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Introductions to PolyGamma2

Introduction to the differentiated gamma functions

General

Almost simultaneously with the development of the mathematical theory of factorials, binomials, and gamma functions in the 18th century, some mathematicians introduced and studied related special functions that are basically derivatives of the gamma function. These functions appeared in coefficients of the series expansions of the solutions in the logarithmic cases of some important differential equations. They appear in the Bessel differential equation for example. So functions in this group are called the differentiated gamma functions.

The harmonic numbers $H_n \left(H_1 == 1, H_2 == 1 + \frac{1}{2}, H_3 == 1 + \frac{1}{2} + \frac{1}{3}, H_4 == 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \ldots\right)$ for integer n have a very long history. The famous Pythagoras of Samos (569–475 B.C.) was the first to encounter the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} == 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots 3$$

in connection with string vibrations and his special interest in music.

Richard Suiseth (14th century) and Nicole d'Oresme (1350) studied the harmonic series and discovered that it diverges. Pietro Mengoli (1647) proved the divergence of the harmonic series. Nicolaus Mercator (1668) studied the harmonic series corresponding to the series of $\log(1+z)$ and Jacob Bernoulli (1689) again proved the divergence of the harmonic series. The harmonic numbers H_n with integer n also appeared in an article of G. W. Leibniz (1673).

In his famous work, J. Stirling (1730) not only found the asymptotic formula for factorial $n!$, but used the digamma psi function $\psi(z)$ (related to the harmonic numbers), which is equal to the derivative of the logarithm from the gamma function ($\psi(z) == \partial \log(\Gamma(z)) / \partial z$). Later L. Euler (1740) also used harmonic numbers and introduced the generalized harmonic numbers $H_n^{(r)}$.

The digamma function $\psi(z)$ and its derivatives $\psi^{(n)}(z)$ of positive integer orders n were widely used in the research of A. M. Legendre (1809), S. Poisson (1811), C. F. Gauss (1810), and others. M. A. Stern (1847) proved the convergence of the Stirling series for the digamma function:

$$\psi(z) == -\gamma + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \prod_{j=1}^{k} (z-j)}{k\, k!}.$$

At the end of the 20th century, mathematicians began to investigate extending the function $\psi^{(n)}(z)$ to all complex values of n (B. Ross (1974), N. Grossman (1976)). R. W. Gosper (1997) defined and studied the cases $n = -2$ and -3. V. S. Adamchik (1998) suggested the definition $\psi^{(n)}(z)$ for complex n using Liouville's fractional integration operator. A natural extension of $\psi^{(n)}(z)$ for the complex order $n = \nu$ was recently suggested by O. I. Marichev (2001) during the development of subsections with fractional integro-differentiation for the Wolfram Functions website and the technical computing system *Mathematica*:

$$\psi^{(\nu)}(z) = z^{-\nu}\left(\frac{\gamma - \log(z) + \psi(-\nu)}{z\,\Gamma(-\nu)} - \frac{\gamma}{\Gamma(1-\nu)}\right) + z^{1-\nu}\sum_{k=1}^{\infty}\frac{1}{k^2}\,{}_2\tilde{F}_1\left(1, 2; 2-\nu; -\frac{z}{k}\right).$$

Definitions of the differentiated gamma functions

The digamma function $\psi(z)$, polygamma function $\psi^{(\nu)}(z)$, harmonic number H_z, and generalized harmonic number $H_z^{(r)}$ are defined by the following formulas (the first formula is a general definition for complex arguments and the second formula is for positive integer arguments):

$$\psi(z) = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z-1}\right) - \gamma$$

$$\psi(n) = \sum_{k=1}^{n-1}\frac{1}{k} - \gamma \;/;\; n \in \mathbb{N}^+.$$

Here γ is the Euler gamma constant $\gamma = 0.577216\ldots$:

$$\psi^{(\nu)}(z) = z^{-\nu}\left(\frac{\gamma - \log(z) + \psi(-\nu)}{z\,\Gamma(-\nu)} - \frac{\gamma}{\Gamma(1-\nu)}\right) + z^{1-\nu}\sum_{k=1}^{\infty}\frac{1}{k^2}\,{}_2\tilde{F}_1\left(1, 2; 2-\nu; -\frac{z}{k}\right).$$

Remark: This formula presents the (not unique) continuation of the classical definition of $\psi^{(\nu)}(z)$ from positive integer values of ν to its arbitrary complex values.

For positive integer n and arbitrary complex z, r, the following definitions are commonly used:

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\sum_{k=0}^{\infty}\frac{1}{(k+z)^{n+1}} \;/;\; n \in \mathbb{N}^+ \land \operatorname{Re}(z) > 0$$

