Introductions to GoldenRatio

Introduction to the classical constants

General

Golden ratio

The division of a line segment whose total length is $a+b$ into two parts a and b where the ratio of $a+b$ to a is equal to the ratio a to b is known as the golden ratio. The two ratios are both approximately equal to 1.618..., which is called the golden ratio constant and usually notated by ϕ:

$$\frac{a+b}{a} == \frac{a}{b} == 1.618\ldots == \frac{1+\sqrt{5}}{2} == \phi.$$

The concept of golden ratio division appeared more than 2400 years ago as evidenced in art and architecture. It is possible that the magical golden ratio divisions of parts are rather closely associated with the notion of beauty in pleasing, harmonious proportions expressed in different areas of knowledge by biologists, artists, musicians, historians, architects, psychologists, scientists, and even mystics. For example, the Greek sculptor Phidias (490–430 BC) made the Parthenon statues in a way that seems to embody the golden ratio; Plato (427–347 BC), in his Timaeus, describes the five possible regular solids, known as the Platonic solids (the tetrahedron, cube, octahedron, dodecahedron, and icosahedron), some of which are related to the golden ratio.

The properties of the golden ratio were mentioned in the works of the ancient Greeks Pythagoras (c. 580–c. 500 BC) and Euclid (c. 325–c. 265 BC), the Italian mathematician Leonardo of Pisa (1170s or 1180s–1250), and the Renaissance astronomer J. Kepler (1571–1630). Specifically, in book VI of the *Elements*, Euclid gave the following definition of the golden ratio: "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less". Therein Euclid showed that the "mean and extreme ratio", the name used for the golden ratio until about the 18th century, is an irrational number.

In 1509 L. Pacioli published the book *De Divina Proportione*, which gave new impetus to the theory of the golden ratio; in particular, he illustrated the golden ratio as applied to human faces by artists, architects, scientists, and mystics. G. Cardano (1545) mentioned the golden ratio in his famous book *Ars Magna*, where he solved quadratic and cubic equations and was the first to explicitly make calculations with complex numbers. Later M. Mästlin (1597) evaluated $1/\phi$ approximately as $0.6180340\ldots$. J. Kepler (1608) showed that the ratios of Fibonacci numbers approximate the value of the golden ratio and described the golden ratio as a "precious jewel". R. Simson (1753) gave a simple limit representation of the golden ratio based on its very simple continued fraction $\phi == 1 + \frac{1}{1+\frac{1}{1+\ldots}}$. M. Ohm (1835) gave the first known use of the term "golden section", believed to have originated earlier in the century from an unknown source. J. Sulley (1875) first used the term "golden ratio" in English and G. Chrystal (1898) first used this term in a mathematical context.

The symbol ϕ (phi) for the notation of the golden ratio was suggested by American mathematician M. Barrwas in 1909. Phi is the first Greek letter in the name of the Greek sculptor Phidias.

Throughout history many people have tried to attribute some kind of magic or cult meaning as a valid description of nature and attempted to prove that the golden ratio was incorporated into different architecture and art objects (like the Great Pyramid, the Parthenon, old buildings, sculptures and pictures). But modern investigations (for example, G. Markowsky (1992), C. Falbo (2005), and A. Olariu (2007)) showed that these are mostly misconceptions: the differences between the golden ratio and real ratios of these objects in many cases reach 20–30% or more.

The golden ratio has many remarkable properties related to its quasi symmetry. It satisfies the quadratic equation $z^2 - z - 1 = 0$, which has solutions $z_1 == \phi$ and $z_2 == 1 - \phi$. The absolute value of the second solution is called the golden ratio conjugate, $\Phi == \phi - 1$. These ratios satisfy the following relations:

$$\phi - 1 == \frac{1}{\phi} \bigwedge \Phi + 1 == \frac{1}{\Phi}.$$

Applications of the golden ratio also include algebraic coding theory, linear sequential circuits, quasicrystals, phyllotaxis, biomathematics, and computer science.

Pi

The constant $\pi == 3.14159\ldots$ is the most frequently encountered classical constant in mathematics and the natural sciences. Initially it was defined as the ratio of the length of a circle's circumference to its diameter. Many further interpretations and applications in practically all fields of qualitative science followed. For instance, the following table illustrates how the constant π is applied to evaluate surface areas and volumes of some simple geometrical objects:

		$\mathbb{R}^3$ (surface area)	$\mathbb{R}^3$ (volume)	$\mathbb{R}^n$ (hyper surface area)	$\mathbb{R}^n$ (hypervolu
sphere (of radius r and diameter d)		$S == 4\pi r^2 == \pi d^2$	$V == \frac{4\pi}{3} r^3 == \frac{\pi}{6} d^3$	$S_{2k} == \frac{2\pi^k}{(k-1)!} r^{2k-1}$ $S_{2k+1} == \frac{\pi^k 2^{2k+1} k!}{(2k)!} r^{2k}$ $S_n == \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1}$	$V_{2k} == \frac{\pi^k}{k!} r$ $V_{2k+1} == \frac{\pi^k}{(}$ $V_n == \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)}$
ellipsoid (spheroid) (of semi-axes a, b, c, or r_j)		containing elliptic integrals	$V == \frac{4\pi}{3} a b c$	containing elliptic integrals	$V_{2k} == \frac{\pi^k}{k!} \Gamma$ $V_{2k+1} == \frac{\pi^k}{(}$ $V_n == \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)}$
cylinder (of height h and radius r)		$S == 2\pi r (r + h)$	$V == \pi r^2 h$	$S_{2k} == \frac{2^{2k} \pi^{k-1} k!}{(2k)!} r^{2k-2} ((2k-1) h + 2r)$ $S_{2k+1} == \frac{2\pi^k}{k!} r^{2k-1} (h k + r)$ $S_n == \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} r^{n-2}((n-1) h + 2r)$	$V_{2k} == \frac{2^{2k} \pi}{(2}$ $V_{2k+1} == \frac{\pi^k}{k!}$ $V_n == \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$
cone (of height h and radius r)		$S == \pi r \left(r + \sqrt{r^2 + h^2}\right)$	$V == \frac{\pi}{3} r^2 h$	$S_{2k} == \frac{4^k \pi^{k-1} k!}{(2k)!} \left(r + \sqrt{h^2 + r^2}\right) r^{2k-2}$ $S_{2k+1} == \frac{\pi^k \left(r + \sqrt{h^2 + r^2}\right)}{k!} r^{2k-1}$ $S_n == \frac{\pi^{\frac{n-1}{2}} r^{n-2} \left(r + \sqrt{h^2 + r^2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$	$V_{2k} == \frac{2^{2k-1}}{}$ $V_{2k+1} == \frac{}{(2}$ $V_n == \frac{\pi^{\frac{n-1}{2}} h}{n \Gamma(}$

	$\mathbb{R}$ (circumference)	$\mathbb{R}^2$ (surface area)
circle (of radius r and diameter d)	$c == 2\pi r == \pi d$	$S == \pi r^2 == \frac{\pi}{4} d^2$
ellipse (of semi-axes a, b)	containing elliptic integrals	$S == \pi a b$

Different approximations of π have been known since antiquity or before when people discovered some basic properties of circles. The design of Egyptian pyramids (c. 3000 BC) incorporated π as $22/7 == 3 + 1/7$ ($\sim 3.142857 \ldots$) in numerous places. The Egyptian scribe Ahmes (Middle Kingdom papyrus, c. 2000 BC) wrote the oldest known text to give an approximate value for π as $(16/9)^2 \sim 3.16045\ldots$. Babylonian mathematicians (19th century BC) were using an estimation of π as $25/8$, which is within 0.53% of the exact value. (China, c. 1200 BC) and the Biblical verse I Kings 7:23 (c. 971–852 BC) gave the estimation of π as 3. Archimedes (Greece, c. 240 BC) knew that $3 + 10/71 < \pi < 3 + 1/7$ and gave the estimation of π as $3.1418\ldots$. Aryabhata (India, 5th century) gave the approximation of π as 62832/20000, correct to four decimal places. Zu Chongzhi (China, 5th century) gave two approximations of π as 355/113 and 22/7 and restricted π between 3.1415926 and 3.1415927.

