

Introductions to Sec

Introduction to the trigonometric functions

General

The six trigonometric functions sine $\sin(z)$, cosine $\cos(z)$, tangent $\tan(z)$, cotangent $\cot(z)$, cosecant $\csc(z)$, and secant $\sec(z)$ are well known and among the most frequently used elementary functions. The most popular functions $\sin(z)$, $\cos(z)$, $\tan(z)$, and $\cot(z)$ are taught worldwide in high school programs because of their natural appearance in problems involving angle measurement and their wide applications in the quantitative sciences.

The trigonometric functions share many common properties.

Definitions of trigonometric functions

All trigonometric functions can be defined as simple rational functions of the exponential function of $i z$:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan(z) = -\frac{i(e^{iz} - e^{-iz})}{e^{iz} + e^{-iz}}$$

$$\cot(z) = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

$$\csc(z) = \frac{2i}{e^{iz} - e^{-iz}}$$

$$\sec(z) = \frac{2}{e^{iz} + e^{-iz}}.$$

The functions $\tan(z)$, $\cot(z)$, $\csc(z)$, and $\sec(z)$ can also be defined through the functions $\sin(z)$ and $\cos(z)$ using the following formulas:

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

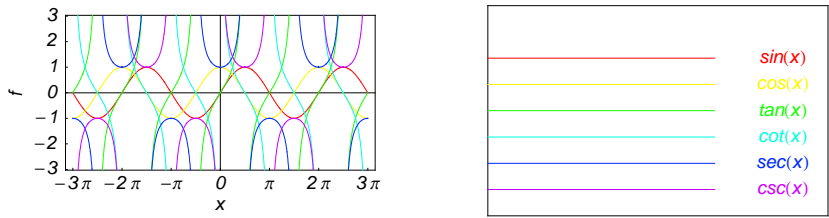
$$\cot(z) = \frac{\cos(z)}{\sin(z)}$$

$$\csc(z) = \frac{1}{\sin(z)}$$

$$\sec(z) = \frac{1}{\cos(z)}.$$

A quick look at the trigonometric functions

Here is a quick look at the graphics for the six trigonometric functions along the real axis.



Connections within the group of trigonometric functions and with other function groups

Representations through more general functions

The trigonometric functions are particular cases of more general functions. Among these more general functions, four different classes of special functions are particularly relevant: Bessel, Jacobi, Mathieu, and hypergeometric functions.

For example, $\sin(z)$ and $\cos(z)$ have the following representations through Bessel, Mathieu, and hypergeometric functions:

$$\sin(z) = \sqrt{\frac{\pi z}{2}} J_{1/2}(z) \quad \sin(z) = -i \sqrt{\frac{\pi iz}{2}} I_{1/2}(iz) \quad \sin(z) = \sqrt{\frac{\pi z}{2}} Y_{-1/2}(z) \quad \sin(z) = \frac{i}{\sqrt{2\pi}} (\sqrt{iz} K_{1/2}(iz) - \sqrt{-iz} K_{1/2}(-iz))$$

$$\cos(z) = \sqrt{\frac{\pi z}{2}} J_{-1/2}(z) \quad \cos(z) = \sqrt{\frac{\pi iz}{2}} I_{-1/2}(iz) \quad \cos(z) = -\sqrt{\frac{\pi z}{2}} Y_{1/2}(z) \quad \cos(z) = \sqrt{\frac{iz}{2\pi}} K_{1/2}(iz) + \sqrt{-\frac{iz}{2\pi}} K_{1/2}(-iz)$$

$$\sin(z) = \text{Se}(1, 0, z) \quad \cos(z) = \text{Ce}(1, 0, z)$$

$$\sin(z) = z {}_0F_1\left(\frac{3}{2}; -\frac{z^2}{4}\right) \quad \cos(z) = {}_0F_1\left(\frac{1}{2}; -\frac{z^2}{4}\right).$$

On the other hand, all trigonometric functions can be represented as degenerate cases of the corresponding doubly periodic Jacobi elliptic functions when their second parameter is equal to 0 or 1:

$$\sin(z) = \text{sd}(z | 0) = \text{sn}(z | 0) \quad \sin(z) = -i \text{sc}(iz | 1) = -i \text{sd}(iz | 1)$$

$$\cos(z) = \text{cd}(z | 0) = \text{cn}(z | 0) \quad \cos(z) = \text{nc}(iz | 1) = \text{nd}(iz | 1)$$

$$\tan(z) = \text{sc}(z | 0) \quad \tan(z) = -i \text{sn}(iz | 1)$$

$$\cot(z) = \text{cs}(z | 0) \quad \cot(z) = i \text{ns}(iz | 1)$$

$$\csc(z) = \text{ds}(z | 0) = \text{ns}(z | 0) \quad \csc(z) = i \text{cs}(iz | 1) = i \text{ds}(iz | 1)$$

$$\sec(z) = \text{dc}(z | 0) = \text{nc}(z | 0) \quad \sec(z) = \text{cn}(iz | 1) = \text{dn}(iz | 1).$$

Representations through related equivalent functions

Each of the six trigonometric functions can be represented through the corresponding hyperbolic function:

$$\sin(z) = -i \sinh(iz) \quad \sin(iz) = i \sinh(z)$$

$$\cos(z) = \cosh(iz) \quad \cos(iz) = \cosh(z)$$

$$\tan(z) = -i \tanh(iz) \quad \tan(iz) = i \tanh(z)$$

$$\cot(z) = i \coth(iz) \quad \cot(iz) = -i \coth(z)$$

$$\csc(z) = i \text{csch}(iz) \quad \csc(iz) = -i \text{csch}(z)$$

$$\sec(z) = \text{sech}(iz) \quad \sec(iz) = \text{sech}(z).$$

Relations to inverse functions

Each of the six trigonometric functions is connected with its corresponding inverse trigonometric function by two formulas. One is a simple formula, and the other is much more complicated because of the multivalued nature of the inverse function:

$$\begin{aligned} \sin(\sin^{-1}(z)) &= z & \sin^{-1}(\sin(z)) &= z /; -\frac{\pi}{2} < \operatorname{Re}(z) < \frac{\pi}{2} \bigvee \operatorname{Re}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Im}(z) \geq 0 \bigvee \operatorname{Re}(z) = \frac{\pi}{2} \bigwedge \operatorname{Im}(z) \leq 0 \\ \cos(\cos^{-1}(z)) &= z & \cos^{-1}(\cos(z)) &= z /; 0 < \operatorname{Re}(z) < \pi \bigvee \operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \geq 0 \bigvee \operatorname{Re}(z) = \pi \wedge \operatorname{Im}(z) \leq 0 \\ \tan(\tan^{-1}(z)) &= z & \tan^{-1}(\tan(z)) &= z /; |\operatorname{Re}(z)| < \frac{\pi}{2} \bigvee \operatorname{Re}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Im}(z) < 0 \bigvee \operatorname{Re}(z) = \frac{\pi}{2} \bigwedge \operatorname{Im}(z) > 0 \\ \cot(\cot^{-1}(z)) &= z & \cot^{-1}(\cot(z)) &= z /; |\operatorname{Re}(z)| < \frac{\pi}{2} \bigvee \operatorname{Re}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Im}(z) < 0 \bigvee \operatorname{Re}(z) = \frac{\pi}{2} \bigwedge \operatorname{Im}(z) \geq 0 \\ \csc(\csc^{-1}(z)) &= z & \csc^{-1}(\csc(z)) &= z /; |\operatorname{Re}(z)| < \frac{\pi}{2} \bigvee \operatorname{Re}(z) = -\frac{\pi}{2} \bigwedge \operatorname{Im}(z) \leq 0 \bigvee \operatorname{Re}(z) = \frac{\pi}{2} \bigwedge \operatorname{Im}(z) \geq 0 \\ \sec(\sec^{-1}(z)) &= z & \sec^{-1}(\sec(z)) &= z /; 0 < \operatorname{Re}(z) < \pi \bigvee \operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \geq 0 \bigvee \operatorname{Re}(z) = \pi \wedge \operatorname{Im}(z) \leq 0. \end{aligned}$$

Representations through other trigonometric functions

Each of the six trigonometric functions can be represented by any other trigonometric function as a rational function of that function with linear arguments. For example, the sine function can be representative as a group-defining function because the other five functions can be expressed as follows:

$$\begin{aligned} \cos(z) &= \sin\left(\frac{\pi}{2} - z\right) & \cos^2(z) &= 1 - \sin^2(z) \\ \tan(z) &= \frac{\sin(z)}{\cos(z)} = \frac{\sin(z)}{\sin\left(\frac{\pi}{2} - z\right)} & \tan^2(z) &= \frac{\sin^2(z)}{1 - \sin^2(z)} \\ \cot(z) &= \frac{\cos(z)}{\sin(z)} = \frac{\sin\left(\frac{\pi}{2} - z\right)}{\sin(z)} & \cot^2(z) &= \frac{1 - \sin^2(z)}{\sin^2(z)} \\ \csc(z) &= \frac{1}{\sin(z)} & \csc^2(z) &= \frac{1}{\sin^2(z)} \\ \sec(z) &= \frac{1}{\cos(z)} = \frac{1}{\sin\left(\frac{\pi}{2} - z\right)} & \sec^2(z) &= \frac{1}{1 - \sin^2(z)}. \end{aligned}$$