$$H_z = \psi(z+1) + \gamma$$

$$H_n = \sum_{k=1}^{n}\frac{1}{k} \;/;\; n \in \mathbb{N}^+$$

$$H_z^{(r)} = \zeta(r) - \zeta(r, z+1)$$

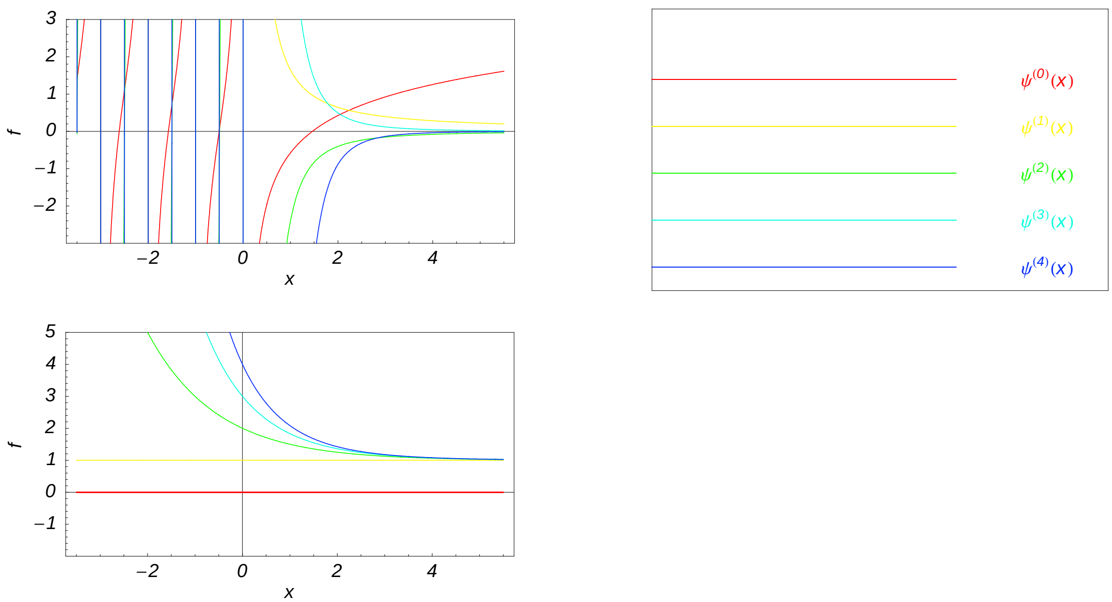
$$H_n^{(r)} = \sum_{k=1}^{n}\frac{1}{k^r} \;/;\; n \in \mathbb{N}.$$

The previous definition for $H_z^{(r)}$ uses the *Mathematica* definition for the Hurwitz zeta function $\zeta(r, z)$. Branch cuts and related properties are thus inherited from $\zeta(r, z)$.

The previous functions are interconnected and belong to the differentiated gamma functions group. These functions are widely used in the coefficients of series expansions for many mathematical functions (especially the so-called logarithmic cases).

A quick look at the differentiated gamma functions

Here is a quick look at the graphics for the differentiated gamma functions along the real axis.



Connections within the group of differentiated gamma functions and with other function groups

Representations through more general functions

The differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ are particular cases of the more general hypergeometric and Meijer G functions. Although the arguments of these functions do not depend on the variable z, it is included in their parameters.

For example, $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ can be represented through generalized hypergeometric functions ${}_{p+1}F_p$ by the following formulas:

$$\psi(z) = (z-1)\,{}_3F_2(1, 1, 2-z; 2, 2; 1) - \gamma$$

$$\psi^{(1)}(z) = (z-1)^2\,{}_3F_2(1, 1, 2-z; 2, 2; 1)^2 - 2\,(z-1)\,{}_4F_3(1, 1, 1, 2-z; 2, 2, 2; 1) + \frac{\pi^2}{6}$$

$$\psi^{(n)}(z) = (-1)^{n+1}\,n!\,z^{-n-1}\,{}_{n+2}F_{n+1}(1, a_1, a_2, \ldots, a_{n+1}; a_1+1, a_2+1, \ldots, a_{n+1}+1; 1) \,/;\, a_1 = a_2 = \ldots = a_{n+1} = z \land n \in \mathbb{N}^+$$

$$H_z = z\,{}_3F_2(1, 1, 1-z; 2, 2; 1)$$

$$H_z^{(2)} = 2\,z\,{}_4F_3(1, 1, 1, 1 - z; 2, 2, 2; 1) - z^2\,{}_3F_2(1, 1, 1 - z; 2, 2; 1)^2$$

$$H_z^{(r)} = {}_{r+1}F_r(1, a_1, a_2, \ldots, a_r; a_1 + 1, a_2 + 1, \ldots, a_r + 1; 1) - (z+1)^{-r}\,{}_{r+1}F_r(1, b_1, b_2, \ldots, b_r; b_1 + 1, b_2 + 1, \ldots, b_r + 1; 1) \,/;\, a_1 = a_2 = \ldots = a_r = 1 \bigwedge b_1 = b_2 = \ldots = b_r = z + 1 \bigwedge r - 1 \in \mathbb{N}^+.$$

The aforementioned general formulas can be rewritten using the classical Meijer G functions as follows:

$$\psi(z) = -\frac{1}{\Gamma(1-z)}\, G_{3,3}^{1,3}\!\left(-1 \,\middle|\, \begin{matrix} 0, 0, z-1 \\ 0, -1, -1 \end{matrix}\right) - \gamma$$

$$\psi^{(n)}(z) = (-1)^{n+1}\, n!\, G_{n+2,n+2}^{1,n+2}\!\left(-1 \,\middle|\, \begin{matrix} 0, 1-z, \ldots, 1-z \\ 0, -z, \ldots, -z \end{matrix}\right) /;\, n \in \mathbb{N}^+$$

$$H_z = -\frac{1}{\Gamma(-z)}\, G_{3,3}^{1,3}\!\left(-1 \,\middle|\, \begin{matrix} 0, 0, z \\ 0, -1, -1 \end{matrix}\right)$$

$$H_z^{(r)} = G_{r+1,r+1}^{1,r+1}\!\left(-1 \,\middle|\, \begin{matrix} 0, 1-a_1, \ldots, 1-a_r \\ 0, -a_1, \ldots, -a_r \end{matrix}\right) - G_{r+1,r+1}^{1,r+1}\!\left(-1 \,\middle|\, \begin{matrix} 0, 1-b_1, \ldots, 1-b_r \\ 0, -b_1, \ldots, -b_r \end{matrix}\right) /;\, a_1 = a_2 = \ldots = a_r = 1 \bigwedge b_1 = b_2 = \ldots = b_r = z + 1 \bigwedge r - 1 \in \mathbb{N}^+.$$