A reinvestigation of π began by building corresponding series and other calculus-related formulas for this constant. Simultaneously, scientists continued to evaluate π with greater and greater accuracy and proved different structural properties of π. Madhava of Sangamagrama (India, 1350–1425) found the infinite series expansion $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$ (currently named the Gregory-Leibniz series or Leibniz formula) and evaluated π with 11 correct digits. Ghyath ad-din Jamshid Kashani (Persia, 1424) evaluated π with 16 correct digits. F. Viete (1593) represented $2/\pi$ as the infinite product $\frac{2}{\pi} = \frac{\sqrt{2}}{2} \frac{\sqrt{2+\sqrt{2}}}{2} \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \ldots$. Ludolph van Ceulen (Germany, 1610) evaluated 35 decimal places of π. J. Wallis (1655) represented π as the infinite product $\frac{\pi}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \frac{6}{5} \frac{6}{7} \frac{8}{7} \frac{8}{9} \ldots$. J. Machin (England, 1706) developed a quickly converging series for π, based on the formula $\pi/4 = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$, and used it to evaluate 100 correct digits. W. Jones (1706) introduced the symbol π for notation of the Pi constant. L. Euler (1737) adopted the symbol π and made it standard. C. Goldbach (1742) also widely used the symbol π. J. H. Lambert (1761) established that π is an irrational number. J. Vega (Slovenia, 1789) improved J. Machin's 1706 formula and calculated 126 correct digits for π. W. Rutherford (1841) calculated 152 correct digits for π. After 20 years of hard work, W. Shanks (1873) presented 707 digits for π, but only 527 digits were correct (as D. F. Ferguson found in 1947). F. Lindemann (1882) proved that π is transcendental. F. C. W. Stormer (1896) derived the formula $\frac{\pi}{4} = 44\tan^{-1}\left(\frac{1}{57}\right) + 7\tan^{-1}\left(\frac{1}{239}\right) - 12\tan^{-1}\left(\frac{1}{682}\right) + 24\tan^{-1}\left(\frac{1}{12943}\right)$, which was used in 2002 for the evaluation of 1,241,100,000,000 digits of π. D. F. Ferguson (1947) recalculated π to 808 decimal places, using a mechanical desk calculator. K. Mahler (1953) proved that π is not a Liouville number.

Modern computer calculation of π was started by D. Shanks (1961), who reported 100000 digits of π. This record was improved many times; Yasumasa Kanada (Japan, December 2002) using a 64-node Hitachi supercomputer evaluated 1,241,100,000,000 digits of π. For this purpose he used the earlier mentioned formula of F. C. W. Stormer (1896) and the formula $\frac{\pi}{4} = 12\tan^{-1}\left(\frac{1}{49}\right) + 32\tan^{-1}\left(\frac{1}{57}\right) - 5\tan^{-1}\left(\frac{1}{239}\right) + 12\tan^{-1}\left(\frac{1}{110443}\right)$. Future improved results are inevitable.

Degree

Babylonians divided the circle into 360 degrees (360°), probably because 360 approximates the number of days in a year. Ptolemy (Egypt, c. 90–168 AD) in *Mathematical Syntaxis* used the symbol sing ° in astronomical calculations. Mathematically, one degree (1 °) has the numerical value $\frac{\pi}{180}$:

$$^\circ = \frac{\pi}{180}.$$

Therefore, all historical and other information about ° can be derived from information about π.

Euler constant

J. Napier in his work on logarithms (1618) mentioned the existence of a special convenient constant for the calculation of logarithms (but he did not evaluate this constant). It is possible that the table of logarithms was written by W. Oughtred, who is credited in 1622 with inventing the slide rule, which is a tool used for multiplication, division, evaluation of roots, logarithms, and other functions. In 1669 I. Newton published the series $2 + 1/2! + 1/3! + \ldots = 2.71828\ldots$, which actually converges to that special constant. At that time J. Bernoulli tried to find the limit of $(1 + 1/n)^n$, when $n \to \infty$. G. W. Leibniz (1690–1691) was the first, in correspondence to C. Huygens, to recognize this limit as a special constant, but he used the notation b to represent it.

L. Euler began using the letter e for that constant in 1727–1728, and introduced this notation in a letter to C. Goldbach (1731). However, the first use of e in a published work appeared in Euler's *Mechanica* (1736). In 1737 L. Euler proved that e and e^2 are irrational numbers and represented e through continued fractions. In 1748 L. Euler represented e as an infinite sum and found its first 23 digits:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots.$$

D. Bernoulli (1760) used e as the base of the natural logarithms. J. Lambert (1768) proved that $e^{p/q}$ is an irrational number, if p/q is a nonzero rational number.

In the 19th century A. Cauchy (1823) determined that $e = \lim_{z\to\infty} (1 + 1/z)^z$; J. Liouville (1844) proved that e does not satisfy any quadratic equation with integral coefficients; C. Hermite (1873) proved that e is a transcendental number; and E. Catalan (1873) represented e through infinite products.

The only constant appearing more frequently than e in mathematics is π. Physical applications of e are very often connected with time-dependent processes. For example, if $w(t)$ is a decreasing value of a quantity at time t, which decreases at a rate proportional to its value with coefficient $-\lambda$, this quantity is subject to exponential decay described by the following differential equation and its solution:

$$w'(t) = -\lambda\, t\ /;\ w(t) = c\, e^{-\lambda t}$$

where $c = w(0)$ is the initial quantity at time $t = 0$. Examples of such processes can be found in the following: a radionuclide that undergoes radioactive decay, chemical reactions (like enzyme-catalyzed reactions), electric charge, vibrations, pharmacology and toxicology, and the intensity of electromagnetic radiation.

Euler gamma

In 1735 the Swiss mathematician L. Euler introduced a special constant that represents the limiting difference between the harmonic series and the natural logarithm:

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right).$$

Euler denoted it using the symbol C, and initially calculated its value to 6 decimal places, which he extended to 16 digits in 1781. L. Mascheroni (1790) first used the symbol γ for the notation of this constant and calculated its value to 19 correct digits. Later J. Soldner (1809) calculated γ to 40 correct digits, which C. Gauss and F. Nicolai (1812) verified. E. Catalan (1875) found the integral representation for this constant $\gamma == 1 - \int_0^1 \frac{t^2+t^4+t^8+\ldots}{t+1}\, dt$.

This constant was named the Euler gamma or Euler-Mascheroni constant in the honor of its founders.

Applications include discrete mathematics and number theory.

Catalan constant

The Catalan constant $C == 1 - 1/3^2 + 1/5^2 - 1/7^2 + \ldots == 0.915966\ldots$ was named in honor of Eu. Ch. Catalan (1814–1894), who introduced a faster convergent equivalent series and expressions in terms of integrals. Based on methods resulting from collaborations with M. Leclert, E. Catalan (1865) computed C up to 9 decimals. M. Bresse (1867) computed 24 decimals of C using a technique from E. Kummer's work. J. Glaisher (1877) evaluated 20 digits of the Catalan constant, which he extended to 32 digits in 1913.

The Catalan constant is applied in number theory, combinatorics, and different areas of mathematical analysis.

Glaisher constant

The works of H. Kinkelin (1860) and J. Glaisher (1877–1878) introduced one special constant:

$$A == \exp\left(\frac{1}{12} - \zeta'(-1)\right),$$

which was later called the Glaisher or Glaisher-Kinkelin constant in honor of its founders. This constant is used in number theory, Bose-Einstein and Fermi-Dirac statistics, analytic approximation and evaluation of integrals and products, regularization techniques in quantum field theory, and the Scharnhorst effect of quantum electrodynamics.

Khinchin constant

The 1934 work of A. Khinchin considered the limit of the geometric mean of continued fraction terms $\lim_{n\to\infty} \left(\prod_{k=1}^{n} q_k\right)^{1/n}$ and found that its value is a constant independent for almost all continued fractions:

$$q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \ldots}}} == x \bigwedge q_k \in \mathbb{N}^+.$$

The constant—named the Khinchin constant in the honor of its founder—established that rational numbers, solutions of quadratic equations with rational coefficients, the golden ratio ϕ, and the Euler number e upon being expanded into continued fractions do not have the previous property. Other site numerical verifications showed that continued fraction expansions of π, the Euler-Mascheroni constant γ, and Khinchin's constant K itself can satisfy that property. But it was still not proved accurately.