All six trigonometric functions can be transformed into any other trigonometric function of this group if the argument z is replaced by $p\pi/2 + qz$ with $q^2 = 1 \wedge p \in \mathbb{Z}$:

$$\begin{aligned} \sin(-z - 2\pi) &= -\sin(z) & \sin(z - 2\pi) &= \sin(z) \\ \sin\left(-z - \frac{3\pi}{2}\right) &= \cos(z) & \sin\left(z - \frac{3\pi}{2}\right) &= \cos(z) \\ \sin(-z - \pi) &= \sin(z) & \sin(z - \pi) &= -\sin(z) \\ \sin\left(-z - \frac{\pi}{2}\right) &= -\cos(z) & \sin\left(z - \frac{\pi}{2}\right) &= -\cos(z) \\ \sin\left(z + \frac{\pi}{2}\right) &= \cos(z) & \sin\left(\frac{\pi}{2} - z\right) &= \cos(z) \\ \sin(z + \pi) &= -\sin(z) & \sin(\pi - z) &= \sin(z) \\ \sin\left(z + \frac{3\pi}{2}\right) &= -\cos(z) & \sin\left(\frac{3\pi}{2} - z\right) &= -\cos(z) \\ \sin(z + 2\pi) &= \sin(z) & \sin(2\pi - z) &= -\sin(z) \end{aligned}$$

$$\begin{aligned} \cos(-z - 2\pi) &= \cos(z) & \cos(z - 2\pi) &= \cos(z) \\ \cos\left(-z - \frac{3\pi}{2}\right) &= \sin(z) & \cos\left(z - \frac{3\pi}{2}\right) &= -\sin(z) \\ \cos(-z - \pi) &= -\cos(z) & \cos(z - \pi) &= -\cos(z) \\ \cos\left(-z - \frac{\pi}{2}\right) &= -\sin(z) & \cos\left(z - \frac{\pi}{2}\right) &= \sin(z) \\ \cos\left(z + \frac{\pi}{2}\right) &= -\sin(z) & \cos\left(\frac{\pi}{2} - z\right) &= \sin(z) \\ \cos(z + \pi) &= -\cos(z) & \cos(\pi - z) &= -\cos(z) \\ \cos\left(z + \frac{3\pi}{2}\right) &= \sin(z) & \cos\left(\frac{3\pi}{2} - z\right) &= -\sin(z) \\ \cos(z + 2\pi) &= \cos(z) & \cos(2\pi - z) &= \cos(z) \end{aligned}$$

$$\begin{aligned} \tan(-z - \pi) &= -\tan(z) & \tan(z - \pi) &= \tan(z) \\ \tan\left(-z - \frac{\pi}{2}\right) &= \cot(z) & \tan\left(z - \frac{\pi}{2}\right) &= -\cot(z) \\ \tan\left(z + \frac{\pi}{2}\right) &= -\cot(z) & \tan\left(\frac{\pi}{2} - z\right) &= \cot(z) \\ \tan(z + \pi) &= \tan(z) & \tan(\pi - z) &= -\tan(z) \end{aligned}$$

$$\begin{aligned} \cot(-z - \pi) &= -\cot(z) & \cot(z - \pi) &= \cot(z) \\ \cot\left(-z - \frac{\pi}{2}\right) &= \tan(z) & \cot\left(z - \frac{\pi}{2}\right) &= -\tan(z) \\ \cot\left(z + \frac{\pi}{2}\right) &= -\tan(z) & \cot\left(\frac{\pi}{2} - z\right) &= \tan(z) \\ \cot(z + \pi) &= \cot(z) & \cot(\pi - z) &= -\cot(z) \end{aligned}$$

$$\begin{aligned} \csc(-z - 2\pi) &= -\csc(z) & \csc(z - 2\pi) &= \csc(z) \\ \csc\left(-z - \frac{3\pi}{2}\right) &= \sec(z) & \csc\left(z - \frac{3\pi}{2}\right) &= \sec(z) \\ \csc(-z - \pi) &= \csc(z) & \csc(z - \pi) &= -\csc(z) \\ \csc\left(-z - \frac{\pi}{2}\right) &= -\sec(z) & \csc\left(z - \frac{\pi}{2}\right) &= -\sec(z) \\ \csc\left(z + \frac{\pi}{2}\right) &= \sec(z) & \csc\left(\frac{\pi}{2} - z\right) &= \sec(z) \\ \csc(z + \pi) &= -\csc(z) & \csc(\pi - z) &= \csc(z) \\ \csc\left(z + \frac{3\pi}{2}\right) &= -\sec(z) & \csc\left(\frac{3\pi}{2} - z\right) &= -\sec(z) \\ \csc(z + 2\pi) &= \csc(z) & \csc(2\pi - z) &= -\csc(z) \end{aligned}$$

$$\begin{aligned} \sec(-z - 2\pi) &= \sec(z) & \sec(z - 2\pi) &= \sec(z) \\ \sec\left(-z - \frac{3\pi}{2}\right) &= \csc(z) & \sec\left(z - \frac{3\pi}{2}\right) &= -\csc(z) \\ \sec(-z - \pi) &= -\sec(z) & \sec(z - \pi) &= -\sec(z) \\ \sec\left(-z - \frac{\pi}{2}\right) &= -\csc(z) & \sec\left(z - \frac{\pi}{2}\right) &= \csc(z) \\ \sec\left(z + \frac{\pi}{2}\right) &= -\csc(z) & \sec\left(\frac{\pi}{2} - z\right) &= \csc(z) \\ \sec(z + \pi) &= -\sec(z) & \sec(\pi - z) &= -\sec(z) \\ \sec\left(z + \frac{3\pi}{2}\right) &= \csc(z) & \sec\left(\frac{3\pi}{2} - z\right) &= -\csc(z) \\ \sec(z + 2\pi) &= \sec(z) & \sec(2\pi - z) &= \sec(z). \end{aligned}$$

The best-known properties and formulas for trigonometric functions

Real values for real arguments

For real values of argument z , the values of all the trigonometric functions are real (or infinity).

In the points $z = 2\pi n/m$; $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, the values of trigonometric functions are algebraic. In several cases they can even be rational numbers or integers (like $\sin(\pi/2) = 1$ or $\sin(\pi/6) = 1/2$). The values of trigonometric functions can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \dots\}$.

Simple values at zero

All trigonometric functions have rather simple values for arguments $z = 0$ and $z = \pi/2$:

$$\begin{aligned}\sin(0) &= 0 & \sin\left(\frac{\pi}{2}\right) &= 1 \\ \cos(0) &= 1 & \cos\left(\frac{\pi}{2}\right) &= 0 \\ \tan(0) &= 0 & \tan\left(\frac{\pi}{2}\right) &= \infty \\ \cot(0) &= \infty & \cot\left(\frac{\pi}{2}\right) &= 0 \\ \csc(0) &= \infty & \csc\left(\frac{\pi}{2}\right) &= 1 \\ \sec(0) &= 1 & \sec\left(\frac{\pi}{2}\right) &= \infty.\end{aligned}$$

Analyticity

All trigonometric functions are defined for all complex values of z , and they are analytical functions of z over the whole complex z -plane and do not have branch cuts or branch points. The two functions $\sin(z)$ and $\cos(z)$ are entire functions with an essential singular point at $z = \infty$. All other trigonometric functions are meromorphic functions with simple poles at points $z = \pi k$; $k \in \mathbb{Z}$ for $\csc(z)$ and $\cot(z)$, and at points $z = \pi/2 + \pi k$; $k \in \mathbb{Z}$ for $\sec(z)$ and $\tan(z)$.

Periodicity

All trigonometric functions are periodic functions with a real period (2π or π):

$$\begin{aligned}\sin(z) &= \sin(z + 2\pi) & \sin(z + 2\pi k) &= \sin(z) /; k \in \mathbb{Z} \\ \cos(z) &= \cos(z + 2\pi) & \cos(z + 2\pi k) &= \cos(z) /; k \in \mathbb{Z} \\ \tan(z) &= \tan(z + \pi) & \tan(z + \pi k) &= \tan(z) /; k \in \mathbb{Z} \\ \cot(z) &= \cot(z + \pi) & \cot(z + \pi k) &= \cot(z) /; k \in \mathbb{Z} \\ \csc(z) &= \csc(z + 2\pi) & \csc(z + 2\pi k) &= \csc(z) /; k \in \mathbb{Z} \\ \sec(z) &= \sec(z + 2\pi) & \sec(z + 2\pi k) &= \sec(z) /; k \in \mathbb{Z}.\end{aligned}$$

Parity and symmetry

All trigonometric functions have parity (either odd or even) and mirror symmetry:

$$\begin{aligned}\sin(-z) &= -\sin(z) & \sin(\bar{z}) &= \overline{\sin(z)} \\ \cos(-z) &= \cos(z) & \cos(\bar{z}) &= \overline{\cos(z)} \\ \tan(-z) &= -\tan(z) & \tan(\bar{z}) &= \overline{\tan(z)} \\ \cot(-z) &= -\cot(z) & \cot(\bar{z}) &= \overline{\cot(z)} \\ \csc(-z) &= -\csc(z) & \csc(\bar{z}) &= \overline{\csc(z)} \\ \sec(-z) &= \sec(z) & \sec(\bar{z}) &= \overline{\sec(z)}.\end{aligned}$$

Simple representations of derivatives

The derivatives of all trigonometric functions have simple representations that can be expressed through other trigonometric functions:

$$\begin{aligned} \frac{\partial \sin(z)}{\partial z} &= \cos(z) & \frac{\partial \cos(z)}{\partial z} &= -\sin(z) & \frac{\partial \tan(z)}{\partial z} &= \sec^2(z) \\ \frac{\partial \cot(z)}{\partial z} &= -\csc^2(z) & \frac{\partial \csc(z)}{\partial z} &= -\cot(z) \csc(z) & \frac{\partial \sec(z)}{\partial z} &= \sec(z) \tan(z). \end{aligned}$$

Simple differential equations

The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through $\sin(z)$ and $\cos(z)$:

$$\begin{aligned} w''(z) + w(z) &= 0 /; w(z) = \cos(z) \wedge w(0) = 1 \wedge w'(0) = 0 \\ w''(z) + w(z) &= 0 /; w(z) = \sin(z) \wedge w(0) = 0 \wedge w'(0) = 1 \\ w''(z) + w(z) &= 0 /; w(z) = c_1 \cos(z) + c_2 \sin(z). \end{aligned}$$

All six trigonometric functions satisfy first-order nonlinear differential equations:

$$\begin{aligned} w'(z) - \sqrt{1 - (w(z))^2} &= 0 /; w(z) = \sin(z) \wedge w(0) = 0 \wedge |\operatorname{Re}(z)| < \frac{\pi}{2} \\ w'(z) - \sqrt{1 - (w(z))^2} &= 0 /; w(z) = \cos(z) \wedge w(0) = 1 \wedge |\operatorname{Re}(z)| < \frac{\pi}{2} \\ w'(z) - w(z)^2 - 1 &= 0 /; w(z) = \tan(z) \wedge w(0) = 0 \\ w'(z) + w(z)^2 + 1 &= 0 /; w(z) = \cot(z) \wedge w\left(\frac{\pi}{2}\right) = 0 \\ w'(z)^2 - w(z)^4 + w(z)^2 &= 0 /; w(z) = \csc(z) \\ w'(z)^2 - w(z)^4 + w(z)^2 &= 0 /; w(z) = \sec(z). \end{aligned}$$

Applications of trigonometric functions

Triangle theorems

The prime application of the trigonometric functions are triangle theorems. In a triangle, a , b , and c represent the lengths of the sides opposite to the angles, Δ the area, R the circumradius, and r the inradius. Then the following identities hold:

$$\begin{aligned} \alpha + \beta + \gamma &= \pi \\ \frac{\sin(\alpha)}{a} &= \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} \\ \sin(\alpha) \sin(\beta) \sin(\gamma) &= \frac{\Delta}{2R^2} & \sin(\alpha) &= \frac{2\Delta}{bc} \\ \cos(\alpha) &= \frac{b^2 + c^2 - a^2}{2bc} & \cot(\alpha) &= \frac{b^2 + c^2 - a^2}{4\Delta} \\ \sin\left(\frac{\alpha}{2}\right) \sin\left(\frac{\beta}{2}\right) \sin\left(\frac{\gamma}{2}\right) &= \frac{r}{4R} & \cos(\alpha) + \cos(\beta) + \cos(\gamma) &= 1 + \frac{r}{R} \end{aligned}$$

$$\cot(\alpha) + \cot(\beta) + \cot(\gamma) = \frac{a^2 + b^2 + c^2}{4\Delta}$$

$$\tan(\alpha) + \tan(\beta) + \tan(\gamma) = \tan(\alpha)\tan(\beta)\tan(\gamma)$$

$$\cot(\alpha)\cot(\beta) + \cot(\alpha)\cot(\gamma) + \cot(\beta)\cot(\gamma) = 1$$

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1 - 2\cos(\alpha)\cos(\beta)\cos(\gamma)$$

$$\frac{\tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right)}{\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right)} = \frac{r}{c}$$

For a right-angle triangle the following relations hold:

$$\sin(\alpha) = \frac{a}{c}; \gamma = \frac{\pi}{2} \quad \cos(\alpha) = \frac{b}{c}; \gamma = \frac{\pi}{2}$$

$$\tan(\alpha) = \frac{a}{b}; \gamma = \frac{\pi}{2} \quad \cot(\alpha) = \frac{b}{a}; \gamma = \frac{\pi}{2}$$

$$\csc(\alpha) = \frac{c}{a}; \gamma = \frac{\pi}{2} \quad \sec(\alpha) = \frac{c}{b}; \gamma = \frac{\pi}{2}$$

Other applications

Because the trigonometric functions appear virtually everywhere in quantitative sciences, it is impossible to list their numerous applications in teaching, science, engineering, and art.