Representations through related equivalent functions

The differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ can be represented through derivatives of the gamma function $\Gamma(z)$:

$$\psi(z) = \frac{1}{\Gamma(z)} \frac{\partial \Gamma(z)}{\partial z}$$

$$\psi^{(n)}(z) = \frac{\partial^n}{\partial z^n}\left(\frac{\frac{\partial \Gamma(z)}{\partial z}}{\Gamma(z)}\right) /;\, n \in \mathbb{N}$$

$$H_z = \frac{1}{\Gamma(z+1)} \frac{\partial \Gamma(z+1)}{\partial z} + \gamma$$

$$H_z^{(r)} = \frac{(-1)^r}{(r-1)!}\left(\left(\frac{\partial^{r-1}}{\partial z^{r-1}} \frac{\frac{\partial \Gamma(z+1)}{\partial z}}{\Gamma(z+1)} \Big|_{z\to 0}\right) - \frac{\partial^{r-1}}{\partial z^{r-1}} \frac{\frac{\partial \Gamma(z+1)}{\partial z}}{\Gamma(z+1)}\right) /;\, r \in \mathbb{N}^+.$$

The differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ can also be represented through derivatives of the logarithm of the gamma function $\log\Gamma(z)$:

$$\psi(z) = \frac{\partial \log\Gamma(z)}{\partial z}$$

$$\psi^{(n)}(z) = \frac{\partial^{n+1} \log\Gamma(z)}{\partial z^{n+1}} /;\, n \in \mathbb{N}$$

$$H_z = \frac{\partial \log\Gamma(z+1)}{\partial z} + \gamma$$

$$H_z^{(r)} = \frac{(-1)^r}{(r-1)!}\left(\left(\frac{\partial^r \log\Gamma(z+1)}{\partial z^r}\,\Big|_{z\to 0}\right) - \frac{\partial^r \log\Gamma(z+1)}{\partial z^r}\right) /;\ r \in \mathbb{N}^+.$$

The functions $\psi^{(n)}(z)$ and $H_z^{(r)}$ are intimately related to the Hurwitz zeta function $\zeta(s, a)$ and the Bernoulli polynomials and numbers $B_m(z+1)$, B_m by the formulas:

$$\psi^{(n)}(z) = (-1)^{n+1}\, n!\, \zeta(n+1, z) /;\ n \in \mathbb{N}^+ \bigwedge \operatorname{Re}(z) > 0$$

$$H_z^{(r)} = \zeta(r) - \zeta(r, z+1)$$

$$H_n^{(-m)} = \frac{B_{m+1}(n+1) - B_{m+1}}{m+1} /;\ m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}.$$

Representations through other differentiated gamma functions

The differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ are interconnected through the following formulas:

$$\psi(z) = \psi^{(0)}(z)$$

$$\psi(z) = H_{z-1} - \gamma$$

$$\psi^{(n)}(z) = \frac{\partial^n \psi(z)}{\partial z^n} /;\ n \in \mathbb{N}^+$$

$$H_n = H_n^{(1)}$$

$$H_z = \psi(z+1) + \gamma$$

$$H_z^{(r)} = \frac{(-1)^r}{(r-1)!}\left(\psi^{(r-1)}(1) - \psi^{(r-1)}(z+1)\right) /;\ r \in \mathbb{N}^+.$$

The best-known properties and formulas for differentiated gamma functions

Real values for real arguments

For real values of the argument z and nonnegative integer n, the differentiated gamma functions $\psi(z)$, $\psi^{(n)}(z)$, H_z, and $H_z^{(n)}$ are real (or infinity). The function $H_z^{(n)}$ is real (or infinity) for real values of argument z and integer n.

Simple values at zero

The differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ have simple values for zero arguments:

$$\psi(0) = \tilde{\infty}$$

$$\psi^{(0)}(0) = \tilde{\infty}$$

$$\psi^{(\nu)}(0) = 0 /;\ \operatorname{Re}(\nu) < -1$$

$$\psi^{(\nu)}(0) = \tilde{\infty} /;\ \operatorname{Re}(\nu) > -1$$

$$\psi^{(0)}(z) = \psi(z)$$

$$H_0 = 0$$

$H_0^{(r)} == 0$

$H_z^{(0)} == z.$

Values at fixed points

The differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ with rational arguments can sometimes be evaluated through classical constants and logarithms, for example:

$\psi(-1) == \tilde{\infty}$

$\psi\left(-\frac{1}{2}\right) == 2 - \gamma - \log(4)$

$\psi(0) == \tilde{\infty}$

$\psi\left(\frac{1}{2}\right) == -\gamma - \log(4)$

$\psi(1) == -\gamma$

$\psi\left(\frac{3}{2}\right) == 2 - \gamma - \log(4)$

$\psi^{(1)}\left(\frac{1}{4}\right) == 8\, C + \pi^2$

$H_{-1} == \tilde{\infty}$

$H_{-\frac{1}{2}} == -\log(4)$

$H_0 == 0$

$H_{\frac{1}{2}} == 2 - \log(4)$

$H_1 == 1$

$H_{\frac{3}{2}} == \frac{8}{3} - \log(4).$

Specific values for specialized variables

The previous relations are particular cases of the following general formulas:

$\psi(n) == \sum_{k=1}^{n-1} \frac{1}{k} - \gamma \,/;\, n \in \mathbb{N}$

$\psi(-n) == \tilde{\infty} \,/;\, n \in \mathbb{N}$

$\psi\left(\frac{1}{2} + n\right) == \sum_{k=1}^{n-1} \frac{1}{k} + \sum_{k=n}^{2n-1} \frac{2}{k} - \log(4) - \gamma \,/;\, n \in \mathbb{N}$

$$\psi\left(\frac{1}{2}-n\right) == \sum_{k=1}^{n-1}\frac{1}{k}+\sum_{k=n}^{2n-1}\frac{2}{k}-\log(4)-\gamma\ /;\ n\in\mathbb{N}$$

$$\psi\left(\frac{p}{q}+n\right) == q\sum_{k=0}^{n-1}\frac{1}{p+k\,q}+2\sum_{k=1}^{\left\lfloor\frac{q-1}{2}\right\rfloor}\cos\left(\frac{2\pi p k}{q}\right)\log\left(\sin\left(\frac{\pi k}{q}\right)\right)-\frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right)-\log(2q)-\gamma\ /;\ n\in\mathbb{N}\bigwedge p\in\mathbb{N}^+\bigwedge q\in\mathbb{N}^+\bigwedge p<q$$

$$\psi\left(\frac{p}{q}-n\right) == q\sum_{k=0}^{n-1}\frac{1}{q(k+1)-p}+2\sum_{k=1}^{\left\lfloor\frac{q-1}{2}\right\rfloor}\cos\left(\frac{2\pi p k}{q}\right)\log\left(\sin\left(\frac{\pi k}{q}\right)\right)-\frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right)-\log(2q)-\gamma\ /;$$
$$n\in\mathbb{N}\bigwedge p\in\mathbb{N}^+\bigwedge q\in\mathbb{N}^+\bigwedge p<q$$

$$H_n == \sum_{k=1}^{n}\frac{1}{k}\ /;\ n\in\mathbb{N}$$

$$H_{-n} == \tilde{\infty}\ /;\ n\in\mathbb{N}^+$$

$$H_{\frac{1}{2}+n} == \sum_{k=1}^{n-1}\frac{1}{k}+\sum_{k=n}^{2n-1}\frac{2}{k}+\frac{2}{2n+1}-\log(4)-\gamma\ /;\ n\in\mathbb{N}$$

$$H_{\frac{1}{2}-n} == \sum_{k=1}^{n-1}\frac{1}{k}+\sum_{k=n}^{2n-1}\frac{2}{k}+\frac{2}{1-2n}-\log(4)\ /;\ n\in\mathbb{N}$$

$$H_{\frac{p}{q}+n} == q\sum_{k=0}^{n}\frac{1}{p+k\,q}+2\sum_{k=1}^{\left\lfloor\frac{q-1}{2}\right\rfloor}\cos\left(\frac{2\pi p k}{q}\right)\log\left(\sin\left(\frac{\pi k}{q}\right)\right)-\frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right)-\log(2q)\ /;\ n\in\mathbb{N}\bigwedge p\in\mathbb{N}^+\bigwedge q\in\mathbb{N}^+\bigwedge p<q$$

$$H_{\frac{p}{q}-n} == q\sum_{k=0}^{n-2}\frac{1}{q(k+1)-p}+2\sum_{k=1}^{\left\lfloor\frac{q-1}{2}\right\rfloor}\cos\left(\frac{2\pi p k}{q}\right)\log\left(\sin\left(\frac{\pi k}{q}\right)\right)-\frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right)-\log(2q)\ /;\ n\in\mathbb{N}^+\bigwedge p\in\mathbb{N}^+\bigwedge q\in\mathbb{N}^+\bigwedge p<q.$$

The differentiated gamma functions $\psi^{(\nu)}(z)$ and $H_z^{(r)}$ with integer parameters ν and r have the following representations:

$$\psi^{(-1)}(z) == \log\Gamma(z)$$

$$\psi^{(0)}(z) == \psi(z)$$

$$\psi^{(n)}(z) == \frac{\partial^n\psi(z)}{\partial z^n}\ /;\ n\in\mathbb{N}^+$$

$$H_z^{(-2)} == \frac{1}{6}z(z+1)(2z+1)$$

$$H_z^{(-1)} == \frac{1}{2}z(z+1)$$

$$H_z^{(0)} == z$$

$$H_z^{(1)} == H_z$$

$H_0^{(r)} == 0$

$H_1^{(r)} == 1$

$H_2^{(r)} == 1 + 2^{-r}$

$H_3^{(r)} == 1 + 2^{-r} + 3^{-r}.$

Analyticity

The digamma function $\psi(z)$ and the harmonic number H_z are defined for all complex values of the variable z. The functions $\psi^{(t)}(z)$ and $H_z^{(r)}$ are analytical functions of r and z over the whole complex r- and z-planes. For fixed z, the generalized harmonic number $H_z^{(r)}$ is an entire function of r.

Poles and essential singularities

The differentiated gamma functions $\psi(z)$ and H_z have an infinite set of singular points $z == -k$, where $k \in \mathbb{N}$ for $\psi(z)$ and $k \in \mathbb{N}^+$ for H_z. These points are the simple poles with residues -1. The point $z == \tilde{\infty}$ is the accumulation point of poles for the functions $\psi(z)$, H_z, and $\psi^{(\nu)}(z)$ (with fixed nonnegative integer ν), which means that $\tilde{\infty}$ is an essential singular point.

For fixed nonnegative integer ν, the function $\psi^{(\nu)}(z)$ has an infinite set of singular points: $z == -m /; \nu == 0 \bigwedge m \in \mathbb{N}$ are the simple poles with residues -1; and $z == -m /; \nu > 0 \bigwedge m \in \mathbb{N}$ are the poles of order $\nu + 1$ with residues 0.

For fixed z, the function $\psi^{(\nu)}(z)$ does not have poles and the function $H_z^{(r)}$ has only one singular point at $r = \tilde{\infty}$, which is an essential singular point.

Branch points and branch cuts

The functions $\psi(z)$ and H_z do not have branch points and branch cuts.

For integer ν, the function $\psi^{(\nu)}(z)$ does not have branch points and branch cuts.

