(a, z)$, the generalized regularized incomplete gamma function $Q(a, z_1, z_2)$, the log-gamma function (almost equal to the logarithm of the gamma function) $\log\Gamma(z)$, the inverse of the regularized incomplete gamma function $Q^{-1}(a, z)$, and the inverse of the generalized regularized incomplete gamma function $Q^{-1}(a, z_1, z_2)$ are defined by the following formulas:

$$\Gamma(z) == \int_0^\infty t^{z-1} e^{-t} \, dt \;/;\; \mathrm{Re}(z) > 0$$

$$\Gamma(z) == \int_1^\infty t^{z-1} e^{-t} \, dt + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+z)}$$

$$\Gamma(a, z) == \int_z^\infty t^{a-1} e^{-t} \, dt$$

$$\Gamma(a, z_1, z_2) == \int_{z_1}^{z_2} t^{a-1} e^{-t} \, dt$$

$$Q(a, z) == \frac{\Gamma(a, z)}{\Gamma(a)}$$

$$Q(a, z_1, z_2) == \frac{\Gamma(a, z_1, z_2)}{\Gamma(a)}$$

$$\log\Gamma(z) == \sum_{k=1}^{\infty} \left(\frac{z}{k} - \log\left(1 + \frac{z}{k}\right) \right) - \gamma z - \log(z).$$

The function $\log\Gamma(z)$ is equivalent to $\log(\Gamma(z))$ as a multivalued analytic function, except that it is conventionally defined with a different branch cut structure and principal sheet. The function $\log\Gamma(z)$ allows a concise formulation of many identities related to the Riemann zeta function $\zeta(z)$:

$z == Q(a, w) /; w == Q^{-1}(a, z)$

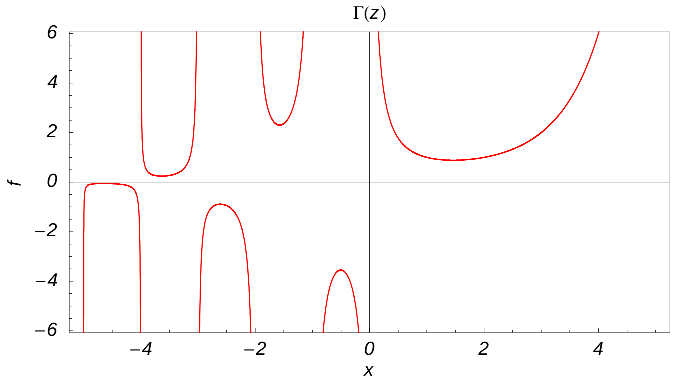
$z_2 == Q(a, z_1, w) /; w == Q^{-1}(a, z_1, z_2)$.

The previous functions comprise the interconnected group called the gamma functions.

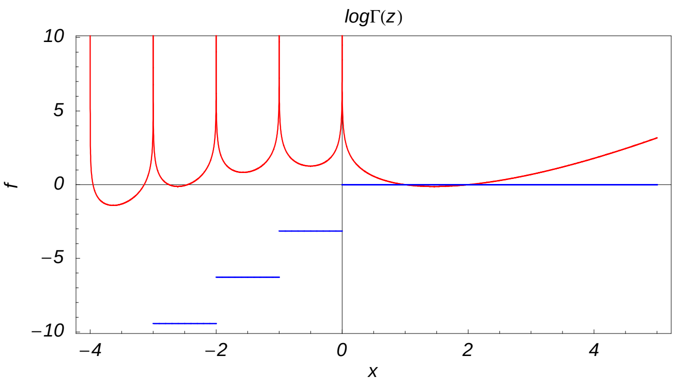
Instead of the first three previous classical definitions using definite integrals, the other equivalent definitions with infinite series can be used.

A quick look at the gamma functions

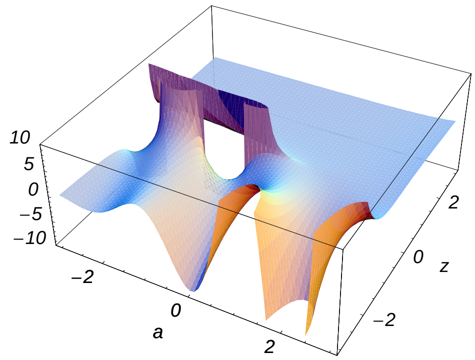
Here is a quick look at graphics for the gamma function and the function $\log\Gamma(z)$ along the real axis. The real parts are shown in red and the imaginary parts are shown in blue.



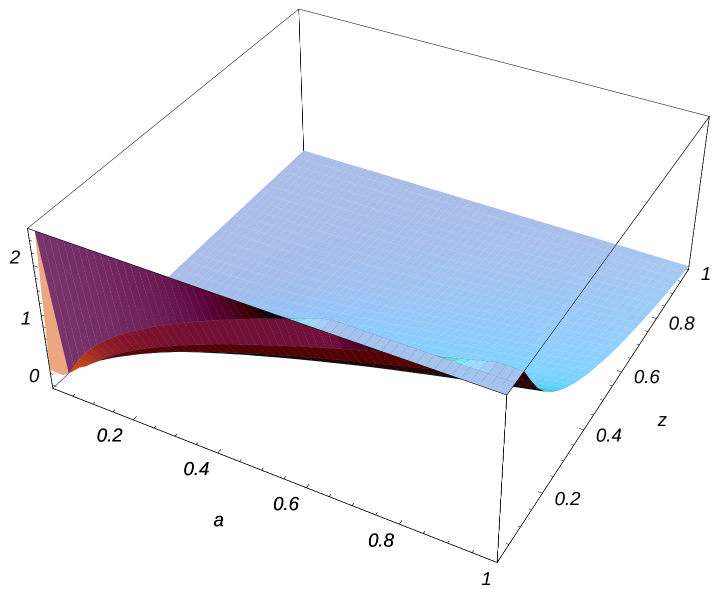
Here is a quick look at the graphics for the gamma function and the function $\log\Gamma(z)$ along the real axis.



These two graphics show the real part (left) and imaginary part (right) of $\Gamma(a, z)$ over the a-z–plane.



The next graphic shows the regularized incomplete gamma function $Q(a, z)$ over the a-z-plane.



Connections within the group of gamma functions and with other function groups

Representations through more general functions

The incomplete gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ are particular cases of the more general hypergeometric and Meijer G functions.

For example, they can be represented through hypergeometric functions ${}_1F_1$ and ${}_1\tilde{F}_1$ or the Tricomi confluent hypergeometric function U:

$$\Gamma(a, z) = \Gamma(a)\left(1 - z^a \, {}_1\tilde{F}_1(a; a+1; -z)\right) /; -a \notin \mathbb{N}$$

$$\Gamma(a, z) = \Gamma(a) - \frac{z^a}{a} \, {}_1F_1(a; a+1; -z) /; -a \notin \mathbb{N}$$

$$\Gamma(a, z) = e^{-z} \, U(1-a, 1-a, z)$$

$$\Gamma(a, z_1, z_2) = \Gamma(a)\left(z_2^a \, {}_1\tilde{F}_1(a; a+1; -z_2) - z_1^a \, {}_1\tilde{F}_1(a; a+1; -z_1)\right) /; -a \notin \mathbb{N}$$

$$\Gamma(a, z_1, z_2) == \frac{z_2^a}{a} \, {}_1F_1(a; a+1; -z_2) - \frac{z_1^a}{a} \, {}_1F_1(a; a+1; -z_1) \,/;\, -a \notin \mathbb{N}$$