Introduction to the Secant Function

Defining the secant function

The word "secant" was introduced by Alhābā Alhāsib around 800. Later on Th. Fincke (1583) used the word "secans" in Latin for notation of the corresponding function. The secant function also appeared in the works of A. Magini (1592) and B. Cavalieri (1643). J. Kresa (1720) used the symbol "sec" that was later widely used by L. Euler (1748).

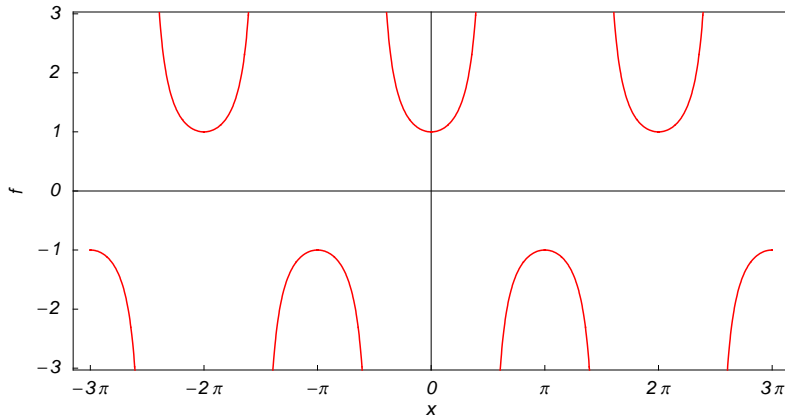
The classical definition of the secant function for real arguments is: "the secant of an angle α in a right-angle triangle is the ratio of the length of the hypotenuse to the adjacent leg." This description of $\sec(\alpha)$ is valid for $0 < \alpha < \pi/2$ when this triangle is nondegenerate. This approach to the secant can be expanded to arbitrary real values of α if consideration is given to the arbitrary point $\{x, y\}$ in the x, y -Cartesian plane and $\sec(\alpha)$ is defined as the ratio $(x^2 + y^2)^{1/2} / x$, assuming that α is the value of the angle between the positive direction of the x -axis and the direction from the origin to the point $\{x, y\}$.

Comparing the $\sec(\alpha)$ definition with the definition of the cosine function shows that the following formula can also be used as a definition of the secant function:

$$\sec(z) = \frac{1}{\cos(z)}$$

A quick look at the secant function

Here is a graphic of the secant function $f(x) = \sec(x)$ for real values of its argument x .



Representation through more general functions

The secant function $\sec(z)$ can be represented using more general mathematical functions. As the ratio of one and the cosine function that is a particular case of the generalized hypergeometric, Bessel, Struve, and Mathieu functions, the secant function can also be represented as ratios of one and those special functions. Here are some examples:

$$\sec(z) = \frac{1}{{}_0F_1\left(\frac{1}{2}; -\frac{z^2}{4}\right)} \quad \sec(z) = \frac{1}{\sqrt{\frac{\pi z}{2}} J_{-\frac{1}{2}}(z)} \quad \sec(z) = \frac{1}{\sqrt{\frac{\pi iz}{2}} I_{-\frac{1}{2}}(iz)}$$

$$\sec(z) = -\frac{1}{\sqrt{\frac{\pi z}{2}} Y_{\frac{1}{2}}(z)} \quad \sec(z) = \frac{1}{1 - \sqrt{\frac{\pi z}{2}} H_{\frac{1}{2}}(z)} \quad \sec(z) = \frac{1}{\text{Ce}(1,0,z)}$$

But these representations are not very useful because they include complicated special functions in the denominators.

It is more useful to write the secant function as particular cases of one special function. That can be done using doubly periodic Jacobi elliptic functions that degenerate into the secant function when their second parameter is equal to 0 or 1:

$$\sec(z) = \text{dc}(z | 0) = \text{nc}(z | 0) = \text{ds}\left(\frac{\pi}{2} - z \mid 0\right) = \text{ns}\left(\frac{\pi}{2} - z \mid 0\right) = \text{cn}(iz | 1) = \text{dn}(iz | 1) = i \text{cs}\left(\frac{\pi i}{2} - iz \mid 1\right) = i \text{ds}\left(\frac{\pi i}{2} - iz \mid 1\right).$$

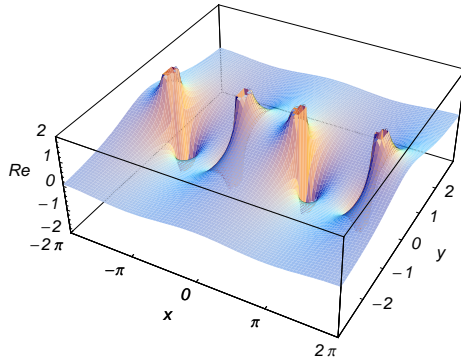
Definition of the secant function for a complex argument

In the complex z -plane, the function $\sec(z)$ is defined using $\cos(z)$ or the exponential function e^w in the points iz and $-iz$ through the formula:

$$\sec(z) = \frac{1}{\cos(z)} = \frac{2}{e^{iz} + e^{-iz}}.$$

In the points $z = \pi/2 + \pi k / ; k \in \mathbb{Z}$, where $\cos(z)$ has zeros, the denominator of the last formula equals zero and $\sec(z)$ has singularities (poles of the first order).

Here are two graphics showing the real and imaginary parts of the secant function over the complex plane.



The best-known properties and formulas for the secant function

Values in points

Using the connection between the cosine and secant functions gives the following table of values of the secant function for angles between 0 and 2π :

$$\begin{array}{llll} \sec(0) = 1 & \sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} & \sec\left(\frac{\pi}{4}\right) = \sqrt{2} & \sec\left(\frac{\pi}{3}\right) = 2 \\ \sec\left(\frac{\pi}{2}\right) = \infty & \sec\left(\frac{2\pi}{3}\right) = -2 & \sec\left(\frac{3\pi}{4}\right) = -\sqrt{2} & \sec\left(\frac{5\pi}{6}\right) = -\frac{2}{\sqrt{3}} \\ \sec(\pi) = -1 & \sec\left(\frac{7\pi}{6}\right) = -\frac{2}{\sqrt{3}} & \sec\left(\frac{5\pi}{4}\right) = -\sqrt{2} & \sec\left(\frac{4\pi}{3}\right) = -2 \\ \sec\left(\frac{3\pi}{2}\right) = \infty & \sec\left(\frac{5\pi}{3}\right) = 2 & \sec\left(\frac{7\pi}{4}\right) = \sqrt{2} & \sec\left(\frac{11\pi}{6}\right) = \frac{2}{\sqrt{3}} \end{array}$$

$$\sec(2\pi) = 1 \quad \sec(\pi m) = (-1)^m \quad ; \quad m \in \mathbb{Z} \quad \sec\left(\pi\left(\frac{1}{2} + m\right)\right) = \infty \quad ; \quad m \in \mathbb{Z}.$$

General characteristics

For real values of argument z , the values of $\sec(z)$ are real.

In the points $z = 2\pi n/m$; $n \in \mathbb{Z} \wedge m \in \mathbb{Z}$, the values of $\sec(z)$ are algebraic. In several cases they can be integers $-2, -1, 1$, or 2 :

$$\sec(0) = 1 \quad \sec\left(\frac{\pi}{3}\right) = 2 \quad \sec\left(\frac{2\pi}{3}\right) = -2 \quad \sec(\pi) = -1.$$

The values of $\sec\left(\frac{n\pi}{m}\right)$ can be expressed using only square roots if $n \in \mathbb{Z}$ and m is a product of a power of 2 and distinct Fermat primes $\{3, 5, 17, 257, \dots\}$.

The function $\sec(z)$ is an analytical function of z that is defined over the whole complex z -plane and does not have branch cuts and branch points. It has an infinite set of singular points:

- (a) $z = \pi/2 + \pi k$; $k \in \mathbb{Z}$ are the simple poles with residues $(-1)^{k-1}$.
- (b) $z = \infty$ is an essential singular point.

It is a periodic function with a real period 2π :

$$\sec(z + 2\pi) = \sec(z)$$

$$\sec(z) = \sec(z + 2\pi k) \ ; \ k \in \mathbb{Z}$$

$$\sec(z) = (-1)^k \sec(z + \pi k) \ ; \ k \in \mathbb{Z}.$$

The function $\sec(z)$ is an even function with mirror symmetry:

$$\sec(-z) = \sec(z) \quad \sec(\bar{z}) = \overline{\sec(z)}.$$

Differentiation

The first derivative of $\sec(z)$ has simple representations using either the $\tan(z)$ function or the $\sec(z)$ function:

$$\frac{\partial \sec(z)}{\partial z} = \tan(z) \sec(z).$$

The n^{th} derivative of $\sec(z)$ has much more complicated representations than symbolic n^{th} derivatives for $\sin(z)$ and $\cos(z)$:

$$\frac{\partial^n \sec(z)}{\partial z^n} = \sec(z) \left(\delta_n + (n+1)! \sum_{k=0}^n \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} \frac{(-1)^k 2^{1-k} (k-2j)^n \sec^k(z)}{(k+1)! j! (k-j)! (n-k)!} \cos\left(\frac{\pi n}{2} + (k-2j)z\right) \right) \ ; \ n \in \mathbb{N},$$

where δ_n is the Kronecker delta symbol: $\delta_0 = 1$ and $\delta_n = 0 \ ; \ n \neq 0$.