For fixed noninteger ν, the function $\psi^{(\nu)}(z)$ has two singular branch points $z == 0$ and $z == \tilde{\infty}$, and it is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above:

$$\lim_{\epsilon \to +0} \psi^{(\nu)}(x - i\,\epsilon) == \psi^{(\nu)}(x) /; x < 0$$

$$\lim_{\epsilon \to +0} \psi^{(\nu)}(x - i\,\epsilon) == e^{2 i \pi \nu} \left(\psi^{(\nu)}(x) + \frac{2\, i\, \pi\, x^{-\nu-1}}{\Gamma(-\nu)}\right) /; x < 0.$$

For fixed z, the functions $\psi^{(\nu)}(z)$ and $H_z^{(r)}$ do not have branch points and branch cuts.

Periodicity

The differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ do not have periodicity.

Parity and symmetry

The differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ have mirror symmetry:

$$\psi(\bar{z}) == \overline{\psi(z)}$$

$$\psi^{(\bar{\nu})}(\bar{z}) == \overline{\psi^{(\nu)}(z)}$$

$$H_{\bar{z}} == \overline{H_z}$$

$$H_{\bar{z}}^{(\bar{r})} == \overline{H_z^{(r)}}.$$

Series representations

The differentiated gamma functions $\psi(z)$, H_z, and $H_z^{(r)}$ have the following series expansions near regular points:

$$\psi(z) \propto \psi(z_0) + \zeta(2, z_0)\,(z - z_0) - \zeta(3, z_0)\,(z - z_0)^2 + \ldots \;/;\; (z \to z_0) \bigwedge \neg\,(z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

$$\psi(z) == \psi(z_0) + \sum_{j=0}^{\infty} (-1)^j\,\zeta(j + 2, z_0)\,(z - z_0)^{j+1} \;/;\; \neg\,(z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

$$\psi(z) == \psi(z_0) + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j\,(z - z_0)^{j+1}}{(k + z_0)^{j+2}} \;/;\; \neg\,(z_0 \in \mathbb{Z} \bigwedge z_0 \leq 0)$$

$$H_z \propto \frac{\pi^2\,z}{6} - \zeta(3)\,z^2 + \frac{\pi^4\,z^3}{90} - \ldots \;/;\; (z \to 0)$$

$$H_z == \sum_{j=0}^{\infty} (-1)^j\,\zeta(j + 2)\,z^{j+1} \;/;\; |z| < 1$$

$$H_z \propto H_{z_0} + \zeta(2, z_0 + 1)\,(z - z_0) - \zeta(3, z_0 + 1)\,(z - z_0)^2 + \ldots \;/;\; (z -> z_0)$$

$$H_z == H_{z_0} + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j\,(z - z_0)^{j+1}}{(k + z_0 + 1)^{j+2}}$$

$$H_z^{(r)} == r\,\zeta(r + 1)\,z - \frac{1}{2}\,r\,(r + 1)\,\zeta(r + 2)\,z^2 + \ldots \;/;\; \mathrm{Re}(r) > 1 \land |z| < 1$$

$$H_z^{(r)} == \sum_{j=1}^{\infty} \frac{(-1)^{j-1}\,(r)_j\,\zeta(j + r)\,z^j}{j!} \;/;\; \mathrm{Re}(r) > 1 \land |z| < 1$$

$$H_z^{(r)} \propto H_z + \sum_{k=1}^{\infty} \left(\frac{\log(k + z)}{k + z} - \frac{\log(k)}{k}\right)(r - 1) + \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{\log^2(k + z)}{k + z} - \frac{\log^2(k)}{k}\right)(r - 1)^2 + \ldots \;/;\; (r \to 1).$$

Near singular points, the differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ can be expanded through the following series:

$$\psi(z) \propto -\frac{1}{z} - \gamma + \frac{\pi^2\,z}{6} - \zeta(3)\,z^2 + \frac{\pi^4\,z^3}{90} - \ldots \;/;\; (z \to 0)$$

$$\psi(z) == -\frac{1}{z} - \gamma + \sum_{j=0}^{\infty} (-1)^j \zeta(j+2)\, z^{j+1} \;/;\; |z| < 1$$

$$\psi(z) \propto -\frac{1}{z+n} + \psi(n+1) + \left(\frac{\pi^2}{3} - \zeta(2, n+1)\right)(z+n) - \zeta(3, n+1)\,(z+n)^2 + \ldots \;/;\; (z \to -n) \land n \in \mathbb{N}$$

$$\psi(z) == -\frac{1}{z+n} + \psi(n+1) + \sum_{k=1}^{\infty} \left(\frac{\psi^{(k)}(1)}{k!} + \zeta(k+1) - \zeta(k+1, n+1)\right)(z+n)^k \;/;\; |z+n| < 1 \land n \in \mathbb{N}$$

$$\psi^{(\nu)}(z) \propto -\mathcal{FC}_{\exp}^{(\nu)}(z, -1)\, z^{-\nu-1} - \frac{\gamma\, z^{-\nu}}{\Gamma(1-\nu)} + \frac{\pi^2\, z^{1-\nu}}{6\,\Gamma(2-\nu)} \left(1 - \frac{12\,\zeta(3)\, z}{\pi^2\,(2-\nu)} + \frac{2\,\pi^2\, z^2}{5\,(2-\nu)\,(3-\nu)} + \ldots\right) /;\; (z \to 0) \land \mathrm{Re}(\nu) > 0$$

$$\psi^{(\nu)}(z) == -\mathcal{FC}_{\exp}^{(\nu)}(z, -1)\, z^{-\nu-1} - \frac{\gamma\, z^{-\nu}}{\Gamma(1-\nu)} + z^{1-\nu} \sum_{k=1}^{\infty} \frac{1}{k^2}\, {}_2\tilde{F}_1\!\left(1, 2; 2-\nu; -\frac{z}{k}\right)$$