$$\Gamma(a, z_1, z_2) == e^{-z_1} \, U(1-a, 1-a, z_1) - e^{-z_2} \, U(1-a, 1-a, z_2)$$

$$Q(a, z) == 1 - z^a \, {}_1\tilde{F}_1(a; a+1; -z) \,/;\, -a \notin \mathbb{N}^+$$

$$Q(a, z) == 1 - \frac{z^a}{\Gamma(a+1)} \, {}_1F_1(a; a+1; -z) \,/;\, -a \notin \mathbb{N}^+$$

$$Q(a, z) == \frac{1}{\Gamma(a)} \, e^{-z} \, U(1-a, 1-a, z)$$

$$Q(a, z_1, z_2) == z_2^a \, {}_1\tilde{F}_1(a; a+1; -z_2) - z_1^a \, {}_1\tilde{F}_1(a; a+1; -z_1)$$

$$Q(a, z_1, z_2) == \frac{z_2^a}{\Gamma(a+1)} \, {}_1F_1(a; a+1; -z_2) - \frac{z_1^a}{\Gamma(a+1)} \, {}_1F_1(a; a+1; -z_1) \,/;\, -a \notin \mathbb{N}$$

$$Q(a, z_1, z_2) == \frac{1}{\Gamma(a)} \left(e^{-z_1} \, U(1-a, 1-a, z_1) - e^{-z_2} \, U(1-a, 1-a, z_2)\right).$$

These functions also have rather simple representations in terms of classical Meijer G functions:

$$\Gamma(a, z) == G_{1,2}^{2,0}\!\left(z \,\middle|\, \begin{matrix} 1 \\ 0, a \end{matrix}\right)$$

$$\Gamma(a, z_1, z_2) == G_{1,2}^{1,1}\!\left(z_2 \,\middle|\, \begin{matrix} 1 \\ a, 0 \end{matrix}\right) - G_{1,2}^{1,1}\!\left(z_1 \,\middle|\, \begin{matrix} 1 \\ a, 0 \end{matrix}\right)$$

$$Q(a, z) == \frac{1}{\Gamma(a)} \, G_{1,2}^{2,0}\!\left(z \,\middle|\, \begin{matrix} 1 \\ 0, a \end{matrix}\right)$$

$$Q(a, z_1, z_2) == \frac{1}{\Gamma(a)} \left(G_{1,2}^{1,1}\!\left(z_2 \,\middle|\, \begin{matrix} 1 \\ a, 0 \end{matrix}\right) - G_{1,2}^{1,1}\!\left(z_1 \,\middle|\, \begin{matrix} 1 \\ a, 0 \end{matrix}\right)\right).$$

The log-gamma function $\log\Gamma(z)$ can be expressed through polygamma and zeta functions by the following formulas:

$$\log\Gamma(z) == \int_1^z \psi(t) \, dt$$

$$\log\Gamma(z) == \frac{\partial^{-\nu-1} \psi^{(\nu)}(z)}{\partial z^{-\nu-1}}$$

$$\log\Gamma(z) == \frac{1}{2} \log(2\pi) + \zeta^{(1,0)}(0, z) \,/;\, \mathrm{Re}(z) > 0.$$

Representations through related equivalent functions

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ can be represented using the related exponential integral $E_\nu(z)$ by the following formulas:

$$\Gamma(a, z) == z^a \, E_{1-a}(z)$$

$$\Gamma(a, z_1, z_2) = z_1^a E_{1-a}(z_1) - z_2^a E_{1-a}(z_2)$$

$$Q(a, z) = \frac{z^a E_{1-a}(z)}{\Gamma(a)}$$

$$Q(a, z_1, z_2) = \frac{1}{\Gamma(a)} (z_1^a E_{1-a}(z_1) - z_2^a E_{1-a}(z_2)).$$

Relations to inverse functions

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ are connected with the inverse of the regularized incomplete gamma function $Q^{-1}(a, z)$ and the inverse of the generalized regularized incomplete gamma function $Q^{-1}(a, z_1, z_2)$ by the following formulas:

$$\Gamma\left(a, Q^{-1}(a, z)\right) = \Gamma(a)\, z$$

$$\Gamma\left(a, z_1, Q^{-1}(a, z_1, z_2)\right) = \Gamma(a)\, z_2$$

$$Q\left(a, Q^{-1}(a, z)\right) = z$$

$$Q\left(a, z_1, Q^{-1}(a, z_1, z_2)\right) = z_2$$

$$Q^{-1}(a, Q(a, z_1) - z_2) = Q^{-1}(a, z_1, z_2)$$

$$Q^{-1}(a, z_1, z_2) = Q^{-1}(a, Q(a, z_1) - z_2).$$

Representations through other gamma functions

The gamma functions $\Gamma(a)$, $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and $\log\Gamma(z)$ are connected with each other by the formulas:

$$\Gamma(a) = \Gamma(a, 0) \;/;\; \mathrm{Re}(a) > 0$$

$$\Gamma(a, z) = \Gamma(a) + \Gamma(a, z, 0) \;/;\; \mathrm{Re}(a) > 0$$

$$\Gamma(a, z) = \Gamma(a)\,(Q(a, z, 0) + 1) \;/;\; \mathrm{Re}(a) > 0$$

$$\Gamma(a, z) = \Gamma(a)\, Q(a, z)$$

$$\Gamma(a, z_1, z_2) = \Gamma(a, z_1) - \Gamma(a, z_2)$$

$$\Gamma(a, z_1, z_2) = \Gamma(a)\, Q(a, z_1, z_2)$$

$$Q(a, z) = \frac{\Gamma(a, z, 0)}{\Gamma(a)} + 1 \;/;\; \mathrm{Re}(a) > 0$$

$$Q(a, z) = Q(a, z, 0) + 1 \;/;\; \mathrm{Re}(a) > 0$$

$$Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)}$$

$$Q(a, z_1, z_2) = Q(a, z_1) - Q(a, z_2)$$

$$Q(a, z_1, z_2) == \frac{\Gamma(a, z_1, z_2)}{\Gamma(a)}$$

$$\log\Gamma(z) == \log(\Gamma(z)) \text{ /; } 0 < \text{Re}(z) \leq 2 \bigwedge |\text{Im}(z)| \leq \frac{7}{2}$$

$$\log\Gamma(z) == 2\, i\, \pi\, k(z) + \log(\Gamma(z)) \text{ /; } k(z) == \int_0^z \theta(-\text{Re}(\Gamma(t)))\, |\text{Im}(\Gamma(t)\, \psi(t))|\, \delta(\text{Im}(\Gamma(t)))\, dt \in \mathbb{Z}.$$

The best-known properties and formulas for exponential integrals

Real values for real arguments

For real values of z, the values of the gamma function $\Gamma(z)$ are real (or infinity). For real values of the parameter a and positive arguments z, z_1, z_2, the values of the gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and $\log\Gamma(z)$ are real (or infinity).