Ordinary differential equation

The function $\sec(z)$ satisfies the following first-order nonlinear differential equation:

$$w'(z)^2 - w(z)^4 + w(z)^2 = 0 \ ; \ w(z) = \sec(z).$$

Series representation

The function $\sec(z)$ has the following series expansion at the origin that converges for all finite values z with $|z| < \frac{\pi}{2}$:

$$\sec(z) = 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k} z^{2k}}{(2k)!},$$

where E_{2k} are the Euler numbers.

The secant function $\sec(z)$ can also be presented using other kinds of series by the following formulas:

$$\sec(z) = \pi \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)}{\left(k + \frac{1}{2}\right)^2 \pi^2 - z^2} \ ; \ \frac{z}{\pi} - \frac{1}{2} \notin \mathbb{Z}$$

$$\sec^2(z) = \sum_{k=-\infty}^{\infty} \frac{1}{\left(z + \pi\left(k + \frac{1}{2}\right)\right)^2} \ ; \ \frac{z}{\pi} - \frac{1}{2} \notin \mathbb{Z}.$$

Integral representation

The function $\sec(z)$ has a well-known integral representation through the following definite integral along the positive part of the real axis:

$$\sec(z) = \frac{2}{\pi} \int_0^\infty \frac{t^{2z}}{t^2 + 1} dt \quad ; \quad |\operatorname{Re}(z)| < \frac{\pi}{2}.$$

Product representation

The famous infinite product representation for $\cos(z)$ can be easily rewritten as the following product representation for the secant function:

$$\sec(z) = \prod_{k=1}^\infty \frac{\pi^2 (2k-1)^2}{\pi^2 (2k-1)^2 - 4z^2}.$$

Limit representation

The secant function has the following limit representation:

$$\sec(z) = \lim_{n \rightarrow \infty} \sum_{k=-n}^n \frac{(-1)^k}{\pi \left(k + \frac{1}{2}\right) - z} \quad ; \quad \frac{z}{\pi} - \frac{1}{2} \notin \mathbb{Z}.$$

Indefinite integration

Indefinite integrals of expressions involving the secant function can sometimes be expressed using elementary functions. However, special functions are frequently needed to express the results even when the integrands have a simple form (if they can be evaluated in closed form). Here are some examples:

$$\int \sec(z) dz = \log\left(\cot\left(\frac{z}{2} - \frac{\pi}{4}\right)\right)$$

$$\int \sqrt{\sec(z)} dz = 2 \cos^{\frac{1}{2}}(z) F\left(\frac{z}{2} \middle| 2\right) \sec^{\frac{1}{2}}(z)$$

$$\int \sec^v(az) dz = \frac{\csc(az) \sec^{v-1}(az) \sqrt{\sin^2(az)}}{av - a} {}_2F_1\left(\frac{1-v}{2}, \frac{1}{2}; \frac{3-v}{2}; \cos^2(az)\right).$$

Definite integration

Definite integrals that contain the secant function are sometimes simple and their values can be expressed through elementary functions. Here is one example:

$$\int_0^{\frac{\pi}{4}} t \sec^2(t) dt = \frac{1}{4} (\pi - \log(4)).$$

Some special functions can be used to evaluate more complicated definite integrals. For example, polygamma and gamma functions and the Catalan constant are needed to express the following integrals:

$$\int_0^{\frac{\pi}{4}} t \sec(t) dt = \frac{1}{32} \left(8\pi (\log(1 - (-1)^{3/4}) - \log(1 + (-1)^{3/4})) + \sqrt[4]{-1} \left(\psi^{(1)}\left(\frac{1}{8}\right) - i \psi^{(1)}\left(\frac{3}{8}\right) - \psi^{(1)}\left(\frac{5}{8}\right) + i \psi^{(1)}\left(\frac{7}{8}\right) \right) - 64 C \right)$$

$$\int_0^{\frac{\pi}{2}} \sec^a(t) dt = \frac{\sqrt{\pi}}{2\Gamma(1-\frac{a}{2})} \Gamma\left(\frac{1-a}{2}\right); \operatorname{Re}(a) < 1.$$

Finite summation

Finite sums that contain the secant function have the following simple values:

$$\sum_{k=0}^n \sec\left(\frac{2k\pi}{2n+1}\right) = \frac{1}{2} (-1)^n (2n+1) + \frac{1}{2}; n \in \mathbb{N}$$

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sec^2\left(\frac{k\pi}{n}\right) = \frac{1}{12} (4n^2 - (-1)^n (2n^2 + 1) - 7); n \in \mathbb{N}^+$$

$$\sum_{k=0}^{n-1} \sec^2\left(\frac{\pi k}{n} + z\right) = n^2 \operatorname{csc}^2\left(\frac{1}{2} n (2z + \pi)\right); n \in \mathbb{N}^+$$

$$\sum_{k=0}^{n-1} \sec^2\left(\frac{2\pi k}{n} + z\right) = \frac{1}{2} (1 - (-1)^n) n^2 \operatorname{csc}^2\left(\frac{1}{2} n (2z + \pi)\right) + \frac{1}{4} (1 + (-1)^n) n^2 \operatorname{csc}^2\left(\frac{1}{4} n (2z + \pi)\right); n \in \mathbb{N}^+$$

$$\sum_{k=1}^n \frac{1}{2^{2k}} \sec^2\left(\frac{z}{2^k}\right) = \operatorname{csc}^2(z) - \frac{1}{2^{2n}} \operatorname{csc}^2\left(\frac{z}{2^n}\right); n \in \mathbb{N}.$$

Infinite summation

The evaluation limit of the last formula in the previous subsection for $n \rightarrow \infty$ gives the following value for the corresponding infinite sum:

$$\sum_{k=1}^{\infty} \frac{1}{2^{2k}} \sec^2\left(\frac{z}{2^k}\right) = \operatorname{csc}^2(z) - \frac{1}{z^2}.$$

Finite products

The following finite product from the secant can be represented through the cosecant function:

$$\prod_{k=0}^{n-1} \sec\left(z + \frac{\pi k}{n}\right) = (-1)^{n-1} 2^{n-1} \operatorname{csc}\left(\frac{n(\pi - 2z)}{2}\right); n \in \mathbb{N}^+.$$

Infinite products

The following infinite product from the secant can be represented through the cosecant function:

$$\prod_{k=1}^{\infty} \sec\left(\frac{z}{2^k}\right) = z \operatorname{csc}(z).$$

Addition formulas

The secants of a sum and a difference can be represented by the following formulas that are derived from the cosines of a sum and a difference:

$$\sec(a + b) = \frac{1}{\cos(b) \cos(a) - \sin(a) \sin(b)}$$

$$\sec(a - b) = \frac{1}{\cos(a) \cos(b) + \sin(a) \sin(b)}$$

Multiple arguments

In the case of multiple arguments $z, 2z, 3z, \dots$, the function $\sec(z)$ can be represented as a rational function including powers of a secant. Here are two examples:

$$\sec(2z) = \frac{\sec^2(z)}{2 - \sec^2(z)}$$

$$\sec(3z) = \frac{\sec^3(z)}{4 - 3 \sec^2(z)}$$

Half-angle formulas

The secant of a half-angle can be represented by the following simple formula that is valid in a vertical strip:

$$\sec\left(\frac{z}{2}\right) = \frac{\sqrt{2}}{\sqrt{1 + \cos(z)}} \quad ; \quad |\operatorname{Re}(z)| < \pi \vee \operatorname{Re}(z) = -\pi \wedge \operatorname{Im}(z) > 0 \vee \operatorname{Re}(z) = \pi \wedge \operatorname{Im}(z) < 0.$$

To make this formula correct for all complex z , a complicated prefactor is needed:

$$\sec\left(\frac{z}{2}\right) = \frac{c(z)}{\sqrt{1 + \cos(z)}} \quad ; \quad c(z) = (-1)^{\lfloor \frac{\operatorname{Re}(z)+\pi}{2\pi} \rfloor} \sqrt{2} \left(1 - \left(1 + (-1)^{\lfloor \frac{\operatorname{Re}(z)+\pi}{2\pi} \rfloor + \lfloor \frac{-\operatorname{Re}(z)+\pi}{2\pi} \rfloor} \right) \theta(-\operatorname{Im}(z)) \right),$$

where $c(z)$ contains the unit step, real part, imaginary part, and the floor functions.

Sums of two direct functions

The sum and difference of two secant functions can be described by the following formulas:

$$\sec(a) + \sec(b) = 2 \cos\left(\frac{a-b}{2}\right) \cos\left(\frac{a+b}{2}\right) \sec(a) \sec(b)$$

$$\sec(a) - \sec(b) = 2 \sin\left(\frac{a-b}{2}\right) \sin\left(\frac{a+b}{2}\right) \sec(a) \sec(b).$$

Products involving the direct function

The product of two secants and the product of a secant and a cosecant have the following representations:

$$\sec(a) \sec(b) = \frac{2}{\cos(a-b) + \cos(a+b)}$$

$$\sec(a) \csc(b) = \frac{2}{\sin(a+b) - \sin(a-b)}$$

Inequalities

One of the most famous inequalities for a secant function is the following:

$$\sec(x) > x \csc(x) /; 0 < x < \frac{\pi}{2} \wedge x \in \mathbb{R}.$$

Relations with its inverse function

There are simple relations between the function $\sec(z)$ and its inverse function $\sec^{-1}(z)$:

$$\sec(\sec^{-1}(z)) = z \quad \sec^{-1}(\sec(z)) = z /; 0 < \operatorname{Re}(z) < \pi \vee \operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \geq 0 \vee \operatorname{Re}(z) = \pi \wedge \operatorname{Im}(z) \leq 0.$$

The second formula is valid at least in the vertical strip $0 < \operatorname{Re}(z) < \pi$. Outside of this strip a much more complicated relation (containing the unit step, real part, and the floor functions) holds:

$$\sec^{-1}(\sec(z)) = \frac{\pi}{2} \left(1 - (-1)^{\lfloor -\frac{\operatorname{Re}(z)}{\pi} \rfloor} \right) + (-1)^{\lfloor -\frac{\operatorname{Re}(z)}{\pi} \rfloor} \left(\left(1 + (-1)^{\lfloor \frac{\operatorname{Re}(z)}{\pi} \rfloor + \lfloor -\frac{\operatorname{Re}(z)}{\pi} \rfloor} \right) \theta(\operatorname{Im}(z)) - 1 \right) \left(z + \pi \left\lfloor -\frac{\operatorname{Re}(z)}{\pi} \right\rfloor \right).$$

Representations through other trigonometric functions

Secant and cosecant functions are connected by a very simple formula that contains the linear function in the argument:

$$\sec(z) = \csc\left(\frac{\pi}{2} - z\right).$$

The secant function can also be represented using other trigonometric functions by the following formulas:

$$\sec(z) = \frac{1}{\sin\left(\frac{\pi}{2} - z\right)} \quad \sec(z) = \frac{1 + \tan^2\left(\frac{z}{2}\right)}{1 - \tan^2\left(\frac{z}{2}\right)} \quad \sec(z) = \frac{\cot^2\left(\frac{z}{2}\right) + 1}{\cot^2\left(\frac{z}{2}\right) - 1}.$$

Representations through hyperbolic functions

The secant function has representations using the hyperbolic functions:

$$\sec(z) = \frac{i}{\sinh\left(\frac{\pi i}{2} - iz\right)} \quad \sec(z) = \frac{1}{\cosh(iz)} \quad \sec(z) = \frac{1 - \tanh^2\left(\frac{iz}{2}\right)}{1 + \tanh^2\left(\frac{iz}{2}\right)} \quad \sec(z) = \frac{\coth^2\left(\frac{iz}{2}\right) - 1}{\coth^2\left(\frac{iz}{2}\right) + 1}$$

$$\sec(z) = i \operatorname{csch}\left(\frac{\pi i}{2} - iz\right) \quad \sec(z) = \operatorname{sech}(iz) \quad \sec(iz) = \operatorname{sech}(z).$$

Applications

The secant function is used throughout mathematics, the exact sciences, and engineering.