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, n!}{z^{n+1}} + \psi^{(n)}(1) + \psi^{(n+1)}(1)\, z + \frac{1}{2}\, \psi^{(n+2)}(1)\, z^2 + \ldots \;/;\; (z \to 0) \bigwedge n \in \mathbb{N}^+$$

$$\psi^{(n)}(z) == \frac{(-1)^{n-1}\, n!}{z^{n+1}} + \sum_{k=0}^{\infty} \frac{\psi^{(k+n)}(1)\, z^k}{k!} \;/;\; |z| < 1 \bigwedge n \in \mathbb{N}^+$$

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}\, n!}{(z+m)^{n+1}} + \Gamma\,(n+1)\, H_m^{(n+1)} + \psi^{(n)}(1) + \left(\Gamma(n+2)\, H_m^{(n+2)} + \psi^{(n+1)}(1)\right)(z+m) + \frac{1}{2}\left(\Gamma(n+3)\, H_m^{(n+3)} + \psi^{(n+2)}(1)\right)(z+m)^2 + \ldots \;/;\; (z \to -m) \bigwedge m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+$$

$$\psi^{(n)}(z) == \frac{(-1)^{n-1}\, n!}{(z+m)^{n+1}} + \sum_{k=0}^{\infty} \frac{\psi^{(k+n)}(1) + \Gamma(k+n+1)\, H_m^{(k+n+1)}}{k!}\,(z+m)^k \;/;\; |z+m| < 1 \bigwedge m \in \mathbb{N} \bigwedge n \in \mathbb{N}^+$$

$$H_z \propto -\frac{1}{z+n} + H_{n-1} + \left(\frac{\pi^2}{3} - \zeta(2, n)\right)(z+n) - \zeta(3, n)\,(z+n)^2 + \ldots \;/;\; (z \to -n) \bigwedge n \in \mathbb{N}^+$$

$$H_z == -\frac{1}{z+n} + H_{n-1} + \sum_{k=1}^{\infty} \left(\frac{\psi^{(k)}(1)}{k!} + \zeta(k+1) - \zeta(k+1, n)\right)(z+n)^k \;/;\; |z+n| < 1 \bigwedge n \in \mathbb{N}^+$$

$$H_z^{(r)} == -\frac{1}{(z+m)^r} + (-1)^{r-1}\, H_m^{(r)} - \frac{(-1)^r}{(r-1)!} \sum_{k=1}^{\infty} \frac{\Gamma(k+r)\, H_m^{(k+r)} + \psi^{(k+r-1)}(1)}{k!}\,(z+m)^k \;/;\; |z+m| < 1 \bigwedge m \in \mathbb{N}^+ \bigwedge r \in \mathbb{N}^+.$$

Here γ is the Euler gamma constant $\gamma == 0.577216\ldots$.

Except for the generalized power series, there are other types of series through which differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ can be represented, for example:

$$\psi(z) == -\frac{1}{z} + z \sum_{k=0}^{\infty} \frac{1}{(k+1)\,(k+z+1)} - \gamma$$

$$\psi^{(n)}(z) = (-1)^{n+1}\, n!\, \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}} \;/;\; n \in \mathbb{N}^+$$

$$H_z = z \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+z+1)}$$

$$H_z^{(r)} = \sum_{k=1}^{\infty} \left(\frac{1}{k^r} - \frac{1}{(k+z)^r} \right) /;\; \mathrm{Re}(r) > 1.$$

Asymptotic series expansions

The asymptotic behavior of the differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ can be described by the following formulas (only the main terms of the asymptotic expansions are given):

$$\psi(z) \propto \log(z) - \frac{1}{2z} - \frac{1}{12z^2}\left(1 + O\left(\frac{1}{z^2}\right)\right) /;\; |\mathrm{Arg}(z)| < \pi \bigwedge (|z| \to \infty)$$

$$\psi^{(n)}(z) \propto \frac{(-1)^{n-1}(n-1)!}{z^n} + \frac{(-1)^{n-1}\, n!}{2z^{n+1}} + \frac{(-1)^{n-1}(n+1)!}{12z^{n+2}}\left(1 + O\left(\frac{1}{z^2}\right)\right) /;\; n \in \mathbb{N}^+ \bigwedge |\mathrm{Arg}(z)| < \pi \bigwedge (|z| \to \infty)$$

$$H_z \propto \log(z) + \gamma + \frac{1}{2z} - \frac{1}{12z^2}\left(1 + O\left(\frac{1}{z^2}\right)\right) /;\; |\mathrm{Arg}(z)| < \pi \bigwedge (|z| \to \infty)$$

$$H_z^{(r)} \propto \frac{(-1)^r\, \psi^{(r-1)}(1)}{(r-1)!} - \frac{z^{1-r}}{r-1}\left(1 + O\left(\frac{1}{z}\right)\right) /;\; |\mathrm{Arg}(z)| < \pi \bigwedge r-1 \in \mathbb{N}^+ \bigwedge (|z| \to \infty).$$

Integral representations

The differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ can also be represented through the following equivalent integrals:

$$\psi(z) = \int_0^1 \frac{1 - t^{z-1}}{1-t}\, dt - \gamma \;/;\; \mathrm{Re}(z) > 0$$

$$\psi(z) = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{(t+1)^{-z}}{t} \right) dt \;/;\; \mathrm{Re}(z) > 0$$

$$\psi(z) = \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{1 - e^{-t}}\, dt - \gamma \;/;\; \mathrm{Re}(z) > 0$$

$$\psi^{(\nu)}(z) = \int_0^{\infty} \frac{1}{1-e^{-t}} \left(\frac{e^{-t} z^{-\nu}}{\Gamma(1-\nu)} - (-t)^{\nu}\, e^{-zt}\, Q(-\nu, 0, -tz) \right) dt - \frac{\gamma z^{-\nu}}{\Gamma(1-\nu)} \;/;\; \mathrm{Re}(z) > 0$$