Applications of the Khinchin constant K include number theory.

Imaginary unit

The imaginary unit constant i allows the real number system $\mathbb{R}$ to be extended to the complex number system $\mathbb{C}$. This system allows for solutions of polynomial equations such as $z^2 + 1 == 0$ and more complicated polynomial equations through complex numbers. Hence $i^2 == -1$ and $(-i)^2 == -1$, and the previous quadratic equation has two solutions as is expected for a quadratic polynomial:

$$z^2 == -1 \;/;\; z == z_1 == i \bigwedge z == z_2 == -i.$$

The imaginary unit has a long history, which started with the question of how to understand and interpret the solution of the simple quadratic equation $z^2 == -1$.

It was clear that $1^2 == (-1)^2 == 1$. But it was not clear how to get -1 from something squared.

In the 16th, 17th, and 18th centuries this problem was intensively discussed together with the problem of solving the cubic, quartic, and other polynomial equations. S. Ferro (Italy, 1465–1526) first discovered a method to solve cubic equations. N. F. Tartaglia (Italy, 1500–1557) independently solved cubic equations. G. Cardano (Italy, 1545) published the solutions to the cubic and quartic equations in his book *Ars Magna*, with one case of this solution communicated to him by N. Tartaglia. He noted the existence of so-called imaginary numbers, but did not describe their properties. L. Ferrari (Italy, 1522–1565) solved the quartic equation, which was mentioned in the book *Ars Magna* by his teacher G. Cardano. R. Descartes (1637) suggested the name "imaginary" for nonreal numbers like $1 + \sqrt{-1}$. J. Wallis (1685) in *De Algebra tractates* published the idea of the graphic representation of complex numbers. J. Bernoulli (1702) used imaginary numbers. R. Cotes (1714) derived the formula:

$$e^{i\,\phi} == \cos(\phi) + i \sin(\phi),$$

which in 1748 was found by L. Euler and hence named for him.

A. Moivre (1730) derived the well-known formula $(\cos(x) + i \sin(x))^n = \cos(n\,x) + i \sin(n\,x) \;/;\; n \in \mathbb{N}$, which bears his name.
Investigations of L. Euler (1727, 1728) gave new imputus to the theory of complex numbers and functions of complex arguments (analytic functions). In a letter to C. Goldbach (1731) L. Euler introduced the notation e for the base of the natural logarithm e==2.71828182… and he proved that e is irrational. Later on L. Euler (1740–1748) found a series expansion for e^z, which lead to the famous and very basic formula connecting exponential and trigonometric functions $\cos(x) + i \sin(x) == e^{i\,x}$ (1748). H. Kühn (1753) used imaginary numbers. L. Euler (1755) used the word "complex" (1777) and first used the letter i to represent $\sqrt{-1}$. C. Wessel (1799) gave a geometrical interpretation of complex numbers.

As a result, mathematicians introduced the use of a special symbol—the imaginary unit i that is equal to $i = \sqrt{-1}$:

$$i^2 == -1.$$

In the 19th century the conception and theory of complex numbers was basically formed. A. Buee (1804) independently came to the idea of J. Wallis about geometrical representations of complex numbers in the plane. J. Argand (1806) introduced the name modulus for $\sqrt{x^2 + y^2}$, and published the idea of geometrical interpretation of complex numbers known as the Argand diagram. C. Mourey (1828) laid the foundations for the theory of directional numbers in a little treatise.

The imaginary unit i was interpreted in a geometrical sense as the point with coordinates {0, 1} in the Cartesian (Euclidean) x, y plane with the vertical y axis upward and the origin {0, 0}. This geometric interpretation establishes the following representations of the complex number z through two real numbers x and y as:

$$z == x + i\, y \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R} \Longleftrightarrow (x,\, y)$$

$$z == r\cos(\phi) + i\, r\sin(\phi) \;/;\; r \in \mathbb{R} \wedge r > 0 \bigwedge \phi \in \mathbb{R},$$

where $r = \sqrt{x^2 + y^2}$ is the distance between points $\{x,\, y\}$ and {0, 0}, and ϕ is the angle between the line connecting points {0, 0} and $\{x,\, y\}$ and the positive x axis direction (the so-called polar representation).

The last formula lead to the following basic relations:

$$r == \sqrt{x^2 + y^2}$$

$$x == r\cos(\phi)$$

$$y == r\sin(\phi)$$

$$\phi == \tan^{-1}\left(\frac{y}{x}\right) /;\; x > 0,$$

which describe the main characteristics of the complex number $z == x + i\, y$—the so-called modulus (absolute value) r, the real part x, the imaginary part y, and the argument ϕ.

The Euler formula $e^{i\phi} == \cos(\phi) + i\sin(\phi)$ allows the representation of the complex number z, using polar coordinates $(r,\ \phi)$ in the more compact form:

$$z == r\, e^{i\phi} \;/;\; r \in \mathbb{R} \wedge r \geq 0 \bigwedge \phi \in [0,\, 2\pi).$$

It also allows the expression of the logarithm of a complex number through the formula:

$$\log(z) == \log(r) + i\,\phi \;/;\; r \in \mathbb{R} \wedge r > 0 \bigwedge \phi \in \mathbb{R}.$$

Taking into account that the cosine and sine have period 2π, it follows that $e^{i\phi}$ has period $2\pi i$:

$$e^{i\phi + 2\pi i} == e^{i(\phi + 2\pi)} == \cos(\phi + 2\pi) + i\sin(\phi + 2\pi) == \cos(\phi) + i\sin(\phi) == e^{i\phi}.$$

Generically, the logarithm function $\log(z)$ is the multivalued function:

$$\log(z) == \log(r) + i\,(\phi + 2\pi k) \;/;\; r \in \mathbb{R} \wedge r > 0 \bigwedge \phi \in \mathbb{R} \bigwedge k \in \mathbb{Z}.$$

For specifying just one value for the logarithm $\log(z)$ and one value of the argument ϕ for a given complex number z, the restriction $-\pi < \phi \leq \pi$ is generally used for the argument ϕ.

C. F. Gauss (1831) introduced the name "imaginary unit" for $\sqrt{-1}$, suggested the term complex number for $x + i y$, and called $x^2 + y^2$ the norm, but mentioned that the theory of complex numbers is quite unknown, and in 1832 published his chief memoir on the subject. A. Cauchy (1789–1857) proved several important basic theorems in complex analysis. N. Abel (1802–1829) was the first to widely use complex numbers with well-known success. K. Weierstrass (1841) introduced the notation $|z|$ for the modulus of complex numbers, which he called the absolute value. E. Kummer (1844), L. Kronecker (1845), Scheffler (1845, 1851, 1880), A. Bellavitis (1835, 1852), Peacock (1845), A. Morgan (1849), A. Mobius (1790–1868), J. Dirichlet (1805–1859), and others made large contributions in developing complex number theory.

Definitions of classical constants and the imaginary unit

Classical constants and the imaginary unit include eight basic constants: golden ratio ϕ, pi π, the number of radians in one degree $°$, Euler number (or Euler constant or base of natural logarithm) e, Euler-Mascheroni constant (Euler gamma) γ, Catalan number (Catalan's constant) C, Glaisher constant (Glaisher-Kinkelin constant) A, Khinchin constant (Khintchine's constant) K, and the imaginary unit i. They are defined by the following formulas:

$$\phi = \frac{1}{2}\left(1+\sqrt{5}\right) = \exp\left(\operatorname{csch}^{-1}(2)\right)$$

$$\pi = 4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}$$

$$° = \frac{\pi}{180}$$

$$e = \sum_{k=0}^{\infty}\frac{1}{k!}$$

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right)$$

$$C = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$$

$$A = \exp\left(\frac{1}{12} - \zeta'(-1)\right)$$

$$K = \prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\log_2(k)}$$

$$i = \sqrt{-1}\,.$$

The number π is the ratio of the circumference of a circle to its diameter.