Introduction to the Trigonometric Functions in *Mathematica*

Overview

The following shows how the six trigonometric functions are realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the trigonometric functions or return them are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

All six trigonometric functions are represented as built-in functions in *Mathematica*. Following *Mathematica*'s general naming convention, the StandardForm function names are simply capitalized versions of the traditional mathematics names. Here is a list `trigFunctions` of the six trigonometric functions in StandardForm.

```
trigFunctions = {Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Csc[z]}
{Sin[z], Cos[z], Tan[z], Cot[z], Sec[z], Csc[z]}
```

Here is a list `trigFunctions` of the six trigonometric functions in TraditionalForm.

```
trigFunctions // TraditionalForm
{sin(z), cos(z), tan(z), cot(z), sec(z), csc(z)}
```

Additional forms of notations

Mathematica also knows the most popular forms of notations for the trigonometric functions that are used in other programming languages. Here are three examples: CForm, TeXForm, and FortranForm.

```
trigFunctions /. {z -> 2 Pi z} // (CForm /@ #) &
{Sin (2 * Pi * z), Cos (2 * Pi * z), Tan (2 * Pi * z),
 Cot (2 * Pi * z), Sec (2 * Pi * z), Csc (2 * Pi * z)}

trigFunctions /. {z -> 2 Pi z} // (TeXForm /@ #) &
{\sin (2\, \pi \, z), \cos (2\, \pi \, z), \tan (2\, \pi \, z), \cot
 (2\, \pi \, z), \sec (2\, \pi \, z), \csc (2\, \pi \, z)}

trigFunctions /. {z -> 2 Pi z} // (FortranForm /@ #) &
{Sin (2 * Pi * z), Cos (2 * Pi * z), Tan (2 * Pi * z),
 Cot (2 * Pi * z), Sec (2 * Pi * z), Csc (2 * Pi * z)}
```

Automatic evaluations and transformations

Evaluation for exact, machine-number, and high-precision arguments

For a simple exact argument, *Mathematica* returns exact results. For instance, for the argument $\pi/6$, the Sin function evaluates to $1/2$.

$$\sin\left[\frac{\pi}{6}\right]$$

$$\frac{1}{2}$$

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z -> Pi/6
```

$$\left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}, \sqrt{3}, 2, \frac{2}{\sqrt{3}}\right\}$$

For a generic machine-number argument (a numerical argument with a decimal point and not too many digits), a machine number is returned.

```
Cos[3.]
```

```
-0.989992
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z -> 2.
```

```
{0.909297, -0.416147, -2.18504, -0.457658, 1.09975, -2.403}
```

The next inputs calculate 100-digit approximations of the six trigonometric functions at $z = 1$.

```
N[Tan[1], 40]
```

```
1.557407724654902230506974807458360173087
```

```
Cot[1] // N[#, 50] &
```

```
0.64209261593433070300641998659426562023027811391817
```

```
N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z -> 1, 100]
```

```
{0.841470984807896506652502321630298999622563060798371065672751709991910404391239668,
 9486397435430526959,
0.540302305868139717400936607442976603732310420617922227670097255381100394774471764,
5179518560871830893,
1.557407724654902230506974807458360173087250772381520038383946605698861397151727289,
555099965202242984,
0.642092615934330703006419986594265620230278113918171379101162280426276856839164672,
1984829197601968047,
1.188395105778121216261599452374551003527829834097962625265253666359184367357190487,
913663568030853023,
1.850815717680925617911753241398650193470396655094009298835158277858815411261596705,
921841413287306671}
```

Within a second, it is possible to calculate thousands of digits for the trigonometric functions. The next input calculates 10000 digits for $\sin(1)$, $\cos(1)$, $\tan(1)$, $\cot(1)$, $\sec(1)$, and $\csc(1)$ and analyzes the frequency of the occurrence of the digit k in the resulting decimal number.

```
Map[Function[w, {First[#, Length[#]} & /@ Split[Sort[First[RealDigits[w]]]}],
  N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z -> 1, 10000]]
```

```
{{{0, 983}, {1, 1069}, {2, 1019}, {3, 983}, {4, 972}, {5, 994},
 {6, 994}, {7, 988}, {8, 988}, {9, 1010}}, {{0, 998}, {1, 1034}, {2, 982},
 {3, 1015}, {4, 1013}, {5, 963}, {6, 1034}, {7, 966}, {8, 991}, {9, 1004}},
 {{0, 1024}, {1, 1025}, {2, 1000}, {3, 969}, {4, 1026}, {5, 944}, {6, 999},
 {7, 1001}, {8, 1008}, {9, 1004}}, {{0, 1006}, {1, 1030}, {2, 986},
 {3, 954}, {4, 1003}, {5, 1034}, {6, 999}, {7, 998}, {8, 1009}, {9, 981}},
 {{0, 1031}, {1, 976}, {2, 1045}, {3, 917}, {4, 1001}, {5, 996}, {6, 964},
 {7, 1012}, {8, 982}, {9, 1076}}, {{0, 978}, {1, 1034}, {2, 1016},
 {3, 974}, {4, 987}, {5, 1067}, {6, 943}, {7, 1006}, {8, 1027}, {9, 968}}}
```


Here are 50-digit approximations to the six trigonometric functions at the complex argument $z = 3 + 5i$.

N[Csc[3 + 5 i], 100]

```
0.0019019704237010899966700172963208058404592525121712743108017196953928700340468202\
96847410109982878354 +
0.013341591397996678721837322466473194390132347157253190972075437462485814431570118\
67262664488519840339 i
```

N[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]} /. z -> 3 + 5 i, 50]

```
{10.472508533940392276673322536853503271126419950388 -
73.460621695673676366791192505081750407213922814475 i,
-73.467292212645262467746454594833950830814859165299 -
10.471557674805574377394464224329537808548330651734 i,
-0.000025368676207676032417806136707426288195560702602478 +
0.99991282015135380828209263013972954140566020462086 i,
-0.000025373100044545977383763346789469656754050037355986 -
1.0000871868058967743285316881045218577131612831891 i,
0.0019019704237010899966700172963208058404592525121713 +
0.013341591397996678721837322466473194390132347157253 i,
-0.013340476530549737487361100811100839468470481725038 +
0.0019014661516951513089519270013254277867588978133499 i}
```

Mathematica always evaluates mathematical functions with machine precision, if the arguments are machine numbers. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

{Sin[2.], N[Sin[2]], N[Sin[2], 16], N[Sin[2], 5], N[Sin[2], 20]}

```
{0.909297, 0.909297, 0.909297, 0.909297, 0.90929742682568169540}
```

% // InputForm

```
{0.9092974268256817, 0.9092974268256817, 0.9092974268256817, 0.9092974268256817,
0.909297426825681695396019865911745`20}
```

Precision[%]

```
16
```

Simplification of the argument

Mathematica uses symmetries and periodicities of all the trigonometric functions to simplify expressions. Here are some examples.

Sin[-z]

```
-Sin[z]
```

Sin[z + π]