Simple values at zero

The gamma functions $\Gamma(z)$, $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, $\log\Gamma(z)$, $Q^{-1}(a, z)$, and $Q^{-1}(a, z_1, z_2)$ have the following values at zero arguments:

$$\Gamma(0) == \tilde{\infty}$$

$$\Gamma(0, 0) == \tilde{\infty}$$

$$\Gamma(0, 0, 0) == ¿$$

$$Q(0, 0) == 0$$

$$Q(0, 0, 0) == 0$$

$$\log\Gamma(0) == \infty$$

$$Q^{-1}(0, 0) == 0$$

$$Q^{-1}(0, 0, 0) == 0.$$

Specific values for specialized variables

If the variable z is equal to 0 and $\text{Re}(a) > 0$, the incomplete gamma function $\Gamma(a, z)$ coincides with the gamma function $\Gamma(a)$ and the corresponding regularized gamma function $Q(a, z)$ is equal to 1:

$$\Gamma(a, 0) == \Gamma(a) \text{ /; } \text{Re}(a) > 0 \quad Q(a, 0) == 1 \text{ /; } \text{Re}(a) > 0.$$

In cases when the parameter a equals $1, 2, 3, \ldots$, the incomplete gamma functions $\Gamma(a, z)$ and $Q(a, z)$ can be expressed as an exponential function multiplied by a polynomial. In cases when the parameter a equals $0, -1, -2, \ldots$, the incomplete gamma function $\Gamma(a, z)$ can be expressed with the exponential integral $\text{Ei}(z)$, exponential, and logarithmic functions, but the regularized incomplete gamma function $Q(a, z)$ is equal to 0. In cases when the parameter a equals $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$, the incomplete gamma functions $\Gamma(a, z)$ and $Q(a, z)$ can be expressed through the complementary error function $\text{erfc}(z)$ and the exponential function, for example:

$$\Gamma(1, z) = e^{-z} \qquad Q(1, z) = e^{-z}$$

$$\Gamma(0, z) = -\text{Ei}(-z) + \frac{1}{2}\left(\log(-z) - \log\left(-\frac{1}{z}\right)\right) - \log(z) \qquad Q(0, z) = 0$$

$$\Gamma(-1, z) = \text{Ei}(-z) + \frac{1}{2}\left(\log\left(-\frac{1}{z}\right) - \log(-z)\right) + \log(z) + \frac{e^{-z}}{z} \qquad Q(-1, z) = 0$$

$$\Gamma\left(\frac{1}{2}, z\right) = \sqrt{\pi}\ \text{erfc}\left(\sqrt{z}\right) \qquad Q\left(\frac{1}{2}, z\right) = \text{erfc}\left(\sqrt{z}\right)$$

$$\Gamma\left(-\frac{1}{2}, z\right) = \frac{2 e^{-z}}{\sqrt{z}} - 2\sqrt{\pi}\ \text{erfc}\left(\sqrt{z}\right) \qquad Q\left(-\frac{1}{2}, z\right) = \text{erfc}\left(\sqrt{z}\right) - \frac{e^{-z}}{\sqrt{\pi}\sqrt{z}}.$$

These formulas are particular cases of the following general formulas:

$$\Gamma(n, z) = \frac{(-1)^{n-1}}{(-n)!}\left(\text{Ei}(-z) - \frac{1}{2}\left(\log(-z) - \log\left(-\frac{1}{z}\right)\right) + \log(z)\right) + e^{-z}\sum_{k=0}^{n-1}\frac{z^k}{(n)_{k-n+1}} - e^{-z}\sum_{k=n}^{-1}\frac{z^k}{(n)_{k-n+1}} \text{ /; } n \in \mathbb{Z}$$

$$\Gamma\left(n + \frac{1}{2}, z\right) = \text{erfc}\left(\sqrt{z}\right)\Gamma\left(n + \frac{1}{2}\right) + e^{-z}\sum_{k=0}^{n-1}\frac{z^{k+\frac{1}{2}}}{\left(n + \frac{1}{2}\right)_{k-n+1}} - e^{-z}\sum_{k=n}^{-1}\frac{z^{k+\frac{1}{2}}}{\left(n + \frac{1}{2}\right)_{k-n+1}} \text{ /; } n \in \mathbb{Z}$$

$$Q(n, z) = e^{-z}\sum_{k=0}^{n-1}\frac{z^k}{k!} \text{ /; } n \in \mathbb{N}^+$$

$$Q(-n, z) = 0 \text{ /; } n \in \mathbb{N}$$

$$Q\left(n + \frac{1}{2}, z\right) = \text{erfc}\left(\sqrt{z}\right) + \frac{1}{\Gamma\left(n + \frac{1}{2}\right)}\left(e^{-z}\sum_{k=0}^{n-1}\frac{z^{k+\frac{1}{2}}}{\left(n + \frac{1}{2}\right)_{k-n+1}} - e^{-z}\sum_{k=n}^{-1}\frac{z^{k+\frac{1}{2}}}{\left(n + \frac{1}{2}\right)_{k-n+1}}\right) \text{ /; } n \in \mathbb{Z}.$$

If the argument $z > 0$, the log-gamma function $\log\Gamma(z)$ can be evaluated at these points where the gamma function can be evaluated in closed form. The log-gamma function $\log\Gamma(z)$ can also be represented recursively in terms of $\Gamma(z)$ for $0 < \text{Re}(z) < 1$:

$$\log\Gamma(1) = 0$$

$$\log\Gamma(n) = \log((n-1)!) \text{ /; } n \in \mathbb{N}^+$$

$$\log\Gamma\left(\frac{n}{2}\right) = \log\left(\frac{2^{1-n}\sqrt{\pi}\ (n-1)!}{\frac{n-1}{2}!}\right) \text{ /; } n \in \mathbb{N}$$

$$\log\Gamma(-n) = \infty \text{ /; } n \in \mathbb{N}$$

$$\log\Gamma\left(\frac{p}{q} + n\right) = \log\left(\Gamma\left(\frac{p}{q}\right)\right) - n\log(q) + \sum_{k=1}^{n}\log(p + k\,q - q) \text{ /; } n \in \mathbb{N} \land p \in \mathbb{N}^+ \land q \in \mathbb{N}^+ \land p < q$$

$$\log\Gamma\left(\frac{p}{q} - n\right) = \log\left(\Gamma\left(\frac{p}{q}\right)\right) + \log(q)\,n - \pi\,i\,n - \sum_{k=1}^{n}\log(q\,k - p) \text{ /; } n \in \mathbb{N} \land p \in \mathbb{N}^+ \land q \in \mathbb{N}^+ \land p < q.$$

The generalized incomplete gamma functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ in particular cases can be represented through incomplete gamma functions $\Gamma(a, z)$ and $Q(a, z)$ and the gamma function $\Gamma(a)$:

$$\Gamma(a, z_1, 0) = \Gamma(a, z_1) - \Gamma(a) \text{ /; } \text{Re}(a) > 0$$

$\Gamma(a, 0, z_2) = \Gamma(a) - \Gamma(a, z_2) /; \operatorname{Re}(a) > 0$

$\Gamma(a, z_1, \infty) = \Gamma(a, z_1)$

$\Gamma(a, \infty, z_2) = -\Gamma(a, z_2)$

$\Gamma(a, 0, \infty) = \Gamma(a) /; \operatorname{Re}(a) > 0$

$Q(-n, z_1, z_2) = 0 /; n \in \mathbb{N}$

$Q(a, z_1, \infty) = Q(a, z_1)$

$Q(a, 0, \infty) = 1 /; \operatorname{Re}(a) > 0$

$Q(a, z_1, 0) = Q(a, z_1) - 1 /; \operatorname{Re}(a) > 0$

$Q(a, 0, z_2) = 1 - Q(a, z_2) /; \operatorname{Re}(a) > 0.$

The inverse of the regularized incomplete gamma functions $Q^{-1}(a, z)$ and $Q^{-1}(a, z_1, z_2)$ for particular values of arguments satisfy the following relations:

$Q^{-1}(a, 0) = \infty /; a > 0$

$Q^{-1}(a, 1) = 0 /; a > 0$

$Q^{-1}(a, \infty, z) = Q^{-1}(a, -z).$

Analyticity

The gamma functions $\Gamma(z)$, $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and $\log\Gamma(z)$ are defined for all complex values of their arguments.