Connections within the group of classical constants and the imaginary unit and with other function groups

Representations through functions

The classical constants ϕ, π, $^\circ$, e, γ, C, A, and the imaginary unit i can be represented as particular values of expressions that include functions (Fibonacci F_ν, algebraic roots, exponential and inverse trigonometric functions, complete elliptic integrals $K(z)$ and $E(z)$, dilogarithm $\mathrm{Li}_2(z)$, gamma function $\Gamma(z)$, hypergeometric functions ${}_p\tilde{F}_q$, ${}_pF_q$, Meijer G function $G_{p,q}^{m,n}$, polygamma function $\psi(z)$, Stieltjes constants γ_n, Lerch transcendent $\Phi(z, s, a)$, and Hurwitz and Riemann zeta functions $\zeta(s, a)$ and $\zeta(s)$), for example:

ϕ	π	$^\circ$	e
$\phi == \frac{1}{2}\left(\sqrt{5}\, F_1 + \sqrt{5 F_1^2 - 4}\right)$	$\pi == 4\left(4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)\right)$	$^\circ == \frac{1}{45}\left(4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)\right)$	$e == {}_0\tilde{F}_0($
$\phi == 2^{-1/\nu}\left(\sqrt{5}\, F_\nu + \sqrt{5 F_\nu^2 + 4\cos(\pi\nu)}\right)^{1/\nu} /; \nu \in \mathbb{R} \land \nu > 0$	$\pi == 4\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{1}{3}\right) == 8\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{1}{7}\right) == 4\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{1}{5}\right) + 4\tan^{-1}\left(\frac{1}{8}\right)$	$^\circ == \frac{1}{45}\left(\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right) == \frac{1}{45}\left(2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right) == \frac{1}{45}\left(\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)\right)$	$e == {}_0F_0($
$\phi == \exp\left(\mathrm{csch}^{-1}(2)\right)$	$\pi == 4\left(6\tan^{-1}\left(\frac{1}{8}\right) + 2\tan^{-1}\left(\frac{1}{57}\right) + \tan^{-1}\left(\frac{1}{239}\right)\right)$	$^\circ == \frac{1}{45}\left(6\tan^{-1}\left(\frac{1}{8}\right) + 2\tan^{-1}\left(\frac{1}{57}\right) + \tan^{-1}\left(\frac{1}{239}\right)\right)$	$e == G_{0,1}^{1,0}$
$\phi == \left(z;\, z^2 - z - 1\right)_2^{-1}$	$\pi == 4\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{1}{5}\right) + 4\tan^{-1}\left(\frac{1}{8}\right)$	$^\circ == \frac{1}{45}\left(\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)\right)$	$e == e^z$ /;
	$\pi == 4\left(6\tan^{-1}\left(\frac{1}{8}\right) + 2\tan^{-1}\left(\frac{1}{57}\right) + \tan^{-1}\left(\frac{1}{239}\right)\right)$	$^\circ == \frac{1}{45}\left(6\tan^{-1}\left(\frac{1}{8}\right) + 2\tan^{-1}\left(\frac{1}{57}\right) + \tan^{-1}\left(\frac{1}{239}\right)\right)$	$\log(e) =$
	$\pi == 88\tan^{-1}\left(\frac{1}{28}\right) + 8\tan^{-1}\left(\frac{1}{443}\right) - 20\tan^{-1}\left(\frac{1}{1393}\right) - 40\tan^{-1}\left(\frac{1}{11018}\right)$	$^\circ == \frac{1}{45}\left(22\tan^{-1}\left(\frac{1}{28}\right) + 2\tan^{-1}\left(\frac{1}{443}\right) - 5\tan^{-1}\left(\frac{1}{1393}\right) - 10\tan^{-1}\left(\frac{1}{11018}\right)\right)$	
	$\pi == 48\tan^{-1}\left(\frac{1}{18}\right) + 12\tan^{-1}\left(\frac{1}{70}\right) + 20\tan^{-1}\left(\frac{1}{99}\right) + 32\tan^{-1}\left(\frac{1}{307}\right)$	$^\circ == \frac{1}{45}\left(12\tan^{-1}\left(\frac{1}{18}\right) + 3\tan^{-1}\left(\frac{1}{70}\right) + 5\tan^{-1}\left(\frac{1}{99}\right) + 8\tan^{-1}\left(\frac{1}{307}\right)\right)$	
	$\pi == 4\left(\tan^{-1}\left(\frac{p}{q}\right) + \tan^{-1}\left(\frac{q-p}{p+q}\right)\right) /; p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+$	$^\circ == \frac{1}{45}\left(\tan^{-1}\left(\frac{p}{q}\right) + \tan^{-1}\left(\frac{q-p}{p+q}\right)\right) /; p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+$	
	$\pi == 2K(0) == 2E(0)$	$^\circ == \frac{K(0)}{90} == \frac{E(0)}{90}$	
	$\pi == \Gamma\left(\frac{1}{2}\right)^2 == \sqrt{6\,\mathrm{Li}_2(1)}$	$^\circ == \frac{1}{180}\Gamma\left(\frac{1}{2}\right)^2 == \frac{1}{180}\sqrt{6\,\mathrm{Li}_2(1)}$	

Representations through related functions

The four classical constants ϕ, π, $^\circ$, and e and the imaginary unit i can sometimes be represented through other classical constants and the imaginary unit by formulas such as the following:

	ϕ	π	$^\circ$	e	i
ϕ		$\phi == 2\cos\left(\frac{\pi}{5}\right)$ $\phi == \frac{1}{2}\sec\left(\frac{2\pi}{5}\right)$ $\phi == \frac{1}{2}\csc\left(\frac{\pi}{10}\right)$	$\phi == 2\cos(36^\circ)$ $\phi == \frac{1}{2}\sec(72^\circ)$ $\phi == \frac{1}{2}\csc(18^\circ)$	$\phi == e^{\frac{\pi i}{5}} + e^{-\frac{\pi i}{5}}$	$\phi == e^{\frac{\pi i}{5}} + e^{-\frac{\pi i}{5}}$
π	$\pi == 5\cos^{-1}\left(\frac{\phi}{2}\right)$ $\pi == -5\, i \log\left(\frac{1}{2}\left(\phi + i\sqrt{4-\phi^2}\right)\right)$		$\pi == 180^\circ$	$e == i^{-\frac{2i}{\pi}}$ $e^{\pi i} == -1$ $e^{2\pi i} == 1$ $e^{\pi i k} == (-1)^k /; k \in \mathbb{Z}$	$\pi == -i\log(-1)$ $\pi == 2\, i\log\left(\frac{1-i}{1+i}\right)$ $e == i^{-\frac{2i}{\pi}}$ $e^{\pi i} == -1$ $e^{2\pi i} == 1$ $e^{\pi i k} == (-1)^k$ /;
$^\circ$	$^\circ == \frac{1}{36}\cos^{-1}\left(\frac{\phi}{2}\right)$ $^\circ == -\frac{i}{36}\log\left(\frac{1}{2}\left(\phi + i\sqrt{4-\phi^2}\right)\right)$	$^\circ == \frac{\pi}{180}$		$e^{-90^\circ i k} == i^i$ $e^{180^\circ i} == -1$ $e^{360^\circ i} == 1$	$^\circ == -\frac{i}{180}\log(-$ $^\circ == \frac{1}{90}\, i\log\left(\frac{1-i}{1+i}\right.$ $e^{-90^\circ i k} == i^i$ $e^{180^\circ i} == -1$ $e^{360^\circ i} == 1$ $e^{180^\circ i k} == (-1)^k$
e		$e == i^{-\frac{2i}{\pi}}$ $e^{\pi i} == -1$ $e^{2\pi i} == 1$ $e^{\pi i k} == (-1)^k /; k \in \mathbb{Z}$	$e^{90^\circ i k} == i^k /; k \in \mathbb{Z}$ $e^{180^\circ i} == -1$ $e^{360^\circ i} == 1$ $e^{180^\circ i k} == (-1)^k /; k \in \mathbb{Z}$		$e == i^{-\frac{2i}{\pi}}$ $e^{\pi i} == -1$ $e^{2\pi i} == 1$ $e^{\pi i k} == (-1)^k$ /;
i		$i == -\frac{\pi}{\log(-1)}$	$i == -\frac{180^\circ}{\log(-1)}$	$i^{-\frac{2i}{\pi}} == e$ $e^{\pi i} == -1$ $e^{2\pi i} == 1$ $e^{\pi i k} == (-1)^k /; k \in \mathbb{Z}$	