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^{\infty} \frac{t^n e^{-tz}}{1 - e^{-t}}\, dt \;/;\; n \in \mathbb{N}^+ \bigwedge \mathrm{Re}(z) > 0$$

$$H_z = \int_0^1 \frac{1 - t^z}{1 - t}\, dt \;/;\; \mathrm{Re}(z) > -1$$

$$H_z = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{(t+1)^{-z-1}}{t} \right) dt + \gamma \text{ /; } \mathrm{Re}(z) > -1$$

$$H_z = \int_0^\infty \frac{e^{-t} - e^{-(z+1)t}}{1 - e^{-t}} dt \text{ /; } \mathrm{Re}(z) > -1$$

$$H_z^{(r)} = \frac{1}{(r-1)!} \left((-1)^r \psi^{(r-1)}(1) - \int_0^\infty \frac{t^{r-1} e^{-t(z+1)}}{1 - e^{-t}} dt \right) \text{ /; } r - 1 \in \mathbb{N}^+ \bigwedge \mathrm{Re}(z) > -1$$

$$H_z^{(r)} = \frac{(-1)^{r-1}}{(r-1)!} \int_0^1 \frac{(t^z - 1) \log^{r-1}(t)}{t-1} dt \text{ /; } \mathrm{Re}(z) > -1 \bigwedge r \in \mathbb{N}^+.$$

Transformations

The following formulas describe some of the transformations that change the differentiated gamma functions into themselves:

$$\psi(-z) = \psi(z) + \pi \cot(\pi z) + \frac{1}{z}$$

$$\psi(z+1) = \psi(z) + \frac{1}{z}$$

$$\psi(z+n) = \psi(z) + \sum_{k=0}^{n-1} \frac{1}{z+k} \text{ /; } n \in \mathbb{N}$$

$$\psi(z-1) = \psi(z) - \frac{1}{z-1}$$

$$\psi(z-n) = \psi(z) - \sum_{k=0}^{n-1} \frac{1}{z-k-1} \text{ /; } n \in \mathbb{N}$$

$$\psi^{(n)}(-z) = (-1)^n \psi^{(n)}(z) + n!\, z^{-n-1} + (-1)^n \pi \frac{\partial^n \cot(\pi z)}{\partial z^n} \text{ /; } n \in \mathbb{N}$$

$$\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n!\, z^{-n-1} \text{ /; } n \in \mathbb{N}$$

$$\psi^{(n)}(z+m) = \psi^{(n)}(z) + (-1)^n n! \sum_{k=0}^{m-1} \frac{1}{(z+k)^{n+1}} \text{ /; } n \in \mathbb{N}$$

$$\psi^{(n)}(z-1) = \psi^{(n)}(z) - (-1)^n n!\, (z-1)^{-n-1} \text{ /; } n \in \mathbb{N}$$

$$\psi^{(n)}(z-m) = \psi^{(n)}(z) - (-1)^n n! \sum_{k=1}^{m} \frac{1}{(z-k)^{n+1}} \text{ /; } n \in \mathbb{N}$$

$$H_{-z} = H_z - \frac{1}{z} + \pi \cot(\pi z)$$

$$H_{z+1} = H_z + \frac{1}{z+1}$$

$$H_{n+z} = H_z + \sum_{k=1}^{n} \frac{1}{k+z} \;/;\; n \in \mathbb{N}$$

$$H_{z-1} = H_z - \frac{1}{z}$$

$$H_{z-n} = H_z - \sum_{k=0}^{n-1} \frac{1}{z-k} \;/;\; n \in \mathbb{N}$$

$$H_{-z}^{(r)} = (-1)^{r-1} H_z^{(r)} + (-1)^r z^{-r} + \frac{(-1)^{\left\lfloor \frac{r}{2} \right\rfloor - 1} (2\pi)^r}{r!} B_r - \pi \delta_{r,1} + \frac{\pi}{(r-1)!} \frac{\partial^{r-1} \cot(z\pi)}{\partial z^{r-1}} \;/;\; r \in \mathbb{N}^+$$

$$H_{z+1}^{(r)} = H_z^{(r)} + \frac{1}{(z+1)^r}$$

$$H_{z+n}^{(r)} = H_z^{(r)} + \sum_{k=1}^{n} \frac{1}{(k+z)^r} \;/;\; n \in \mathbb{N}$$

$$H_{z-1}^{(r)} = H_z^{(r)} - \frac{1}{z^r}$$

$$H_{z-n}^{(r)} = H_z^{(r)} - \sum_{k=0}^{n-1} \frac{1}{(z-k)^r} \;/;\; n \in \mathbb{N}.$$

Transformations with arguments that are integer multiples take the following forms:

$$\psi(2z) = \log(2) + \frac{1}{2}\left(\psi\left(z + \frac{1}{2}\right) + \psi(z)\right)$$

$$\psi(mz) = \log(m) + \frac{1}{m} \sum_{k=0}^{m-1} \psi\left(z + \frac{k}{m}\right) \;/;\; m \in \mathbb{N}^+$$

$$\psi^{(n)}(2z) = 2^{-n-1} \left(\psi^{(n)}(z) + \psi^{(n)}\left(z + \frac{1}{2}\right)\right) \;/;\; n \in \mathbb{N}^+$$

$$\psi^{(n)}(mz) = m^{-n-1} \sum_{k=0}^{m-1} \psi^{(n)}\left(z + \frac{k}{m}\right) \;/;\; n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+$$

$$H_{2z} = \frac{1}{2}\left(H_{z-\frac{1}{2}} + H_z\right) + \log(2)$$

$$H_{mz} = \frac{1}{m} \sum_{k=0}^{m-1} H_{z-\frac{k}{m}} + \log(m) \;/;\; m \in \mathbb{N}^+$$

$$H_{2z}^{(r)} = 2^{-r} \left(H_{z-1}^{(r)} + H_{z-\frac{1}{2}}^{(r)}\right) + 2^{-r} z^{-r} + \left(1 - 2^{1-r}\right) \zeta(r) \;/;\; \mathrm{Re}(z) > 0$$

$$H_{mz}^{(r)} = m^{-r} \sum_{k=0}^{m-1} H_{z-\frac{k}{m}}^{(r)} + \left(1 - m^{1-r}\right) \zeta(r) \;/;\; \mathrm{Re}(z) > 0 \bigwedge m \in \mathbb{N}^+.$$