The functions $\Gamma(a, z)$ and $Q(a, z)$ are analytic functions of a and z over the whole complex a- and z-planes excluding the branch cut on the z-plane. For fixed z, they are entire functions of a. The functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ are analytic functions of a, z_1, and z_2 over the whole complex a-, z_1-, and z_2-planes excluding the branch cuts on the z_1- and z_2-planes. For fixed z_1 and z_2, they are entire functions of a.

The function $\log\Gamma(z)$ is an analytical function of z over the whole complex z-plane excluding the branch cut.

Poles and essential singularities

For fixed a, the functions $\Gamma(a, z)$ and $Q(a, z)$ have an essential singularity at $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is a branch point for generic a. For fixed z, the functions $\Gamma(a, z)$ and $Q(a, z)$ have only one singular point at $a = \tilde{\infty}$. It is an essential singularity.

For fixed a, the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ have an essential singularity at $z_1 = \tilde{\infty}$ (for fixed z_2) and at $z_2 = \tilde{\infty}$ (for fixed z_1). At the same time, the points $z_k = \tilde{\infty} /; k = 1, 2$ are branch points for generic a. For fixed z_1 and z_2, the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ have only one singular point at $a = \tilde{\infty}$. It is an essential singularity.

The function $\log\Gamma(z)$ does not have poles or essential singularities.

Branch points and branch cuts

For fixed a, not a positive integer, the functions $\Gamma(a, z)$ and $Q(a, z)$ have two branch points: $z == 0$ and $z == \tilde{\infty}$.

For fixed a, not a positive integer, the functions $\Gamma(a, z)$ and $Q(a, z)$ are single-valued functions on the z-plane cut along the interval $(-\infty, 0)$, where they are continuous from above:

$$\lim_{\epsilon \to +0} \Gamma(a, x + i\epsilon) == \Gamma(a, x) \;/;\; x < 0$$

$$\lim_{\epsilon \to +0} \Gamma(a, x - i\epsilon) == \Gamma(a) - e^{-2 i \pi a} (\Gamma(a) - \Gamma(a, x)) \;/;\; x < 0$$

$$\lim_{\epsilon \to +0} Q(a, x + i\epsilon) == Q(a, x) \;/;\; x < 0$$

$$\lim_{\epsilon \to +0} Q(a, x - i\epsilon) == 1 - e^{-2 i \pi a} (1 - Q(a, x)) \;/;\; x < 0.$$

For fixed z, the functions $\Gamma(a, z)$ and $Q(a, z)$ do not have branch points and branch cuts.

For fixed a, z_1 or fixed a, z_2 (with $a \notin \mathbb{N}^+$), the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ have two branch points with respect to z_2 or z_1: $z_k == 0$, $z_k == \tilde{\infty}$, $k = 1, 2$.

For fixed z_1 and $a \notin \mathbb{N}^+$, the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ are single-valued functions on the z_2-plane cut along the interval $(-\infty, 0)$, where they are continuous from above:

$$\lim_{\epsilon \to +0} \Gamma(a, z_1, x_2 + i\epsilon) == \Gamma(a, z_1, x_2) \;/;\; x_2 < 0$$

$$\lim_{\epsilon \to +0} \Gamma(a, z_1, x_2 - i\epsilon) == \Gamma(a, z_1, x_2) + \left(1 - e^{-2 i \pi a}\right) \Gamma(a, x_2, 0) \;/;\; x_2 < 0$$

$$\lim_{\epsilon \to +0} Q(a, z_1, x_2 + i\epsilon) == Q(a, z_1, x_2) \;/;\; x_2 < 0$$

$$\lim_{\epsilon \to +0} Q(a, z_1, x_2 - i\epsilon) == Q(a, z_1, x_2) + \left(1 - e^{-2 i \pi a}\right) Q(a, x_2, 0) \;/;\; x_2 < 0.$$

For fixed z_2 and $a \notin \mathbb{N}^+$, the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ are single-valued functions on the z_1-plane cut along the interval $(-\infty, 0)$, where they are continuous from above:

$$\lim_{\epsilon \to +0} \Gamma(a, x_1 + i\epsilon, z_2) == \Gamma(a, x_1, z_2) \;/;\; x_1 < 0$$

$$\lim_{\epsilon \to +0} \Gamma(a, x_1 - i\epsilon, z_2) == \left(1 - e^{-2 i \pi a}\right) \Gamma(a, 0, x_1) + \Gamma(a, x_1, z_2) \;/;\; x_1 < 0$$

$$\lim_{\epsilon \to +0} Q(a, x_1 + i\epsilon, z_2) == Q(a, x_1, z_2) \;/;\; x_1 < 0$$

$$\lim_{\epsilon \to +0} Q(a, x_1 - i\epsilon, z_2) == \left(1 - e^{-2 i \pi a}\right) Q(a, 0, x_1) + Q(a, x_1, z_2) \;/;\; x_1 < 0.$$

For fixed z_1 and z_2, the functions $\Gamma(a, z_1, z_2)$ and $Q(a, z_1, z_2)$ do not have branch points and branch cuts.

The function $\log\Gamma(z)$ has two branch points: $z == 0$ and $z = \tilde{\infty}$.

The function $\log\Gamma(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above:

$$\lim_{\epsilon \to +0} \log\Gamma(x + i\,\epsilon) = \log\Gamma(x) \;/;\; x < 0$$

$$\lim_{\epsilon \to +0} \log\Gamma(x - i\,\epsilon) = \log\Gamma(x) - 2\,i\,\pi\,\lfloor x \rfloor \;/;\; x < 0.$$

Periodicity

The gamma functions $\Gamma(z)$, $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, the log-gamma function $\log\Gamma(z)$, and their inverses $Q^{-1}(a, z)$ and $Q^{-1}(a, z_1, z_2)$ do not have periodicity.

Parity and symmetry

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and the log-gamma function $\log\Gamma(z)$ have mirror symmetry (except on the branch cut intervals):

$$\Gamma(\overline{a}, \overline{z}) = \overline{\Gamma(a, z)} \;/;\; z \notin (-\infty, 0)$$

$$\Gamma(\overline{a}, \overline{z_1}, \overline{z_2}) = \overline{\Gamma(a, z_1, z_2)} \;/;\; z_1 \notin (-\infty, 0) \bigwedge z_2 \notin (-\infty, 0)$$

$$Q(\overline{a}, \overline{z}) = \overline{Q(a, z)} \;/;\; z \notin (-\infty, 0)$$

$$Q(\overline{a}, \overline{z_1}, \overline{z_2}) = \overline{Q(a, z_1, z_2)} \;/;\; z_1 \notin (-\infty, 0) \bigwedge z_2 \notin (-\infty, 0)$$

$$\log\Gamma(\overline{z}) = \overline{\log\Gamma(z)} \;/;\; z \notin (-\infty, 0).$$