The best-known properties and formulas for classical constants and the imaginary unit

Values

	first 50 digits	number of digits computed in year
ϕ	1.6180339887498948482045868343656381177203091798057…	2002, near 3 141 000 000 digits
π	3.1415926535897932384626433832795028841971693993751…	December 2002, near 1 241 100 000 000 digits
$^\circ$	0.017453292519943295769236907684886127134428718885417…	December 2002, near 1 241 100 000 000 digits
e	2.7182818284590452353602874713526624977572470936999…	2003, near 50 100 000 000 digits
γ	0.57721566490153286060651209008240243104215933593992…	December 2006, near 116 580 000 digits
C	0.91596559417721901505460351493238411077414937428167…	2002, near 201 000 000 digits
A	1.2824271291006226368753425688697917277676889273250…	December 2004, near 5000 digits
K	2.6854520010653064453097148354817956938203822939944…	1998, near 110 000 digits

The imaginary unit i satisfies the following relation:

$i^2 == -1.$

Evaluation of specific values

For evaluation of the eight classical constants ϕ, π, $^\circ$, e, γ, C, A, and K, *Mathematica* uses procedures that are based on the following formulas or methods:

	basic formula or method
$\boldsymbol{\phi}$	Karatsuba's modifications of Newton's methods for evaluations $\sqrt{z}$ $\left(\text{because } \phi == \frac{1}{2}\left(1+\sqrt{5}\right)\right)$
$\boldsymbol{\pi}$	$\frac{1}{\pi} == 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (545140134k+13591409)}{k!^3 (3k)! \left(640320^3\right)^{k+1/2}}$
$\boldsymbol{^\circ}$	$^\circ == \frac{\pi}{180}$
$\boldsymbol{e}$	$e == \lim_{n\to\infty} \left(\sum_{k=0}^{n} \frac{1}{k!}\right)$
$\boldsymbol{\gamma}$	$\gamma == \lim_{x\to\infty} \left(\frac{1}{\sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}} \sum_{k=0}^{\infty} \sum_{i=1}^{k} \frac{x^k}{i\,(k!)^2} - \frac{\log(x)}{2}\right)$
$\boldsymbol{C}$	$C == \frac{\pi}{8} \log\left(\sqrt{3}+2\right) + \frac{3}{8} \sum_{k=0}^{\infty} \frac{k!^2}{(2k)!\,(2k+1)^2}$
$\boldsymbol{A}$	$A == \lim_{n\to\infty} \exp\left(\frac{1}{12}\log(2\pi) - \frac{1}{2\pi^2}\left(2^{1-\lceil n\log_8(10)+1\rceil} \sum_{j=0}^{2\lceil n\log_8(10)+1\rceil-1} \frac{1}{(j+1)^2} \log(2(j+1))\,(-1)^j \left(\sum_{k=0}^{j-\lceil n\log_8(10)+1\rceil} \binom{\lceil n\log_8(10)+1\rceil}{k} - 2^{\lceil n\log_8(10)+1\rceil}\right)\right) + \frac{\gamma}{12}\right)$
$\boldsymbol{K}$	$K == \lim_{n\to\infty} \exp\left(\frac{1}{\log(2)} \sum_{m=1}^{\lfloor n\log_4(10)\rfloor+1} \frac{1}{m} (\zeta(2m)-1) \sum_{k=1}^{2m-1} \frac{(-1)^{k+1}}{k}\right)$

The formula for $1/\pi$ is called Chudnovsky's formula.

Analyticity

The eight classical constants ϕ, π, $^\circ$, e, γ, C, A, and K are positive real numbers. The constant ϕ is a quadratic irrational number. The constants π, $^\circ$, and e are irrational and transcendental over $\mathbb{Q}$. Whether C and γ are irrational is not known.

The imaginary unit i is an algebraic number.

Series representations

The five classical constants π, e, γ, C, and K have numerous series representations, for example, the following:

$$\pi == 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} == \frac{4}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{1}{2k+1} (-1)^{\left\lfloor \frac{k}{2} \right\rfloor} == 3\sqrt{3} \sum_{k=0}^{\infty} \frac{(-1)^k}{3k+1} - \log(2)\sqrt{3}$$

$$\pi == 2\log(2) + 4 \sum_{k=0}^{\infty} \frac{1}{k+1} (-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor}$$

$$\pi == \sum_{k=0}^{\infty} \frac{1}{16^k} \left(-\frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} + \frac{4}{8k+1}\right) == \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \left(\frac{2}{4k+2} + \frac{1}{4k+3} + \frac{2}{4k+1}\right)$$

$$\pi == \frac{5}{4}\sqrt{5} \sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{2k+1} F_{2k+1}\, 16^{-k} == 4 \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2k+1}}\right)$$

$$\pi = \sum_{k=1}^{\infty} \frac{3^k - 1}{4^k} \zeta(k+1)$$

$$\pi = 2 \sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!} = 3 \sqrt{2 \sum_{k=1}^{\infty} \frac{k!^2}{k^2 (2k)!}} = 6 \sqrt{\sum_{k=0}^{\infty} \frac{(2k)!!}{(2k+1)!! \, 2^{2k+2} (k+1)}}$$

$$\pi = a_n + b_n \sum_{k=1}^{\infty} \frac{k^n}{\binom{2k}{k}} \;/;$$

n	1	2	3	4	5	6	7	8	9	10
a_n	$-3\sqrt{3}$	$-\frac{18\sqrt{3}}{5}$	$-\frac{135\sqrt{3}}{37}$	$-\frac{432\sqrt{3}}{119}$	$-\frac{243\sqrt{3}}{67}$	$-\frac{23\,814\sqrt{3}}{6565}$	$-\frac{42\,795\sqrt{3}}{11\,797}$	$-\frac{2\,355\,156\sqrt{3}}{649\,231}$	$-\frac{48\,314\,475\sqrt{3}}{13\,318\,583}$	$-\frac{365\,306\,274\sqrt{3}}{100\,701\,965}$
b_n	$\frac{9}{2}\sqrt{3}$	$\frac{27}{10}\sqrt{3}$	$\frac{81}{74}\sqrt{3}$	$\frac{81}{238}\sqrt{3}$	$\frac{81}{938}\sqrt{3}$	$\frac{243}{13\,130}\sqrt{3}$	$\frac{81}{23\,594}\sqrt{3}$	$\frac{729}{1\,298\,462}\sqrt{3}$	$\frac{2187}{26\,637\,166}\sqrt{3}$	$\frac{2187}{201\,403\,930}\sqrt{3}$

$$\pi = c_n \left(\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \right)^{\frac{1}{2n}} \;/;$$

n	1	2	3	4	5	…	n /; $n \in \mathbb{N}^+$
c_n	$\sqrt{6}$	$\sqrt{3}\; 10^{1/4}$	$\sqrt{3}\; 35^{1/6}$	$3^{3/8}\, 5^{1/4}\, 14^{1/8}$	$\sqrt{3}\; 385^{1/10}$	…	$\left(\frac{(-1)^{n-1} (2n)!}{2^{2n-1} B_{2n}} \right)^{\frac{1}{2n}}$

$$\pi = c_n \left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2n}} \right)^{\frac{1}{2n}} \;/;$$

n	1	2	3	4	5	…	n /; $n \in \mathbb{N}^+$
c_n	$2\sqrt{3}$	$2\left(\frac{5}{7}\right)^{1/4}\sqrt{3}$	$\left(\frac{35}{31}\right)^{1/6} 2^{5/6}\sqrt{3}$	$2\left(\frac{7}{127}\right)^{1/8} 3^{3/8}\, 5^{1/4}$	$\left(\frac{55}{73}\right)^{1/10} 2^{9/10}\sqrt{3}$	…	$\left(\frac{(-1)^{n} (2n)!}{2^{2n-1} B_{2n}\left(\frac{1}{2}\right)} \right)^{\frac{1}{2n}}$