```
-Sin[z]
```

Sin[z + 2 π]

Sin[z]

Sin[z + 34 π]

Sin[z]

{Sin[-z], Cos[-z], Tan[-z], Cot[-z], Csc[-z], Sec[-z]}

{-Sin[z], Cos[z], -Tan[z], -Cot[z], -Csc[z], Sec[z]}

{Sin[z + π], Cos[z + π], Tan[z + π], Cot[z + π], Csc[z + π], Sec[z + π]}

{-Sin[z], -Cos[z], Tan[z], Cot[z], -Csc[z], -Sec[z]}

{Sin[z + 2 π], Cos[z + 2 π], Tan[z + 2 π], Cot[z + 2 π], Csc[z + 2 π], Sec[z + 2 π]}

{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}

{Sin[z + 342 π], Cos[z + 342 π], Tan[z + 342 π], Cot[z + 342 π], Csc[z + 342 π], Sec[z + 342 π]}

{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}

Mathematica automatically simplifies the composition of the direct and the inverse trigonometric functions into the argument.

**{Sin[ArcSin[z]], Cos[ArcCos[z]], Tan[ArcTan[z]],
Cot[ArcCot[z]], Csc[ArcCsc[z]], Sec[ArcSec[z]]}**

{z, z, z, z, z, z}

Mathematica also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

**{Sin[ArcSin[z]], Sin[ArcCos[z]], Sin[ArcTan[z]],
Sin[ArcCot[z]], Sin[ArcCsc[z]], Sin[ArcSec[z]]}**

$$\left\{ z, \sqrt{1-z^2}, \frac{z}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}z}}, \frac{1}{z}, \sqrt{1-\frac{1}{z^2}} \right\}$$

**{Cos[ArcSin[z]], Cos[ArcCos[z]], Cos[ArcTan[z]],
Cos[ArcCot[z]], Cos[ArcCsc[z]], Cos[ArcSec[z]]}**

$$\left\{ \sqrt{1-z^2}, z, \frac{1}{\sqrt{1+z^2}}, \frac{1}{\sqrt{1+\frac{1}{z^2}}}, \sqrt{1-\frac{1}{z^2}}, \frac{1}{z} \right\}$$

**{Tan[ArcSin[z]], Tan[ArcCos[z]], Tan[ArcTan[z]],
Tan[ArcCot[z]], Tan[ArcCsc[z]], Tan[ArcSec[z]]}**

$$\left\{ \frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, z, \frac{1}{z}, \frac{1}{\sqrt{1-\frac{1}{z^2}z}}, \sqrt{1-\frac{1}{z^2}z} \right\}$$

$$\{\text{Cot}[\text{ArcSin}[z]], \text{Cot}[\text{ArcCos}[z]], \text{Cot}[\text{ArcTan}[z]], \text{Cot}[\text{ArcCot}[z]], \text{Cot}[\text{ArcCsc}[z]], \text{Cot}[\text{ArcSec}[z]]\}$$

$$\left\{ \frac{\sqrt{1-z^2}}{z}, \frac{z}{\sqrt{1-z^2}}, \frac{1}{z}, z, \sqrt{1-\frac{1}{z^2}z}, \frac{1}{\sqrt{1-\frac{1}{z^2}z}} \right\}$$

$$\{\text{Csc}[\text{ArcSin}[z]], \text{Csc}[\text{ArcCos}[z]], \text{Csc}[\text{ArcTan}[z]], \text{Csc}[\text{ArcCot}[z]], \text{Csc}[\text{ArcCsc}[z]], \text{Csc}[\text{ArcSec}[z]]\}$$

$$\left\{ \frac{1}{z}, \frac{1}{\sqrt{1-z^2}}, \frac{\sqrt{1+z^2}}{z}, \sqrt{1+\frac{1}{z^2}z}, z, \frac{1}{\sqrt{1-\frac{1}{z^2}}} \right\}$$

$$\{\text{Sec}[\text{ArcSin}[z]], \text{Sec}[\text{ArcCos}[z]], \text{Sec}[\text{ArcTan}[z]], \text{Sec}[\text{ArcCot}[z]], \text{Sec}[\text{ArcCsc}[z]], \text{Sec}[\text{ArcSec}[z]]\}$$

$$\left\{ \frac{1}{\sqrt{1-z^2}}, \frac{1}{z}, \sqrt{1+z^2}, \sqrt{1+\frac{1}{z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}}, z \right\}$$

In cases where the argument has the structure $\pi k/2 + z$ or $\pi k/2 - z$, and $\pi k/2 + iz$ or $\pi k/2 - iz$ with integer k , trigonometric functions can be automatically transformed into other trigonometric or hyperbolic functions. Here are some examples.

$$\text{Tan}\left[\frac{\pi}{2} - z\right]$$

$$\text{Cot}[z]$$

$$\text{Csc}[iz]$$

$$-i \text{Csch}[z]$$

$$\left\{ \text{Sin}\left[\frac{\pi}{2} - z\right], \text{Cos}\left[\frac{\pi}{2} - z\right], \text{Tan}\left[\frac{\pi}{2} - z\right], \text{Cot}\left[\frac{\pi}{2} - z\right], \text{Csc}\left[\frac{\pi}{2} - z\right], \text{Sec}\left[\frac{\pi}{2} - z\right] \right\}$$

$$\{\text{Cos}[z], \text{Sin}[z], \text{Cot}[z], \text{Tan}[z], \text{Sec}[z], \text{Csc}[z]\}$$

$$\{\text{Sin}[iz], \text{Cos}[iz], \text{Tan}[iz], \text{Cot}[iz], \text{Csc}[iz], \text{Sec}[iz]\}$$

$$\{i \text{Sinh}[z], \text{Cosh}[z], i \text{Tanh}[z], -i \text{Coth}[z], -i \text{Csch}[z], \text{Sech}[z]\}$$

Simplification of simple expressions containing trigonometric functions

Sometimes simple arithmetic operations containing trigonometric functions can automatically produce other trigonometric functions.

1 / Sec[z]

Cos[z]

{1 / Sin[z], 1 / Cos[z], 1 / Tan[z], 1 / Cot[z], 1 / Csc[z], 1 / Sec[z],
Sin[z] / Cos[z], Cos[z] / Sin[z], Sin[z] / Sin[π / 2 - z], Cos[z] / Sin[z]^2}

{Csc[z], Sec[z], Cot[z], Tan[z], Sin[z], Cos[z], Tan[z], Cot[z], Tan[z], Cot[z] Csc[z]}

Trigonometric functions arising as special cases from more general functions

All trigonometric functions can be treated as particular cases of some more advanced special functions. For example, sin(z) and cos(z) are sometimes the results of auto-simplifications from Bessel, Mathieu, Jacobi, hypergeometric, and Meijer functions (for appropriate values of their parameters).

BesselJ[1/2, z]

$$\frac{\sqrt{\frac{2}{\pi}} \text{Sin}[z]}{\sqrt{z}}$$

MathieuC[1, 0, z]

Cos[z]

JacobiSC[z, 0]

Tan[z]

In[14]:= {BesselJ[1/2, z], MathieuS[1, 0, z], JacobiSN[z, 0],
HypergeometricPFQ[{}, {3/2}, -z^2/4], MeijerG[{{}, {}], {{1/2}, {0}}, z^2/4]}

Out[14]= { $\frac{\sqrt{\frac{2}{\pi}} \text{Sin}[z]}{\sqrt{z}}$, Sin[z], Sin[z], $\frac{\text{Sin}[\sqrt{z^2}]}{\sqrt{z^2}}$, $\frac{\sqrt{z^2} \text{Sin}[z]}{\sqrt{\pi} z}$ }

In[15]:= {BesselJ[-1/2, z], MathieuC[1, 0, z], JacobiCD[z, 0],
Hypergeometric0F1[1/2, -z^2/4], MeijerG[{{}, {}], {{0}, {1/2}}, z^2/4]}

Out[15]= { $\frac{\sqrt{\frac{2}{\pi}} \text{Cos}[z]}{\sqrt{z}}$, Cos[z], Cos[z], $\text{Cos}[\sqrt{z^2}]$, $\frac{\text{Cos}[z]}{\sqrt{\pi}}$ }

In[16]:= {JacobiSC[z, 0], JacobiCS[z, 0], JacobiDS[z, 0], JacobiDC[z, 0]}

Out[16]= {Tan[z], Cot[z], Csc[z], Sec[z]}

Equivalence transformations carried out by specialized *Mathematica* functions

General remarks

Almost everybody prefers using $\sin(z)/2$ instead of $\cos(\pi/2 - z)\sin(\pi/6)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\sin(2z)\sin(\pi/16)$ should not be automatically expanded into the more complicated expression $\sin(z)\cos(z)\left(2 - (2 + 2^{1/2})^{1/2}\right)^{1/2}$. *Mathematica* has special commands that produce these types of expansions. Some of them are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of the trigonometric and hyperbolic functions into sums of powers, using the trigonometric and hyperbolic identities where possible. Here are some examples.

`TrigExpand[Sin[x - y]]`

$\text{Cos}[y] \text{Sin}[x] - \text{Cos}[x] \text{Sin}[y]$

`Cos[4 z] // TrigExpand`

$\text{Cos}[z]^4 - 6 \text{Cos}[z]^2 \text{Sin}[z]^2 + \text{Sin}[z]^4$

`TrigExpand[{{Sin[x + y], Sin[3 z]},
{Cos[x + y], Cos[3 z]},
{Tan[x + y], Tan[3 z]},
{Cot[x + y], Cot[3 z]},
{Csc[x + y], Csc[3 z]},
{Sec[x + y], Sec[3 z]}}`

$\left\{ \left\{ \text{Cos}[y] \text{Sin}[x] + \text{Cos}[x] \text{Sin}[y], 3 \text{Cos}[z]^2 \text{Sin}[z] - \text{Sin}[z]^3 \right\}, \right.$
 $\left\{ \text{Cos}[x] \text{Cos}[y] - \text{Sin}[x] \text{Sin}[y], \text{Cos}[z]^3 - 3 \text{Cos}[z] \text{Sin}[z]^2 \right\},$
 $\left\{ \frac{\text{Cos}[y] \text{Sin}[x]}{\text{Cos}[x] \text{Cos}[y] - \text{Sin}[x] \text{Sin}[y]} + \frac{\text{Cos}[x] \text{Sin}[y]}{\text{Cos}[x] \text{Cos}[y] - \text{Sin}[x] \text{Sin}[y]}, \right.$
 $\left. \frac{3 \text{Cos}[z]^2 \text{Sin}[z]}{\text{Cos}[z]^3 - 3 \text{Cos}[z] \text{Sin}[z]^2} - \frac{\text{Sin}[z]^3}{\text{Cos}[z]^3 - 3 \text{Cos}[z] \text{Sin}[z]^2} \right\},$
 $\left\{ \frac{\text{Cos}[x] \text{Cos}[y]}{\text{Cos}[y] \text{Sin}[x] + \text{Cos}[x] \text{Sin}[y]} - \frac{\text{Sin}[x] \text{Sin}[y]}{\text{Cos}[y] \text{Sin}[x] + \text{Cos}[x] \text{Sin}[y]}, \right.$
 $\left. \frac{\text{Cos}[z]^3}{3 \text{Cos}[z]^2 \text{Sin}[z] - \text{Sin}[z]^3} - \frac{3 \text{Cos}[z] \text{Sin}[z]^2}{3 \text{Cos}[z]^2 \text{Sin}[z] - \text{Sin}[z]^3} \right\},$
 $\left\{ \frac{1}{\text{Cos}[y] \text{Sin}[x] + \text{Cos}[x] \text{Sin}[y]}, \frac{1}{3 \text{Cos}[z]^2 \text{Sin}[z] - \text{Sin}[z]^3} \right\},$
 $\left. \left\{ \frac{1}{\text{Cos}[x] \text{Cos}[y] - \text{Sin}[x] \text{Sin}[y]}, \frac{1}{\text{Cos}[z]^3 - 3 \text{Cos}[z] \text{Sin}[z]^2} \right\} \right\}$