Two of the gamma functions have the following permutation symmetry:

$$\Gamma(a, z_1, z_2) = -\Gamma(a, z_2, z_1)$$

$$Q(a, z_1, z_2) = -Q(a, z_2, z_1).$$

Series representations

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, the log-gamma function $\log\Gamma(z)$, and the inverse $Q^{-1}(a, z)$ have the following series expansions:

$$\Gamma(a, z) \propto \Gamma(a) - \frac{z^a}{a}\left(1 - \frac{a\,z}{a+1} + \frac{a\,z^2}{2\,(a+2)} - \ldots\right) \;/;\; (z \to 0)$$

$$\Gamma(a, z) = \Gamma(a) - z^a \sum_{k=0}^{\infty} \frac{(-z)^k}{(a+k)\,k!}$$

$$\Gamma(n, z) = (n-1)!\,e^{-z} \sum_{k=0}^{n-1} \frac{z^k}{k!} \;/;\; n \in \mathbb{N}^+$$

$$\Gamma(-n, z) = \frac{(-1)^n}{n!}\,(\psi(n+1) - \log(z)) - z^{-n} \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \frac{(-z)^k}{(k-n)\,k!} \;/;\; n \in \mathbb{N}$$

$$\Gamma(a, z_1, z_2) \propto z_2^a \left(\frac{1}{a} - \frac{z_2}{a+1} + \frac{z_2^2}{2\,(a+2)} + \ldots\right) - z_1^a \left(\frac{1}{a} - \frac{z_1}{a+1} + \frac{z_1^2}{2\,(a+2)} + \ldots\right) /; (z_1 \to 0) \bigwedge (z_2 \to 0)$$

$$\Gamma(a, z_1, z_2) = z_2^a \sum_{k=0}^{\infty} \frac{(-z_2)^k}{(a+k)\,k!} - z_1^a \sum_{k=0}^{\infty} \frac{(-z_1)^k}{(a+k)\,k!}$$

$$\Gamma(n, z_1, z_2) = (n-1)! \left(e^{-z_1} \sum_{k=0}^{n-1} \frac{z_1^k}{k!} - e^{-z_2} \sum_{k=0}^{n-1} \frac{z_2^k}{k!}\right) /; n \in \mathbb{N}^+$$

$$\Gamma(-n, z_1, z_2) = \frac{(-1)^{n-1}}{n!} (\log(z_1) - \log(z_2)) + \sum_{\substack{k=0 \\ k\neq n}}^{\infty} \frac{(-1)^k \left(z_2^{k-n} - z_1^{k-n}\right)}{(k-n)\,k!} /; n \in \mathbb{N}$$

$$Q(a, z) \propto 1 - z^a \left(\frac{1}{\Gamma(a+1)} - \frac{a\,z}{\Gamma(a+2)} + \frac{a\,(a+1)\,z^2}{2\,\Gamma(a+3)} - \ldots\right) /; (z \to 0)$$

$$Q(a, z) = 1 - z^a \sum_{k=0}^{\infty} \frac{(a)_k\,(-z)^k}{\Gamma(a+k+1)\,k!}$$

$$Q(n, z) = e^{-z} \sum_{k=0}^{n-1} \frac{z^k}{k!} /; n \in \mathbb{N}^+$$

$$Q(a, z_1, z_2) \propto z_2^a \left(\frac{1}{\Gamma(a+1)} - \frac{a\,z_2}{\Gamma(a+2)} + \frac{a\,(a+1)\,z_2^2}{2\,\Gamma(a+3)} - \ldots\right) - z_1^a \left(\frac{1}{\Gamma(a+1)} - \frac{a\,z_1}{\Gamma(a+2)} + \frac{a\,(a+1)\,z_1^2}{2\,\Gamma(a+3)} - \ldots\right) /; (z_1 \to 0) \bigwedge (z_2 \to 0)$$

$$Q(a, z_1, z_2) = z_2^a \sum_{k=0}^{\infty} \frac{(a)_k\,(-z_2)^k}{\Gamma(a+k+1)\,k!} - z_1^a \sum_{k=0}^{\infty} \frac{(a)_k\,(-z_1)^k}{\Gamma(a+k+1)\,k!}$$

$$Q(n, z_1, z_2) = e^{-z_1} \sum_{k=0}^{n-1} \frac{z_1^k}{k!} - e^{-z_2} \sum_{k=0}^{n-1} \frac{z_2^k}{k!} /; n \in \mathbb{N}^+$$

$$\log\Gamma(z) \propto -\log(z) - \gamma\,z + \frac{\pi^2\,z^2}{12} - \frac{\zeta(3)\,z^3}{3} + \frac{\pi^4\,z^4}{360} - \ldots /; (z \to 0)$$

$$\log\Gamma(z) = -\log(z) - \gamma\,z + \sum_{j=0}^{\infty} \frac{(-1)^j\,\zeta(j+2)\,z^{j+2}}{j+2} /; |z| < 1$$

$$\log\Gamma(z) \propto \log\Gamma(z_0) + \psi(z_0)\,(z-z_0) + \frac{\zeta(2, z_0)}{2}\,(z-z_0)^2 - \frac{\zeta(3, z_0)}{3}\,(z-z_0)^3 + \ldots /; (z \to z_0) \bigwedge \neg (z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

$$\log\Gamma(z) = \log\Gamma(z_0) + \psi(z_0)\,(z-z_0) + \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j\,(z-z_0)^{j+2}}{(j+2)\,(k+z_0)^{j+2}} /; \neg (z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

$$\log\Gamma(z) = -\log(z+n) + \log\Gamma\,(z+n+1) - \sum_{k=0}^{n-1} \log(z+k) /; (z \to -n) \land n \in \mathbb{N}$$

$$\log\Gamma(z) \propto -\log(z+n) - \sum_{k=0}^{n-1} \log(z+k)\,(1+O(z+n)) \;/;\; (z \to -n) \land n \in \mathbb{N}$$

$$Q^{-1}(a, z) = (-(z-1)\,\Gamma(a+1))^{1/a} + \frac{\left((-(z-1)\,\Gamma(a+1))^{1/a}\right)^2}{a+1} + \frac{(3\,a+5)\left((-(z-1)\,\Gamma(a+1))^{1/a}\right)^3}{2\,(a+1)^2\,(a+2)} + O\left((z-1)^{4/a}\right).$$

Asymptotic series expansions

The asymptotic behavior of the gamma functions $\Gamma(a, z)$ and $Q(a, z)$, the log-gamma function $\log\Gamma(z)$, and the inverse $Q^{-1}(a, z)$ can be described by the following formulas (only the main terms of asymptotic expansion are given):

$$\Gamma(a, z) \propto e^{-z}\, z^{a-1} \left(1 + O\left(\frac{1}{z}\right)\right) /;\; (|z| \to \infty)$$

$$Q(a, z) \propto \frac{e^{-z}\, z^{a-1}}{\Gamma(a)} \left(1 + O\left(\frac{1}{z}\right)\right) /;\; (|z| \to \infty)$$

$$\log\Gamma(z) \propto \left(z - \frac{1}{2}\right)\log(z) - z + \frac{\log(2\,\pi)}{2} + \frac{1}{12\,z}\left(1 + O\left(\frac{1}{z^2}\right)\right) /;\; |\mathrm{Arg}(z)| < \pi \bigwedge (|z| \to \infty)$$

$$Q^{-1}(a, z) \propto -(a-1)\, W_{-1}\left(-\frac{z^{\frac{1}{a-1}}\,\Gamma(a)^{\frac{1}{a-1}}}{a-1}\right) /;\; (z \to 0).$$