$$\pi = c_n \left(\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n}} \right)^{\frac{1}{2n}} \;/;$$

n	1	2	3	4	5	…	n /; $n \in \mathbb{N}^+$
c_n	$2\sqrt{2}$	$2\; 6^{1/4}$	$2\; 15^{1/6}$	$2\; 3^{1/4}\left(\frac{70}{17}\right)^{1/8}$	$2\left(\frac{35}{31}\right)^{1/10} 3^{2/5}$	…	$\left(\frac{(-1)^{n-1}\, 2\, (2n)!}{\left(2^{2n}-1\right) B_{2n}} \right)^{\frac{1}{2n}}$

$$\pi = c_n \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n-1}} \right)^{\frac{1}{2n-1}} \;/;$$

n	1	2	3	4	5	…	n /; $n \in \mathbb{N}^+$
c_n	4	$2\; 2^{2/3}$	$2\left(\frac{3}{5}\right)^{1/5} 2^{4/5}$	$2\left(\frac{5}{61}\right)^{1/7} 2^{5/7}\, 3^{2/7}$	$2\left(\frac{7}{277}\right)^{1/9} 2^{8/9}\, 3^{2/9}$	…	$\left(\frac{(-1)^{n-1}\, 2^{2n} (2n-2)!}{E_{2n-2}} \right)^{\frac{1}{2n-1}}$

e	γ	C	K
$e == \sum_{k=0}^{\infty} \frac{1}{k!}$	$\gamma == \frac{\log(2)}{2} + \frac{1}{\log(2)} \sum_{k=2}^{\infty} \frac{(-1)^k \log(k)}{k}$	$C == 2 \sum_{k=0}^{\infty} \frac{1}{(4k+1)^2} - \frac{\pi^2}{8}$	$K == \exp\left(\frac{1}{\log(2)} \sum_{k=2}^{\infty} \log\right.$
$e == \sum_{k=0}^{\infty} \frac{2k+1}{(2k)!}$	$\gamma == \sum_{k=2}^{\infty} \left(\log\left(1 - \frac{1}{k}\right) + \frac{1}{k}\right) + 1$	$C == \frac{\pi^2}{8} - 2 \sum_{k=0}^{\infty} \frac{1}{(4k+3)^2}$	$K == \exp\left(\log(2) + \frac{1}{2\log(2}\right.$
$e == 2 \sum_{k=0}^{\infty} \frac{k+1}{(2k+1)!}$	$\gamma == \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k)$	$C == \frac{\pi}{2} \log(2) - \frac{1}{32} \pi \sum_{k=0}^{\infty} \frac{(2k+1)!^2}{16^k k!^4 (k+1)^3}$	$K == \exp\left(\frac{1}{\log(2)} \sum_{k=2}^{\infty} \frac{(-1}{}\right.$
$e == \frac{1}{2} \sum_{k=0}^{\infty} \frac{k+1}{k!}$	$\gamma == 1 - \sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k} ==$	$C == \frac{\pi}{2} \log(2) - \frac{1}{32} \pi \sum_{k=0}^{\infty} \frac{(2k+1)!^2}{16^k k!^4 (k+1)^3}$	$K == \exp\left(\frac{1}{\log(2)} \left(\log^2(2\right.\right.$
$e == \sum_{k=0}^{\infty} \frac{3-4k^2}{(2k+1)!}$	$\gamma == \log(2) - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{4^k (2k+1)}$	$C == \frac{\pi}{2} \log(2) + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} (\psi(k+1) + \gamma)$	$K == \exp\left(\frac{1}{\log(2)} \left(\frac{\pi^2}{6} - \frac{1}{2}\right.\right.$
$e == \sum_{k=0}^{\infty} \frac{(3k)^2+1}{(3k)!}$		$C == \frac{\pi}{4} \log(2) + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \left(\psi\left(k + \frac{3}{2}\right) + \gamma\right)$	$K == \exp\left(\frac{1}{\log(2)} \sum_{k=1}^{\infty} \frac{\zeta(2}{}\right.$
$e == \frac{2}{3} \sum_{k=0}^{\infty} \frac{(k+3)^{k \bmod 2}}{2^{k \bmod 2} k!}$		$C == 1 - \sum_{k=1}^{\infty} \frac{k \zeta(2k+1)}{4^{2k}} == \frac{1}{8} \sum_{k=2}^{\infty} \frac{k}{2^k} \zeta\left(k+1, \frac{3}{4}\right)$	

$$\log(A) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} (j+1)^2 \log(j+1) + \frac{1}{8}.$$

Product representations

The four classical constants π, e, γ, and A can be represented by the following formulas:

π	e	γ	A
$\pi == 2 \prod_{k=1}^{\infty} \frac{4k^2}{(2k-1)(2k+1)}$	$e == 2 \prod_{k=0}^{\infty} \left(\frac{1}{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}+2^{k+1}\right)} 2^{2^{k+1}} \Gamma\left(\frac{1}{2} + 2^k\right)^2\right)^{2^{-k-1}}$	$\gamma == \log\left(\prod_{n=0}^{\infty} \left(\prod_{k=0}^{n} (k+1)^{(-1)^{k+1} \binom{n}{k}}\right)^{\frac{1}{n+1}}\right)$	$A == e$
$\pi == 2 \prod_{k=2}^{\infty} \sec\left(\frac{\pi}{2^k}\right) == 3 \prod_{k=0}^{\infty} \sec\left(\frac{\pi}{12\, 2^k}\right)$	$e == 2 \prod_{j=1}^{\infty} \prod_{k=0}^{2^{j-1}-1} \left(\frac{2k+2^j+2}{2k+2^j+1}\right)^{\frac{1}{2^j}}$		$A == \left(\prod_{k=1}^{\infty}\right.$
$\pi == 2\, e \prod_{k=1}^{\infty} \left(1 + \frac{2}{k}\right)^{(-1)^{k+1} k}$	$e == \prod_{k=1}^{\infty} k^{-\frac{\mu(k)}{k}}$		
$\pi == \sqrt{6 \prod_{k=1}^{\infty} \left(1 - p_k^{-2}\right)^{-1}} \;/;\; p_k \in \mathbb{P}$			

Integral representations

The five classical constants π, (and $\sqrt{\pi}$), γ, C, A, and K have numerous integral representations, for example:

π	γ	C	A
$\pi == 2\int_0^\infty \frac{1}{t^2+1}\,dt$	$\gamma == -\int_0^\infty e^{-t}\log(t)\,dt$	$C == \frac{1}{4}\int_0^\infty \frac{e^{t/2}\,t}{e^t+1}\,dt$	$A == \frac{2^{7/36}}{\sqrt[6]{\pi}}\exp\left(\frac{2}{3}\int_0^{1/2}\log($
$\pi == 4\int_0^1 \sqrt{1-t^2}\,dt$	$\gamma == -\int_0^1 \log(-\log(t))\,dt$	$C == \frac{1}{2}\int_0^\infty t\,\mathrm{sech}(t)\,dt == \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{t}{\sin(t)}\,dt$	
$\pi == 2\int_0^1 \frac{1}{\sqrt{1-t^2}}\,dt$	$\gamma == -\int_0^1 \log\left(\log\left(\frac{1}{t}\right)\right)dt$	$C == -\int_0^1 \frac{\log(t)}{t^2+1}\,dt == -\int_0^1 \frac{1}{t^2+1}\log\left(\frac{1-t}{\sqrt{2}}\right)dt$	
$\pi == 2\,e\int_0^\infty \frac{\cos(t)}{t^2+1}\,dt$	$\gamma == -\int_0^1 \frac{e^{1-\frac{1}{t}}-t}{t\,(1-t)}\,dt$	$C == -2\int_0^{\frac{\pi}{4}} \log(2\sin(t))\,dt == 2\int_0^{\frac{\pi}{4}} \log(2\cos(t))\,dt$	
	$\gamma == \int_0^1 \frac{1-e^{-t}-e^{-1/t}}{t}\,dt$	$C == -\int_0^{\frac{\pi}{4}} \log(\tan(t))\,dt == \int_0^{\frac{\pi}{4}} \log(\cot(t))\,dt$	
	$\gamma == \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t\,e^t}\right)dt$	$C == \int_0^1 \frac{\tan^{-1}(t)}{t}\,dt == \int_0^{\frac{\pi}{2}} \sinh^{-1}(\sin(t))\,dt$	
	$\gamma == 2\int_0^\infty \frac{e^{-t^2}-e^{-t}}{t}\,dt$	$C == \int_0^{\frac{\pi}{2}} \sinh^{-1}(\cos(t))\,dt == \int_0^{\frac{\pi}{2}} \mathrm{csch}^{-1}(\csc(t))\,dt$	
	$\gamma == \frac{\alpha\,\beta}{\alpha-\beta}\int_0^\infty \frac{e^{-t^\alpha}-e^{-t^\beta}}{t}\,dt\ /;$ $\alpha > 0 \land \beta > 0$	$C == \int_0^{\frac{\pi}{2}} \mathrm{csch}^{-1}(\sec(t))\,dt$	
	$\gamma == -\int_{-\infty}^\infty t\,e^{t-e^t}\,dt$	$C == \frac{1}{4}\int_0^1 \frac{K(t)}{\sqrt{t}}\,dt == \frac{1}{2}\int_0^1 K\left(t^2\right)dt$	
	$\gamma == \frac{1}{2} + 2\int_0^\infty \frac{t}{\left(t^2+1\right)\left(e^{2\pi t}-1\right)}\,dt$	$C == \frac{3}{4}\int_0^{\frac{\pi}{6}} \frac{t}{\sin(t)}\,dt + \frac{\pi}{8}\log\left(2+\sqrt{3}\right)$	
	$\gamma == -\int_0^\infty \frac{1}{t}\left(\cos(t) - \frac{1}{t^2+1}\right)dt$	$C == -\frac{\pi^2}{4}\int_0^1 \left(t-\frac{1}{2}\right)\sec(\pi t)\,dt$	
	$\gamma == \int_0^1 \left(\frac{1}{\log(t)} + \frac{1}{1-t}\right)dt$	$C == \int_0^{\frac{\pi}{2}} \frac{t\csc(t)}{\cos(t)+\sin(t)}\,dt - \frac{\pi}{4}\log(2)$	
	$\gamma == \int_0^1 \left(\frac{1}{\log(1-t)} + \frac{1}{t}\right)dt$	$C == \frac{\pi}{4}\log(2) - \frac{1}{2}\int_0^1 \frac{\log(1-t)}{\sqrt{t}\,(t+1)}\,dt$	
		$C == \frac{\pi}{8}\log(2) - \int_0^1 \frac{\log(1-t)}{t^2+1}\,dt$	

$$\pi == c_n \int_0^\infty \frac{\sin^n(t)}{t^n}\,dt\ /;$$

n	**1**	**2**	3	4	5	6	…	$n\ /;\ n \in \mathbb{N}^+$
c_n	2	2	$\frac{8}{3}$	3	$\frac{384}{115}$	$\frac{40}{11}$	…	$2^n \Big/ \left(n\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{(-1)^k\,(n-2k)^{n-1}}{k!\,(n-k)!}\right)$

The following integral is called the Gaussian probability density integral:

$$\sqrt{\pi} == 2\int_0^\infty e^{-t^2}\,dt.$$

The following integrals are called the Fresnel integrals:

$$\sqrt{\pi} == 2\sqrt{2}\int_0^\infty \sin\left(t^2\right)dt$$

$$\sqrt{\pi} == 2\sqrt{2}\int_0^\infty \cos\left(t^2\right)dt.$$

Limit representations

The six classical constants ϕ, π, e, γ, A, and K have numerous limit representations, for example:

ϕ	π	e	γ
$\phi == \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}$	$\pi == \lim_{n\to\infty} \left(2^{4n} \Big/ \left(n \binom{2n}{n}^2\right)\right)$	$e == \lim_{z\to 0} (z+1)^{1/z}$	$\gamma == \lim_{s\to 1} \left(\zeta(s) - \frac{1}{s-}\right.$
$\phi == \lim_{n\to\infty} z_n$ /; $z_{n+1} = \sqrt{1+z_n} \bigwedge z_0 == 1$	$\pi == \lim_{n\to\infty} \frac{4}{n^2} \sum_{k=0}^{n} \sqrt{n^2 - k^2}$	$e == \lim_{z\to\infty} \frac{z}{z!^{1/z}}$	$\gamma == \lim_{s\to\infty} \left(s - \Gamma\left(\frac{1}{s}\right.\right.$
$\phi == \lim_{\nu\to\infty} \frac{F_\nu}{F_{\nu-1}}$	$\pi == 4 \lim_{n\to\infty} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \frac{\sqrt{n-k^2}}{n}$	$e == \lim_{n\to\infty} \prod_{k=1}^{n} \frac{n^2+k}{n^2-k}$	$\gamma == \lim_{x\to 1^+} \left(\sum_{k=1}^{\infty} \left(k\right.\right.$
	$\pi == \lim_{n\to\infty} \frac{2^{4n+1} n!^4}{(2n+1)(2n)!^2}$	$e == \lim_{n\to\infty} \left(2 \sum_{k=0}^{n} \frac{n^k}{k!}\right)^{1/n}$	$\gamma == \lim_{n\to\infty} (H_{n-1} -$
	$\pi == \lim_{n\to\infty} \frac{2(2n)!!^2}{(2n+1)(2n-1)!!^2}$	$e == \lim_{z\to\infty} \left(\frac{z^z}{(z-1)^{z-1}} - \frac{(z-1)^{z-1}}{(z-2)^{z-2}}\right)$	$\gamma == \lim_{\alpha\to 0} (\mathrm{li}(e^{\alpha x}) -$ $\log(\alpha)) -$
	$\pi == \lim_{n\to\infty} \frac{n!^2 (n+1)^{2n^2+n}}{2 n^{2n^2+3n+1}}$	$e == \lim_{z\to\infty} \frac{4z}{{(z)_{z+1}}^{1/z}}$	$\gamma == \lim_{n\to\infty} \left(\log(p_n)\right.$ $\left.\sum_{k=1}^{n} \frac{\log(p_k)}{p_k - 1}\right)$
	$\pi == 16 \lim_{n\to\infty} (n+1) \prod_{k=1}^{n} \frac{k^2}{(2k+1)^2}$	$e == \lim_{n\to\infty} \left(\prod_{k=1}^{\pi(n)} p_k\right)^{\frac{1}{p_n}}$ /; $p_n ==$ prime(n)	$\gamma = \lim_{n\to\infty} \frac{\sum_{k=1}^{n}\left(\left\lceil \frac{n}{k} \right\rceil - \right.}{n}$
			$\gamma == \lim_{x\to 0} \left(\mathrm{Ei}(\log(x\right.$ $\mathrm{Ei}(\log(x+1))$ $\log\left(1 - \frac{1}{x}\right) + \pi$