The following transformations represent summation theorems:

$$\psi(z)+\psi\left(z+\frac{1}{2}\right)=2\,\psi(2\,z)-2\log(2)$$

$$\psi^{(n)}(z)+\psi^{(n)}\left(z+\frac{1}{2}\right)=2^{n+1}\,\psi^{(n)}(2\,z)\;/;\;n\in\mathbb{N}^{+}$$

$$H_z+H_{z+\frac{1}{2}}=2\,H_{2\,z+1}-2\log(2)$$

$$H_z^{(r)}+H_{z+\frac{1}{2}}^{(r)}=2^r\,H_{2\,z+1}^{(r)}+(2-2^r)\,\zeta(r)\;/;\;\mathrm{Re}(z)>-\frac{1}{2}.$$

Identities

The differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ satisfy the following recurrence identities:

$$\psi(z)=\psi(z+1)-\frac{1}{z}$$

$$\psi(z)=\psi(z-1)+\frac{1}{z-1}$$

$$\psi^{(n)}(z)=\psi^{(n)}(z+1)-(-1)^n\,n!\,z^{-n-1}\;/;\;n\in\mathbb{N}$$

$$\psi^{(n)}(z)=\psi^{(n)}(z-1)+(-1)^n\,n!\,(z-1)^{-n-1}\;/;\;n\in\mathbb{N}$$

$$H_z=H_{z+1}-\frac{1}{z+1}$$

$$H_z=H_{z-1}+\frac{1}{z}$$

$$H_z^{(r)}=H_{z+1}^{(r)}-\frac{1}{(z+1)^r}$$

$$H_z^{(r)}=H_{z-1}^{(r)}+\frac{1}{z^r}.$$

The previous formulas can be generalized to the following recurrence identities with a jump of length n:

$$\psi(z)=\psi(z+n)-\sum_{k=0}^{n-1}\frac{1}{z+k}\;/;\;n\in\mathbb{N}$$

$$\psi(z)=\psi(z-n)+\sum_{k=1}^{n}\frac{1}{z-k}\;/;\;n\in\mathbb{N}$$

$$\psi^{(n)}(z)=\psi^{(n)}(z+m)-(-1)^n\,n!\sum_{k=0}^{m-1}\frac{1}{(z+k)^{n+1}}\;/;\;n\in\mathbb{N}$$

$$\psi^{(n)}(z) == \psi^{(n)}(z-m) + (-1)^n\, n!\sum_{k=1}^{m} \frac{1}{(z-k)^{n+1}} \;/;\; n \in \mathbb{N}$$

$$H_z == H_{z+n} - \sum_{k=1}^{n} \frac{1}{z+k} \;/;\; n \in \mathbb{N}$$

$$H_z == H_{z-n} + \sum_{k=0}^{n-1} \frac{1}{z-k} \;/;\; n \in \mathbb{N}$$

$$H_z^{(r)} == H_{z+n}^{(r)} - \sum_{k=1}^{n} \frac{1}{(z+k)^r} \;/;\; n \in \mathbb{N}$$

$$H_z^{(r)} == H_{z-n}^{(r)} + \sum_{k=0}^{n-1} \frac{1}{(z-k)^r} \;/;\; n \in \mathbb{N}.$$

Representations of derivatives

The derivatives of the differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ have rather simple representations:

$$\frac{\partial \psi(z)}{\partial z} == \psi^{(1)}(z)$$

$$\frac{\partial \psi^{(\nu)}(z)}{\partial z} == \psi^{(\nu+1)}(z)$$

$$\frac{\partial H_z}{\partial z} == \frac{\pi^2}{6} - H_z^{(2)}$$

$$\frac{\partial H_z^{(r)}}{\partial z} == r\left(\zeta(r+1) - H_z^{(r+1)}\right)$$

$$\frac{\partial H_n^{(r)}}{\partial r} == -\sum_{k=2}^{n} \frac{\log(k)}{k^r} \;/;\; n \in \mathbb{N}.$$

The corresponding symbolic n^{th}-order derivatives of all the differentiated gamma functions $\psi(z)$, $\psi^{(\nu)}(z)$, H_z, and $H_z^{(r)}$ can be expressed by the following formulas:

$$\frac{\partial^n \psi(z)}{\partial z^n} == \psi^{(n)}(z) \;/;\; n \in \mathbb{N}$$

$$\frac{\partial^m \psi^{(\nu)}(z)}{\partial z^m} == \psi^{(\nu+m)}(z) \;/;\; m \in \mathbb{N}$$

$$\frac{\partial^n H_z}{\partial z^n} == (-1)^n\, n!\left(H_z^{(n+1)} - \zeta(n+1)\right) \;/;\; n \in \mathbb{N}^+$$

$$\frac{\partial^n H_z^{(r)}}{\partial z^n} == \delta_n\, \zeta(r) + (-1)^n\, (r)_n \left(H_z^{(n+r)} - \zeta(n+r)\right) \;/;\; n \in \mathbb{N}$$

$$\frac{\partial^n H_m^{(r)}}{\partial r^n} == (-1)^n \delta_n + (-1)^n \sum_{k=2}^{m} \frac{\log^n(k)}{k^r} \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}.$$

Applications of differentiated gamma functions

Applications of differentiated gamma functions include discrete mathematics, number theory, and calculus.

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