Integral representations

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and the log-gamma function $\log\Gamma(z)$ can also be represented through the following integrals:

$$\Gamma(a, z) = \int_z^\infty t^{a-1}\, e^{-t}\, dt$$

$$\Gamma(a, z_1, z_2) = \int_{z_1}^{z_2} t^{a-1}\, e^{-t}\, dt$$

$$Q(a, z) = \frac{1}{\Gamma(a)} \int_z^\infty t^{a-1}\, e^{-t}\, dt$$

$$Q(a, z_1, z_2) = \frac{1}{\Gamma(a)} \int_{z_1}^{z_2} t^{a-1}\, e^{-t}\, dt$$

$$\log\Gamma(z) = -\int_0^\infty \frac{e^{-t}}{t}\left(\frac{e^{t\,z}-1}{1-e^{-t}} - z\right) dt + \log(\pi) - \log(\sin(\pi\, z)) /;\; \mathrm{Re}(z) < 1$$

$$\log\Gamma(z) = \int_0^\infty \frac{1}{t}\left((z-1)\, e^{-t} + \frac{e^{-t\,z} - e^{-t}}{1-e^{-t}}\right) dt /;\; \mathrm{Re}(z) > 0$$

$$\log\Gamma(z) = 2\int_0^\infty \frac{\tan^{-1}\left(\frac{t}{z}\right)}{e^{2\,\pi\,t}-1}\, dt + \frac{\log(2\,\pi)}{2} + \left(z - \frac{1}{2}\right)\log(z) - z /;\; \mathrm{Re}(z) > 0.$$

Transformations

The argument of the log-gamma function $\log\Gamma(a-z)$ can be simplified if $a == 1$ or 0:

$$\log\Gamma(1-z) == \log(\pi) - \log(\sin(\pi z)) - \log\Gamma(z) \;/;\; -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{\pi}{2}$$

$$\log\Gamma(1-z) == \log(\pi) - \log(\sin(\pi z)) - \log\Gamma(z) + 2 i \pi \left\lfloor \frac{2\,\mathrm{Re}(z)+1}{4} \right\rfloor \left(\mathrm{sgn}(\mathrm{Im}(z)) + \left(\mathrm{sgn}(\mathrm{Im}(z))^2 - 1\right) \mathrm{sgn}(\mathrm{Re}(z))\right) /;\; \frac{2\,\mathrm{Re}(z)+1}{4} \notin \mathbb{Z}$$

$$\log\Gamma(-z) == \log(\pi) - \log(-z) - \log(\sin(\pi z)) - \log\Gamma(z) \;/;\; -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{\pi}{2}$$

$$\log\Gamma(-z) == \log(\pi) - \log(-z) - \log(\sin(\pi z)) - \log\Gamma(z) + 2 i \pi \left\lfloor \frac{2\,\mathrm{Re}(z)+1}{4} \right\rfloor \left(\mathrm{sgn}(\mathrm{Im}(z)) + \left(\mathrm{sgn}(\mathrm{Im}(z))^2 - 1\right) \mathrm{sgn}(\mathrm{Re}(z))\right) /;\; \frac{2\,\mathrm{Re}(z)+1}{4} \notin \mathbb{Z}.$$

Multiple arguments

The log-gamma function $\log\Gamma(m z)$ with $m == 2, 3, \ldots$ can be represented by a formula that follows from the corresponding multiplication formula for the gamma function $\Gamma(z)$:

$$\log\Gamma(2 z) == \log\Gamma\left(z + \frac{1}{2}\right) + \log\Gamma(z) + (2 z - 1) \log(2) - \frac{\log(\pi)}{2}$$

$$\log\Gamma(m z) == \sum_{k=0}^{m-1} \log\Gamma\left(z + \frac{k}{m}\right) + m z \log(m) - \frac{1}{2} (\log(m) + (m-1) \log(2 \pi)) \;/;\; m \in \mathbb{N}^+.$$

Identities

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and the log-gamma function $\log\Gamma(z)$ satisfy the following recurrence identities:

$$\Gamma(a, z) == \frac{1}{a} \left(\Gamma(a+1, z) - e^{-z} z^a\right)$$

$$\Gamma(a, z) == (a-1)\,\Gamma(a-1, z) + e^{-z} z^{a-1}$$

$$\Gamma(a, z_1, z_2) == \frac{1}{a} \left(\Gamma(a+1, z_1, z_2) - e^{-z_1} z_1^a + e^{-z_2} z_2^a\right)$$

$$\Gamma(a, z_1, z_2) == (a-1)\,\Gamma(a-1, z_1, z_2) + e^{-z_1} z_1^{a-1} - z_2^{a-1} e^{-z_2}$$

$$Q(a, z) == Q(a+1, z) - \frac{e^{-z} z^a}{\Gamma(a+1)}$$

$$Q(a, z) == Q(a-1, z) + \frac{e^{-z} z^{a-1}}{\Gamma(a)}$$

$$Q(a, z_1, z_2) == Q(a+1, z_1, z_2) + \frac{e^{-z_2} z_2^a - e^{-z_1} z_1^a}{\Gamma(a+1)}$$

$$Q(a, z_1, z_2) = Q(a-1, z_1, z_2) + \frac{z_1^{a-1} e^{-z_1} - z_2^{a-1} e^{-z_2}}{\Gamma(a)}$$

$$\log\Gamma(z) = \log\Gamma(z+1) - \log(z)$$

$$\log\Gamma(z) = \log\Gamma(z-1) + \log(z-1).$$

The previous formulas can be generalized to the following recurrence identities with a jump of length n:

$$\Gamma(a, z) = \frac{1}{(a)_n} \Gamma(a+n, z) - z^{a-1} e^{-z} \sum_{k=1}^{n} \frac{z^k}{(a)_k} \text{ /; } n \in \mathbb{N}$$

$$\Gamma(a, z) = (-1)^n (1-a)_n \left(\Gamma(a-n, z) + z^{a-n-1} e^{-z} \sum_{k=1}^{n} \frac{z^k}{(a-n)_k} \right) \text{ /; } n \in \mathbb{N}$$

$$\Gamma(a, z_1, z_2) = \frac{1}{(a)_n} \Gamma(a+n, z_1, z_2) - e^{-z_1} \sum_{k=1}^{n} \frac{z_1^{a+k-1}}{(a)_k} + e^{-z_2} \sum_{k=1}^{n} \frac{z_2^{a+k-1}}{(a)_k} \text{ /; } n \in \mathbb{N}$$

$$\Gamma(a, z_1, z_2) = (-1)^n (1-a)_n \left(\Gamma(a-n, z_1, z_2) + e^{-z_1} \sum_{k=1}^{n} \frac{z_1^{a+k-n-1}}{(a-n)_k} - e^{-z_2} \sum_{k=1}^{n} \frac{z_2^{a+k-n-1}}{(a-n)_k} \right) \text{ /; } n \in \mathbb{N}$$

$$Q(a, z) = Q(a+n, z) - z^{a-1} e^{-z} \sum_{k=1}^{n} \frac{z^k}{\Gamma(a+k)} \text{ /; } n \in \mathbb{N}$$

$$Q(a, z) = Q(a-n, z) + z^{a-1} e^{-z} \sum_{k=0}^{n-1} \frac{z^{-k}}{\Gamma(a-k)} \text{ /; } n \in \mathbb{N}$$

$$Q(a, z_1, z_2) = Q(a+n, z_1, z_2) - e^{-z_1} \sum_{k=1}^{n} \frac{z_1^{a+k-1}}{\Gamma(a+k)} + e^{-z_2} \sum_{k=1}^{n} \frac{z_2^{a+k-1}}{\Gamma(a+k)} \text{ /; } n \in \mathbb{N}$$

$$Q(a, z_1, z_2) = Q(a-n, z_1, z_2) + e^{-z_1} \sum_{k=0}^{n-1} \frac{z_1^{a-k-1}}{\Gamma(a-k)} - e^{-z_2} \sum_{k=0}^{n-1} \frac{z_2^{a-k-1}}{\Gamma(a-k)} \text{ /; } n \in \mathbb{N}$$

$$\log\Gamma(z) = \log\Gamma(z+n) - \sum_{k=0}^{n-1} \log(z+k) \text{ /; } n \in \mathbb{N}$$

$$\log\Gamma(z) = \log\Gamma(z-n) + \sum_{k=1}^{n} \log(z-k) \text{ /; } n \in \mathbb{N}.$$