```

TableForm[ (# == TrigExpand[#]) & /@
  Flatten[ {{Sin[x+y], Sin[3 z]}, {Cos[x+y], Cos[3 z]}, {Tan[x+y], Tan[3 z]},
    {Cot[x+y], Cot[3 z]}, {Csc[x+y], Csc[3 z]}, {Sec[x+y], Sec[3 z]}] ] ]
Sin[x+y] == Cos[y] Sin[x] + Cos[x] Sin[y]
Sin[3 z] == 3 Cos[z]^2 Sin[z] - Sin[z]^3
Cos[x+y] == Cos[x] Cos[y] - Sin[x] Sin[y]
Cos[3 z] == Cos[z]^3 - 3 Cos[z] Sin[z]^2
Tan[x+y] ==  $\frac{\text{Cos}[y] \text{Sin}[x]}{\text{Cos}[x] \text{Cos}[y] - \text{Sin}[x] \text{Sin}[y]} + \frac{\text{Cos}[x] \text{Sin}[y]}{\text{Cos}[x] \text{Cos}[y] - \text{Sin}[x] \text{Sin}[y]}$ 
Tan[3 z] ==  $\frac{3 \text{Cos}[z]^2 \text{Sin}[z]}{\text{Cos}[z]^3 - 3 \text{Cos}[z] \text{Sin}[z]^2} - \frac{\text{Sin}[z]^3}{\text{Cos}[z]^3 - 3 \text{Cos}[z] \text{Sin}[z]^2}$ 
Cot[x+y] ==  $\frac{\text{Cos}[x] \text{Cos}[y]}{\text{Cos}[y] \text{Sin}[x] + \text{Cos}[x] \text{Sin}[y]} - \frac{\text{Sin}[x] \text{Sin}[y]}{\text{Cos}[y] \text{Sin}[x] + \text{Cos}[x] \text{Sin}[y]}$ 
Cot[3 z] ==  $\frac{\text{Cos}[z]^3}{3 \text{Cos}[z]^2 \text{Sin}[z] - \text{Sin}[z]^3} - \frac{3 \text{Cos}[z] \text{Sin}[z]^2}{3 \text{Cos}[z]^2 \text{Sin}[z] - \text{Sin}[z]^3}$ 
Csc[x+y] ==  $\frac{1}{\text{Cos}[y] \text{Sin}[x] + \text{Cos}[x] \text{Sin}[y]}$ 
Csc[3 z] ==  $\frac{1}{3 \text{Cos}[z]^2 \text{Sin}[z] - \text{Sin}[z]^3}$ 
Sec[x+y] ==  $\frac{1}{\text{Cos}[x] \text{Cos}[y] - \text{Sin}[x] \text{Sin}[y]}$ 
Sec[3 z] ==  $\frac{1}{\text{Cos}[z]^3 - 3 \text{Cos}[z] \text{Sin}[z]^2}$ 

```

TrigFactor

The function TrigFactor factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials in the trigonometric and hyperbolic functions, using the corresponding identities where possible. Here are some examples.

TrigFactor[**Sin**[x] + **Cos**[y]]

$$\left(\text{Cos}\left[\frac{x}{2} - \frac{y}{2}\right] + \text{Sin}\left[\frac{x}{2} - \frac{y}{2}\right] \right) \left(\text{Cos}\left[\frac{x}{2} + \frac{y}{2}\right] + \text{Sin}\left[\frac{x}{2} + \frac{y}{2}\right] \right)$$

Tan[x] - **Cot**[y] // **TrigFactor**

$$-\text{Cos}[x+y] \text{Csc}[y] \text{Sec}[x]$$