Continued fraction representations

The four classical constants ϕ, π, e, and C have numerous closed-form continued fraction representations, for example:

$$\phi == 1 + \mathrm{K}_k(1, 1)_1^\infty == 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}$$

$$\pi == 3 + \mathrm{K}_k\left((2k-1)^2, 6\right)_1^\infty == 3 + \cfrac{1}{6 + \cfrac{9}{6 + \cfrac{25}{6 + \cfrac{49}{6 + \cfrac{81}{6 + \cfrac{121}{6 + \ldots}}}}}}$$

$$\frac{\pi}{2} = 1 - \frac{1}{3 + \mathrm{K}_k\left(-\left(k - (-1)^k\right)\left(k - (-1)^k + 1\right), 2 + (-1)^k\right)_1^\infty} = 1 - \cfrac{1}{3 - \cfrac{6}{1 - \cfrac{2}{3 - \cfrac{20}{1 - \cfrac{12}{3 - \cfrac{42}{1 - \cfrac{30}{3 - \ldots}}}}}}}$$

$$\frac{4}{\pi} = 1 + \mathrm{K}_k\left(k^2, 2k + 1\right)_1^\infty = 1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cfrac{16}{9 + \cfrac{25}{11 + \cfrac{36}{13 + \ldots}}}}}}$$

$$e = 2 + \mathrm{K}_k\left(1, \left(\frac{2(k+1)}{3}\right)^{\frac{1}{2}\left(1 - (-1)^{(k+2) \bmod 3}\right)}\right)_1^\infty = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \ldots}}}}}}$$

$$e = 1 + \frac{1}{\mathrm{K}_k(k, k)_1^\infty} = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \cfrac{6}{6 + \cfrac{7}{7 + \ldots}}}}}}$$

$$e = 1 + \frac{2}{1 + \mathrm{K}_k(1, 4k + 2)_1^\infty} = 1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{22 + \cfrac{1}{26 + \ldots}}}}}}}$$

$$C = 1 - \frac{1}{2\left(3 + \mathrm{K}_k\left(\left(2\left\lfloor\frac{k+1}{2}\right\rfloor\right)^2, 3^{(k-1) \bmod 2}\right)_1^{\infty}\right)} = 1 - \cfrac{1/2}{3 + \cfrac{4}{1 + \cfrac{4}{3 + \cfrac{16}{1 + \cfrac{16}{3 + \cfrac{36}{1 + \cfrac{36}{3 + \ldots}}}}}}}$$

$$C = \frac{1}{2} + \frac{1}{1 + 2\,\mathrm{K}_k\left(\frac{1}{16}\left(\left((-1)^k - 1\right)^2 (k+1)^2 + 2\left(1 + (-1)^k\right) k\,(k+2)\right), \frac{1}{2}\right)_1^{\infty}}. = \frac{1}{2} + \cfrac{1/2}{\cfrac{1}{2} + \cfrac{1}{\cfrac{1}{2} + \cfrac{2}{\cfrac{1}{2} + \cfrac{4}{\cfrac{1}{2} + \cfrac{6}{\cfrac{1}{2} + \cfrac{9}{\cfrac{1}{2} + \cfrac{12}{\cfrac{1}{2} + \ldots}}}}}}}.$$

Functional identities

The golden ratio ϕ satisfies the following special functional identities:

$$\phi^2 - \phi - 1 = 0$$

$$\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1 + \frac{1}{\phi}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}}} = \ldots$$

$$\phi^n = \phi^{n-1} + \phi^{n-2} \;/;\; n \in \mathbb{N}^+$$

$$\phi^{\phi^{2+\phi}} = \phi^{\phi^{\phi\,(1+\phi)}} = \phi^{\phi^{1+\phi^2}} = \phi^{\phi^{2+\phi}}.$$

Complex characteristics

The eight classical constants (ϕ, π, $°$, e, γ, C, A, and K) and the imaginary unit i have the following complex characteristics:

	Abs	**Arg**	**Re**	**Im**	**Conjugate**	**Sign**
$\boldsymbol{\xi}$	$\lvert\xi\rvert = \xi$	$\mathrm{Arg}(\xi) = 0$	$\mathrm{Re}(\xi) = \xi$	$\mathrm{Im}(\xi) = 0$	$\overline{\xi} = \xi$	$\mathrm{sgn}(\xi) = 1$
$\boldsymbol{i}$	$\lvert i\rvert = 1$	$\mathrm{Arg}(i) = \frac{\pi}{2}$	$\mathrm{Re}(i) = 0$	$\mathrm{Im}(i) = 1$	$\overline{i} = -i$	$\mathrm{sgn}(i) = i$

$/;\; \xi \in \{\phi, \pi, °, e, \gamma, C, A, K\}$

Differentiation

Derivatives of the eight classical constants (ϕ, π, $°$, e, γ, C, A, and K) and imaginary unit constant i satisfy the following relations:

	$\frac{\partial}{\partial z}$ (in classical sense)	$\frac{\partial^{\alpha}}{\partial z^{\alpha}}$ (in fractional sense)
ξ	$\frac{\partial \xi}{\partial z} == 0$	$\frac{\partial^{\alpha} \xi}{\partial z^{\alpha}} == \frac{\xi z^{-\alpha}}{\Gamma(1-\alpha)}$
i	$\frac{\partial i}{\partial z} == 0$	$\frac{\partial^{\alpha} i}{\partial z^{\alpha}} == \frac{i z^{-\alpha}}{\Gamma(1-\alpha)}$

/; $\xi \in \{\phi, \pi, \circ, e, \gamma, C, A, K\}$

Integration

Simple indefinite integrals of the eight classical constants (ϕ, π, $\circ$, e, γ, C, A, and K) and imaginary unit constant i have the following values:

	$\int f(z)\, dz$	$\int z^{\alpha-1} f(z)\, dz$
ξ	$\int \xi\, dz == \xi z$	$\int z^{\alpha-1} \xi\, dz == \frac{\xi z^{\alpha}}{\alpha}$
i	$\int i\, dz == i z$	$\int z^{\alpha-1} i\, dz == \frac{i z^{\alpha}}{\alpha}$

/; $\xi \in \{\phi, \pi, \circ, e, \gamma, C, A, K\}$

Integral transforms

All Fourier integral transforms and Laplace direct and inverse integral transforms of the eight classical constants (ϕ, π, $\circ$, e, γ, C, A, and K) and the imaginary unit i can be evaluated in a distributional or classical sense and can include the Dirac delta function:

$f(t)$	$F_t[f(t)](z)$	$F_t^{-1}[f(t)](z)$	$Fc_t[f(t)](z)$	$Fs_t[f(t)](z)$	$L_t[f(t)](z)$	$L_t^{-1}[$
ξ	$\mathcal{F}_t[\xi](z) == \sqrt{2\pi}\, \xi\, \delta(z)$	$\mathcal{F}_t^{-1}[\xi](z) == \sqrt{2\pi}\, \xi\, \delta(z)$	$\mathcal{F}c_t[\xi](z) == \sqrt{\frac{\pi}{2}}\, \xi\, \delta(z)$	$\mathcal{F}s_t[\xi](z) == \sqrt{\frac{2}{\pi}}\, \frac{\xi}{z}$	$\mathcal{L}_t[\xi](z) == \frac{\xi}{z}$	$\mathcal{L}_t^{-1}[\xi$
i	$\mathcal{F}_t[i](z) == \sqrt{2\pi}\, i\, \delta(z)$	$\mathcal{F}_t^{-1}[i](z) == \sqrt{2\pi}\, i\, \delta(z)$	$\mathcal{F}c_t[i](z) == \sqrt{\frac{\pi}{2}}\, i\, \delta(z)$	$\mathcal{F}s_t[i](z) == \sqrt{\frac{2}{\pi}}\, \frac{i}{z}$	$\mathcal{L}_t[i](z) == \frac{i}{z}$	$\mathcal{L}_t^{-1}[i$

/; $\xi \in \{\phi, \pi, \circ, e, \gamma, C, A, K\}$

Inequalities

The eight classical constants (ϕ, π, $\circ$, e, γ, C, A, and K) satisfy numerous inequalities, for example:

$$1 + \frac{3}{5} < \phi < 1 + \frac{31}{50}$$

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}$$

$$\frac{1}{60} < \circ < \frac{1}{57}$$

$$2 + \frac{7}{10} < e < 2 + \frac{3}{4}$$

$$e^{\pi} \geq \pi^{e}$$

$$\left(1 + \frac{1}{n}\right)^{n + \frac{1}{\log(2)} - 1} \leq e \leq \left(1 + \frac{1}{n}\right)^{n + \frac{1}{2}} /; n \in \mathbb{N}^{+}$$

$$\frac{1}{2} < \gamma < \frac{3}{5}$$

$$\frac{9}{10} < C < 1$$

$$1 + \frac{1}{4} < \boldsymbol{A} < 1 + \frac{3}{10}$$

$$2 + \frac{13}{20} < K < 2 + \frac{7}{10}.$$

Applications of classical constants and the imaginary unit

All classical constants and the imaginary unit are used throughout mathematics, the exact sciences, and engineering.

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