Representations of derivatives

The derivatives of the gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ with respect to the variables z, z_1, and z_2 have simple representations in terms of elementary functions:

$$\frac{\partial \Gamma(a, z)}{\partial z} = -e^{-z} z^{a-1}$$

$$\frac{\partial \Gamma(a, z_1, z_2)}{\partial z_1} = -e^{-z_1} z_1^{a-1}$$

$$\frac{\partial \Gamma(a, z_1, z_2)}{\partial z_2} = e^{-z_2} z_2^{a-1}$$

$$\frac{\partial Q(a, z)}{\partial z} = -\frac{e^{-z} z^{a-1}}{\Gamma(a)}$$

$$\frac{\partial Q(a, z_1, z_2)}{\partial z_1} = -\frac{e^{-z_1} z_1^{a-1}}{\Gamma(a)}$$

$$\frac{\partial Q(a, z_1, z_2)}{\partial z_2} = \frac{e^{-z_2} z_2^{a-1}}{\Gamma(a)}.$$

The derivatives of the log-gamma function $\log\Gamma(z)$ and the inverses of the regularized incomplete gamma functions $Q^{-1}(a, z)$, and $Q^{-1}(a, z_1, z_2)$ with respect to the variables z, z_1, and z_2 have more complicated representations by the formulas:

$$\frac{\partial \log\Gamma(z)}{\partial z} = \psi(z)$$

$$\frac{\partial Q^{-1}(a, z)}{\partial z} = -e^{Q^{-1}(a,z)} Q^{-1}(a, z)^{1-a} \Gamma(a)$$

$$\frac{\partial Q^{-1}(a, z_1, z_2)}{\partial z_1} = e^{Q^{-1}(a,z_1,z_2)-z_1} \left(\frac{Q^{-1}(a, z_1, z_2)}{z_1}\right)^{1-a}$$

$$\frac{\partial Q^{-1}(a, z_1, z_2)}{\partial z_2} = e^{Q^{-1}(a,z_1,z_2)} \Gamma(a) Q^{-1}(a, z_1, z_2)^{1-a}.$$

The derivative of the exponential integral $E_\nu(z)$ by its parameter ν can be represented in terms of the regularized hypergeometric function ${}_2\tilde{F}_2$:

$$\frac{\partial E_\nu(z)}{\partial \nu} = z^{\nu-1} \Gamma(1-\nu) (\log(z) - \psi(1-\nu)) - \Gamma(1-\nu)^2 \, {}_2\tilde{F}_2(1-\nu, 1-\nu; 2-\nu, 2-\nu; -z).$$

The derivatives of the gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$, and their inverses $Q^{-1}(a, z)$ and $Q^{-1}(a, z_1, z_2)$ with respect to the parameter a can be represented in terms of the regularized hypergeometric function ${}_2\tilde{F}_2$:

$$\frac{\partial \Gamma(a, z)}{\partial a} = \Gamma(a)^2 z^a \, {}_2\tilde{F}_2(a, a; a+1, a+1; -z) - \Gamma(a, 0, z) \log(z) + \Gamma(a) \psi(a)$$

$$\frac{\partial \Gamma(a, z_1, z_2)}{\partial a} = \Gamma(a)^2 \, {}_2\tilde{F}_2(a, a; a+1, a+1; -z_1) z_1^a - \Gamma(a)^2 \, {}_2\tilde{F}_2(a, a; a+1, a+1; -z_2) z_2^a - \Gamma(a, 0, z_1) \log(z_1) + \Gamma(a, 0, z_2) \log(z_2)$$

$$\frac{\partial Q(a,z)}{\partial a} == \Gamma(a)\, z^a\, {}_2\tilde{F}_2(a, a; a+1, a+1; -z) + Q(a, z, 0)\, (\log(z) - \psi(a))$$

$$\frac{\partial Q(a,z_1,z_2)}{\partial a} == \Gamma(a)\, z_1^a\, {}_2\tilde{F}_2(a, a; a+1, a+1; -z_1) - \Gamma(a)\, z_2^a\, {}_2\tilde{F}_2(a, a; a+1, a+1; -z_2) + Q(a, z_1, 0)\log(z_1) - Q(a, z_2, 0)\log(z_2) - \psi(a)\, Q(a, z_1, z_2)$$

$$\frac{\partial Q^{-1}(a,z)}{\partial a} == e^w\, w^{1-a} \left(\Gamma(a)^2\, {}_2\tilde{F}_2(a, a; a+1, a+1; -w)\, w^a + (z-1)\,\Gamma(a)\log(w) + (\Gamma(a) - \Gamma(a, w))\,\psi(a)\right) /;\ w == Q^{-1}(a, z)$$

$$\frac{\partial Q^{-1}(a,z_1,z_2)}{\partial a} == e^w\, w^{1-a} \left(\frac{1}{a^2}\left(w^a\, {}_2F_2(a, a; a+1, a+1; -w) - z_1^a\, {}_2F_2(a, a; a+1, a+1; -z_1)\right) + \Gamma(a, w, 0)\log(w) + \Gamma(a, 0, z_1)\log(z_1) + \Gamma(a, z_1, w)\,\psi(a)\right) /;\ w == Q^{-1}(a, z_1, z_2).$$

The symbolic n^{th}-order derivatives of all gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, $Q(a, z_1, z_2)$, and their inverses $Q^{-1}(a, z)$, and $Q^{-1}(a, z_1, z_2)$ have the following representations:

$$\frac{\partial^n \Gamma(a,z)}{\partial z^n} == z^{-n} \sum_{k=0}^{n} (-1)^n \binom{n}{k} (-a)_k\, \Gamma(a-k+n, z) /;\ n \in \mathbb{N}$$

$$\frac{\partial^n \Gamma(a,z)}{\partial a^n} == \Gamma^{(n)}(a) - z^a \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (n-j)!\, \Gamma(a)^{n-j+1} \log^j(z)\, {}_{n-j+1}\tilde{F}_{n-j+1}\left(a_1, a_2, \ldots, a_{n-j+1}; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; -z\right) /;\ a_1 == a_2 == \ldots == a_{n+1} == a \bigwedge n \in \mathbb{N}$$

$$\frac{\partial^n \Gamma(a,z_1,z_2)}{\partial z_1^n} == z_1^{-n} \sum_{k=0}^{n} (-1)^n \binom{n}{k} (-a)_k\, \Gamma(a-k+n, z_1) /;\ n \in \mathbb{N}^+$$

$$\frac{\partial^n \Gamma(a,z_1,z_2)}{\partial z_2^n} == -z_2^{-n} \sum_{k=0}^{n} (-1)^n \binom{n}{k} (-a)_k\, \Gamma(a-k+n, z_2) /;\ n \in \mathbb{N}^+$$

$$\frac{\partial^n \Gamma(a,z_1,z_2)}{\partial a^n} == z_2^a \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (n-j)!\, \Gamma(a)^{n-j+1} \log^j(z_2)\, {}_{n-j+1}\tilde{F}_{n-j+1}\left(a_1, a_2, \ldots, a_{n-j+1}; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; -z_2\right) - z_1^a \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (n-j)!\, \Gamma(a)^{n-j+1} \log^j(z_1)\, {}_{n-j+1}\tilde{F}_{n-j+1}\left(a_1, a_2, \ldots, a_{n-j+1}; a_1+1, a_2+1, \ldots, a_{n-j+1}+1; -z_1\right) /;\ a_1 == a_2 == \ldots == a_{n+1} == a \bigwedge n \in \mathbb{N}$$

$$\frac{\partial^n Q(a,z)}{\partial z^n} == -a\, z^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1-a-k)_{n-1}\, Q(a+k, z) /;\ n \in \mathbb{N}$$

$$\frac{\partial^n Q(a,z)}{\partial a^n} = \frac{\Gamma^{(n)}(a)}{\Gamma(a)} - \frac{1}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(-1)^{n-k}\, \Gamma(n+1, -(a+k)\log(z))}{(a+k)^{n+1}\, k!} \;/;\; n \in \mathbb{N}$$

$$\frac{\partial^n Q(a, z_1, z_2)}{\partial z_1^n} = -a\, z_1^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1-a-k)_{n-1}\, Q(a+k, z_1) \;/;\; n \in \mathbb{N}^+$$

$$\frac{\partial^n Q(a, z_1, z_2)}{\partial z_2^n} = a\, z_2^{-n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1-a-k)_{n-1}\, Q(a+k, z_2) \;/;\; n \in \mathbb{N}^+$$

$$\frac{\partial^n Q(a, z_1, z_2)}{\partial a^n} = n! \sum_{k=0}^{n} \left(z_2^a \sum_{i=0}^{n-k} (-1)^{n-i-k} \binom{n-k}{i} (n-i-k)!\, \Gamma(a)^{n-i-k+1} \log^i(z_2)\; {}_{n-k-i+1}\tilde{F}_{n-k-i+1}(a_1, a_2, \ldots, a_{n-k-i+1}; a_1+1, a_2+1, \ldots, a_{n-k-i+1}+1; -z_2) - z_1^a \sum_{i=0}^{n-k} (-1)^{n-i-k} \binom{n-k}{i} (n-i-k)!\, \Gamma(a)^{n-i-k+1} \log^i(z_1)\; {}_{n-k-i+1}\tilde{F}_{n-k-i+1}(a_1, a_2, \ldots, a_{n-k-i+1}; a_1+1, a_2+1, \ldots, a_{n-k-i+1}+1; -z_1) \right) \sum_{j=0}^{k} \frac{(-1)^j (k+1)\, \Gamma(a)^{-j-1}}{(j+1)!\,(n-k)!\,(k-j)!} \frac{\partial^k \Gamma(a)^j}{\partial a^k} \;/;\; a_1 = a_2 = \ldots = a_{n+1} = a \bigwedge n \in \mathbb{N}$$

$$\frac{\partial^\alpha \log\Gamma(z)}{\partial z^\alpha} = \psi^{(\alpha-1)}(z)$$

$$\frac{\partial^n Q^{-1}(a,z)}{\partial z^n} = w\,\delta_n + \left(-\frac{\Gamma(a)\, e^w}{w^{a-1}}\right)^n \sum_{j_2=0}^{n} \cdots \sum_{j_n=0}^{n} \delta_{\sum_{i=2}^n (i-1)\, j_i, n-1}\, (-1)^{\sum_{i=2}^n j_i} \left(n + \sum_{i=2}^{n} j_i - 1\right)! \prod_{i=2}^{n} \frac{1}{j_i!} \left(\frac{\Gamma(a+1)\, e^w\, w^{-a-i+1}}{i!}\right)^{j_i} \left(\sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} (-a-k+1)_{i-1}\, Q(a+k, w)\right)^{j_i} \;/;\; w = Q^{-1}(a,z) \bigwedge n \in \mathbb{N}$$

$$\frac{\partial^n Q^{-1}(a, z_1, z_2)}{\partial z_2^n} = w\,\delta_n + \grave{} \left(\frac{\Gamma(a)\, e^w}{w^{a-1}}\right)^n \sum_{j_2=0}^{n} \cdots \sum_{j_n=0}^{n} \delta_{\sum_{i=2}^n (i-1)\, j_i, n-1}\, (-1)^{\sum_{i=2}^n j_i} \left(n + \sum_{i=2}^{n} j_i - 1\right)! \prod_{i=2}^{n} \frac{1}{j_i!} \left(\frac{\Gamma(a+1)\, e^w\, w^{1-a}}{i!}\right)^{j_i} \left(a\, w^{-i} \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} (1-a-k)_{i-1}\, Q(a+k, w) + Q(a, z_1)\,\delta_i\right)^{j_i} \;/;\; w = Q^{-1}(a, z_1, z_2) \bigwedge n \in \mathbb{N}.$$

Differential equations

The gamma functions $\Gamma(a, z)$, $\Gamma(a, z_1, z_2)$, $Q(a, z)$, and $Q(a, z_1, z_2)$ satisfy the following second-order linear differential equations:

$$z\, w''(z) + (1-a+z)\, w'(z) = 0 \;/;\; w(z) = c_1\, \Gamma(a, z) + c_2$$

$$z_1\, w''(z_1) + (1-a+z_1)\, w'(z_1) = 0 \;/;\; w(z_1) = c_1\, \Gamma(a, z_1, z_2) + c_2$$

$$z_2\, w''(z_2) + (1-a+z_2)\, w'(z_2) = 0 \;/;\; w(z_2) = c_1\, \Gamma(a, z_1, z_2) + c_2$$

$$z\, w''(z) + (1 - a + z)\, w'(z) == 0 \;/;\; w(z) == c_1\, Q(a, z) + c_2$$

$$z_1\, w''(z_1) + (1 - a + z_1)\, w'(z_1) == 0 \;/;\; w(z_1) == c_1\, Q(a, z_1, z_2) + c_2$$

$$z_2\, w''(z_2) + (1 - a + z_2)\, w'(z_2) == 0 \;/;\; w(z_2) == c_1\, Q(a, z_1, z_2) + c_2,$$

where c_1 and c_2 are arbitrary constants.

The log-gamma function $\log\Gamma(z)$ satisfies the following simple first-order linear differential equation:

$$\frac{\partial w(z)}{\partial z} == \psi(z) \;/;\; w(z) == \log\Gamma(z).$$

The inverses of the regularized incomplete gamma functions $Q^{-1}(a, z)$ and $Q^{-1}(a, z_1, z_2)$ satisfy the following ordinary nonlinear second-order differential equation:

$$w(z)\, w''(z) - w'(z)^2\, (w(z) + 1 - a) == 0 \;/;\; w(z) == Q^{-1}(a, z)$$

$$w(z_2)\, w''(z_2) - w'(z_2)^2\, (-a + w(z_2) + 1) == 0 \;/;\; w(z_2) == Q^{-1}(a, z_1, z_2).$$

Applications of gamma functions

The gamma functions are used throughout mathematics, the exact sciences, and engineering. In particular, the incomplete gamma function is used in solid state physics and statistics, and the logarithm of the gamma function is used in discrete mathematics, number theory, and other fields of sciences.

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