```

TrigFactor[ {Sin[x] + Sin[y],
  Cos[x] + Cos[y],
  Tan[x] + Tan[y],
  Cot[x] + Cot[y],
  Csc[x] + Csc[y],
  Sec[x] + Sec[y]} ] ]

```

$$\left. \begin{aligned} & \left\{ 2 \text{Cos}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Sin}\left[\frac{x}{2} + \frac{y}{2}\right], 2 \text{Cos}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Cos}\left[\frac{x}{2} + \frac{y}{2}\right], \text{Sec}[x] \text{Sec}[y] \text{Sin}[x+y], \right. \\ & \text{Csc}[x] \text{Csc}[y] \text{Sin}[x+y], \frac{1}{2} \text{Cos}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Csc}\left[\frac{x}{2}\right] \text{Csc}\left[\frac{y}{2}\right] \text{Sec}\left[\frac{x}{2}\right] \text{Sec}\left[\frac{y}{2}\right] \text{Sin}\left[\frac{x}{2} + \frac{y}{2}\right], \\ & \left. \frac{2 \text{Cos}\left[\frac{x}{2} - \frac{y}{2}\right] \text{Cos}\left[\frac{x}{2} + \frac{y}{2}\right]}{\left(\text{Cos}\left[\frac{x}{2}\right] - \text{Sin}\left[\frac{x}{2}\right]\right) \left(\text{Cos}\left[\frac{x}{2}\right] + \text{Sin}\left[\frac{x}{2}\right]\right) \left(\text{Cos}\left[\frac{y}{2}\right] - \text{Sin}\left[\frac{y}{2}\right]\right) \left(\text{Cos}\left[\frac{y}{2}\right] + \text{Sin}\left[\frac{y}{2}\right]\right)} \right\} \end{aligned}$$

TrigReduce

The function `TrigReduce` rewrites products and powers of trigonometric and hyperbolic functions in terms of those functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. `TrigReduce` is approximately inverse to `TrigExpand` and `TrigFactor`. Here are some examples.

`TrigReduce[Sin[z]^3]`

$$\frac{1}{4} (3 \sin[z] - \sin[3z])$$

`Sin[x] Cos[y] // TrigReduce`

$$\frac{1}{2} (\sin[x-y] + \sin[x+y])$$

`TrigReduce[{Sin[z]^2, Cos[z]^2, Tan[z]^2, Cot[z]^2, Csc[z]^2, Sec[z]^2}]`

$$\left\{ \frac{1}{2} (1 - \cos[2z]), \frac{1}{2} (1 + \cos[2z]), \frac{1 - \cos[2z]}{1 + \cos[2z]}, \frac{-1 - \cos[2z]}{-1 + \cos[2z]}, -\frac{2}{-1 + \cos[2z]}, \frac{2}{1 + \cos[2z]} \right\}$$

`TrigReduce[TrigExpand[{{Sin[x+y], Sin[3z], Sin[x] Sin[y]},
{Cos[x+y], Cos[3z], Cos[x] Cos[y]},
{Tan[x+y], Tan[3z], Tan[x] Tan[y]},
{Cot[x+y], Cot[3z], Cot[x] Cot[y]},
{Csc[x+y], Csc[3z], Csc[x] Csc[y]},
{Sec[x+y], Sec[3z], Sec[x] Sec[y]}]]]`

$$\left\{ \left\{ \sin[x+y], \sin[3z], \frac{1}{2} (\cos[x-y] - \cos[x+y]) \right\}, \right. \\ \left\{ \cos[x+y], \cos[3z], \frac{1}{2} (\cos[x-y] + \cos[x+y]) \right\}, \\ \left\{ \tan[x+y], \tan[3z], \frac{\cos[x-y] - \cos[x+y]}{\cos[x-y] + \cos[x+y]} \right\}, \\ \left\{ \cot[x+y], \cot[3z], \frac{\cos[x-y] + \cos[x+y]}{\cos[x-y] - \cos[x+y]} \right\}, \\ \left\{ \csc[x+y], \csc[3z], \frac{2}{\cos[x-y] - \cos[x+y]} \right\}, \\ \left. \left\{ \sec[x+y], \sec[3z], \frac{2}{\cos[x-y] + \cos[x+y]} \right\} \right\}$$

`TrigReduce[TrigFactor[{{Sin[x]+Sin[y], Cos[x]+Cos[y],
Tan[x]+Tan[y], Cot[x]+Cot[y], Csc[x]+Csc[y], Sec[x]+Sec[y]}]]]`

$$\left\{ \sin[x] + \sin[y], \cos[x] + \cos[y], \frac{2 \sin[x+y]}{\cos[x-y] + \cos[x+y]}, \right. \\ \left. \frac{2 \sin[x+y]}{\cos[x-y] - \cos[x+y]}, \frac{2 (\sin[x] + \sin[y])}{\cos[x-y] - \cos[x+y]}, \frac{2 (\cos[x] + \cos[y])}{\cos[x-y] + \cos[x+y]} \right\}$$

TrigToExp

The function `TrigToExp` converts direct and inverse trigonometric and hyperbolic functions to exponential or logarithmic functions. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

`TrigToExp[Sin[2 z]]`

$$\frac{1}{2} i e^{-2 i z} - \frac{1}{2} i e^{2 i z}$$

`Sin[z] Tan[2 z] // TrigToExp`

$$-\frac{(e^{-i z} - e^{i z})(e^{-2 i z} - e^{2 i z})}{2(e^{-2 i z} + e^{2 i z})}$$

`TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]`

$$\left\{ \frac{1}{2} i e^{-i z} - \frac{1}{2} i e^{i z}, \frac{e^{-i z}}{2} + \frac{e^{i z}}{2}, \frac{i(e^{-i z} - e^{i z})}{e^{-i z} + e^{i z}}, -\frac{i(e^{-i z} + e^{i z})}{e^{-i z} - e^{i z}}, -\frac{2 i}{e^{-i z} - e^{i z}}, \frac{2}{e^{-i z} + e^{i z}} \right\}$$

ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric or hyperbolic functions. It tries, where possible, to give results that do not involve explicit complex numbers. It is approximately inverse to `TrigToExp`. Here are some examples.

`ExpToTrig[e^{i x \beta}]`

$$\cos[x \beta] + i \sin[x \beta]$$

`\frac{e^{i x \alpha} - e^{i x \beta}}{e^{i x \gamma} + e^{i x \delta}} // ExpToTrig`

$$\frac{\cos[x \alpha] - \cos[x \beta] + i \sin[x \alpha] - i \sin[x \beta]}{\cos[x \gamma] + \cos[x \delta] + i \sin[x \gamma] + i \sin[x \delta]}$$

`ExpToTrig[TrigToExp[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}]]`

$$\{\sin[z], \cos[z], \tan[z], \cot[z], \csc[z], \sec[z]\}$$

`ExpToTrig[{ \alpha e^{-i x \beta} + \alpha e^{i x \beta}, \alpha e^{-i x \beta} + \gamma e^{i x \beta} }]`

$$\{2 \alpha \cos[x \beta], \alpha \cos[x \beta] + \gamma \cos[x \beta] - i \alpha \sin[x \beta] + i \gamma \sin[x \beta]\}$$

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the occurring variables are real. The value option `TargetFunctions` is a list of functions from the set `{Re, Im, Abs, Arg, Conjugate, Sign}`. `ComplexExpand` tries to give results in terms of the specified functions. Here are some examples

`ComplexExpand[Sin[x + i y] Cos[x - i y]]`

$$\begin{aligned} & \text{Cos}[x] \text{Cosh}[y]^2 \text{Sin}[x] - \text{Cos}[x] \text{Sin}[x] \text{Sinh}[y]^2 + \\ & i \left(\text{Cos}[x]^2 \text{Cosh}[y] \text{Sinh}[y] + \text{Cosh}[y] \text{Sin}[x]^2 \text{Sinh}[y] \right) \end{aligned}$$

Csc[x + i y] Sec[x - i y] // ComplexExpand

$$\begin{aligned} & -\frac{4 \text{Cos}[x] \text{Cosh}[y]^2 \text{Sin}[x]}{(\text{Cos}[2x] - \text{Cosh}[2y]) (\text{Cos}[2x] + \text{Cosh}[2y])} + \frac{4 \text{Cos}[x] \text{Sin}[x] \text{Sinh}[y]^2}{(\text{Cos}[2x] - \text{Cosh}[2y]) (\text{Cos}[2x] + \text{Cosh}[2y])} + \\ & i \left(\frac{4 \text{Cos}[x]^2 \text{Cosh}[y] \text{Sinh}[y]}{(\text{Cos}[2x] - \text{Cosh}[2y]) (\text{Cos}[2x] + \text{Cosh}[2y])} + \right. \\ & \left. \frac{4 \text{Cosh}[y] \text{Sin}[x]^2 \text{Sinh}[y]}{(\text{Cos}[2x] - \text{Cosh}[2y]) (\text{Cos}[2x] + \text{Cosh}[2y])} \right) \end{aligned}$$

In[17]:= **li1 = {Sin[x + i y], Cos[x + i y], Tan[x + i y], Cot[x + i y], Csc[x + i y], Sec[x + i y]}**

Out[17]= {Sin[x + i y], Cos[x + i y], Tan[x + i y], Cot[x + i y], Csc[x + i y], Sec[x + i y]}

In[18]:= **ComplexExpand[li1]**

$$\begin{aligned} \text{Out[18]= } & \left\{ \text{Cosh}[y] \text{Sin}[x] + i \text{Cos}[x] \text{Sinh}[y], \text{Cos}[x] \text{Cosh}[y] - i \text{Sin}[x] \text{Sinh}[y], \right. \\ & \frac{\text{Sin}[2x]}{\text{Cos}[2x] + \text{Cosh}[2y]} + \frac{i \text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}, -\frac{\text{Sin}[2x]}{\text{Cos}[2x] - \text{Cosh}[2y]} + \frac{i \text{Sinh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \\ & \left. -\frac{2 \text{Cosh}[y] \text{Sin}[x]}{\text{Cos}[2x] - \text{Cosh}[2y]} + \frac{2 i \text{Cos}[x] \text{Sinh}[y]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{2 \text{Cos}[x] \text{Cosh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]} + \frac{2 i \text{Sin}[x] \text{Sinh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]} \right\} \end{aligned}$$

In[19]:= **ComplexExpand[Re[#] & /@ li1, TargetFunctions -> {Re, Im}]**

$$\begin{aligned} \text{Out[19]= } & \left\{ \text{Cosh}[y] \text{Sin}[x], \text{Cos}[x] \text{Cosh}[y], \frac{\text{Sin}[2x]}{\text{Cos}[2x] + \text{Cosh}[2y]}, \right. \\ & \left. -\frac{\text{Sin}[2x]}{\text{Cos}[2x] - \text{Cosh}[2y]}, -\frac{2 \text{Cosh}[y] \text{Sin}[x]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{2 \text{Cos}[x] \text{Cosh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]} \right\} \end{aligned}$$

In[20]:= **ComplexExpand[Im[#] & /@ li1, TargetFunctions -> {Re, Im}]**

$$\begin{aligned} \text{Out[20]= } & \left\{ \text{Cos}[x] \text{Sinh}[y], -\text{Sin}[x] \text{Sinh}[y], \frac{\text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}, \right. \\ & \left. \frac{\text{Sinh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{2 \text{Cos}[x] \text{Sinh}[y]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{2 \text{Sin}[x] \text{Sinh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]} \right\} \end{aligned}$$

In[21]:= **ComplexExpand[Abs[#] & /@ li1, TargetFunctions -> {Re, Im}]**

$$\text{Out[21]} = \left\{ \sqrt{\text{Cosh}[y]^2 \text{Sin}[x]^2 + \text{Cos}[x]^2 \text{Sinh}[y]^2}, \sqrt{\text{Cos}[x]^2 \text{Cosh}[y]^2 + \text{Sin}[x]^2 \text{Sinh}[y]^2}, \right. \\ \sqrt{\frac{\text{Sin}[2x]^2}{(\text{Cos}[2x] + \text{Cosh}[2y])^2} + \frac{\text{Sinh}[2y]^2}{(\text{Cos}[2x] + \text{Cosh}[2y])^2}}, \\ \sqrt{\frac{\text{Sin}[2x]^2}{(\text{Cos}[2x] - \text{Cosh}[2y])^2} + \frac{\text{Sinh}[2y]^2}{(\text{Cos}[2x] - \text{Cosh}[2y])^2}}, \\ \sqrt{\frac{4 \text{Cosh}[y]^2 \text{Sin}[x]^2}{(\text{Cos}[2x] - \text{Cosh}[2y])^2} + \frac{4 \text{Cos}[x]^2 \text{Sinh}[y]^2}{(\text{Cos}[2x] - \text{Cosh}[2y])^2}}, \\ \left. \sqrt{\frac{4 \text{Cos}[x]^2 \text{Cosh}[y]^2}{(\text{Cos}[2x] + \text{Cosh}[2y])^2} + \frac{4 \text{Sin}[x]^2 \text{Sinh}[y]^2}{(\text{Cos}[2x] + \text{Cosh}[2y])^2}} \right\}$$

In[22]:= % // Simplify[#, {x, y} ∈ Reals] &

$$\text{Out[22]} = \left\{ \frac{\sqrt{-\text{Cos}[2x] + \text{Cosh}[2y]}}{\sqrt{2}}, \frac{\sqrt{\text{Cos}[2x] + \text{Cosh}[2y]}}{\sqrt{2}}, \frac{\sqrt{\text{Sin}[2x]^2 + \text{Sinh}[2y]^2}}{\text{Cos}[2x] + \text{Cosh}[2y]}, \right. \\ \left. \sqrt{-\frac{\text{Cos}[2x] + \text{Cosh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}}, \frac{\sqrt{2}}{\sqrt{-\text{Cos}[2x] + \text{Cosh}[2y]}}, \frac{\sqrt{2}}{\sqrt{\text{Cos}[2x] + \text{Cosh}[2y]}} \right\}$$

In[23]:= ComplexExpand[Arg[#] & /@ l11, TargetFunctions → {Re, Im}]

$$\text{Out[23]} = \left\{ \text{ArcTan}[\text{Cosh}[y] \text{Sin}[x], \text{Cos}[x] \text{Sinh}[y]], \text{ArcTan}[\text{Cos}[x] \text{Cosh}[y], -\text{Sin}[x] \text{Sinh}[y]], \right. \\ \text{ArcTan}\left[\frac{\text{Sin}[2x]}{\text{Cos}[2x] + \text{Cosh}[2y]}, \frac{\text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}\right], \\ \text{ArcTan}\left[-\frac{\text{Sin}[2x]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{\text{Sinh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}\right], \\ \text{ArcTan}\left[-\frac{2 \text{Cosh}[y] \text{Sin}[x]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \frac{2 \text{Cos}[x] \text{Sinh}[y]}{\text{Cos}[2x] - \text{Cosh}[2y]}\right], \\ \left. \text{ArcTan}\left[\frac{2 \text{Cos}[x] \text{Cosh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]}, \frac{2 \text{Sin}[x] \text{Sinh}[y]}{\text{Cos}[2x] + \text{Cosh}[2y]}\right]\right\}$$

In[24]:= ComplexExpand[Conjugate[#] & /@ l11, TargetFunctions → {Re, Im}] // Simplify

$$\text{Out[24]} = \left\{ \text{Cosh}[y] \text{Sin}[x] - i \text{Cos}[x] \text{Sinh}[y], \text{Cos}[x] \text{Cosh}[y] + i \text{Sin}[x] \text{Sinh}[y], \right. \\ \frac{\text{Sin}[2x] - i \text{Sinh}[2y]}{\text{Cos}[2x] + \text{Cosh}[2y]}, -\frac{\text{Sin}[2x] + i \text{Sinh}[2y]}{\text{Cos}[2x] - \text{Cosh}[2y]}, \\ \left. \frac{1}{\text{Cosh}[y] \text{Sin}[x] - i \text{Cos}[x] \text{Sinh}[y]}, \frac{1}{\text{Cos}[x] \text{Cosh}[y] + i \text{Sin}[x] \text{Sinh}[y]} \right\}$$

Simplify

The function `Simplify` performs a sequence of algebraic transformations on its argument, and returns the simplest form it finds. Here are two examples.

`Simplify[Sin[2 z] / Sin[z]]`

`2 Cos[z]`

`Sin[2 z] / Cos[z] // Simplify`

`2 Sin[z]`

Here is a large collection of trigonometric identities. All are written as one large logical conjunction.

$$\text{Simplify}[\#] \ \& \ /@ \left(\begin{aligned} & \text{Cos}[z]^2 + \text{Sin}[z]^2 == 1 \wedge \\ & \text{Sin}[z]^2 == \frac{1 - \text{Cos}[2z]}{2} \wedge \text{Cos}[z]^2 == \frac{1 + \text{Cos}[2z]}{2} \wedge \\ & \text{Tan}[z]^2 == \frac{1 - \text{Cos}[2z]}{1 + \text{Cos}[2z]} \wedge \text{Cot}[z]^2 == \frac{1 + \text{Cos}[2z]}{1 - \text{Cos}[2z]} \wedge \\ & \text{Sin}[2z] == 2 \text{Sin}[z] \text{Cos}[z] \wedge \text{Cos}[2z] == \text{Cos}[z]^2 - \text{Sin}[z]^2 == 2 \text{Cos}[z]^2 - 1 \wedge \\ & \text{Sin}[a + b] == \text{Sin}[a] \text{Cos}[b] + \text{Cos}[a] \text{Sin}[b] \wedge \text{Sin}[a - b] == \text{Sin}[a] \text{Cos}[b] - \text{Cos}[a] \text{Sin}[b] \wedge \\ & \text{Cos}[a + b] == \text{Cos}[a] \text{Cos}[b] - \text{Sin}[a] \text{Sin}[b] \wedge \text{Cos}[a - b] == \text{Cos}[a] \text{Cos}[b] + \text{Sin}[a] \text{Sin}[b] \wedge \\ & \text{Sin}[a] + \text{Sin}[b] == 2 \text{Sin}\left[\frac{a + b}{2}\right] \text{Cos}\left[\frac{a - b}{2}\right] \wedge \text{Sin}[a] - \text{Sin}[b] == 2 \text{Cos}\left[\frac{a + b}{2}\right] \text{Sin}\left[\frac{a - b}{2}\right] \wedge \\ & \text{Cos}[a] + \text{Cos}[b] == 2 \text{Cos}\left[\frac{a + b}{2}\right] \text{Cos}\left[\frac{a - b}{2}\right] \wedge \text{Cos}[a] - \text{Cos}[b] == 2 \text{Sin}\left[\frac{a + b}{2}\right] \text{Sin}\left[\frac{b - a}{2}\right] \wedge \\ & \text{Tan}[a] + \text{Tan}[b] == \frac{\text{Sin}[a + b]}{\text{Cos}[a] \text{Cos}[b]} \wedge \text{Tan}[a] - \text{Tan}[b] == \frac{\text{Sin}[a - b]}{\text{Cos}[a] \text{Cos}[b]} \wedge \\ & A \text{Sin}[z] + B \text{Cos}[z] == A \sqrt{1 + \frac{B^2}{A^2}} \text{Sin}\left[z + \text{ArcTan}\left[\frac{B}{A}\right]\right] \wedge \\ & \text{Sin}[a] \text{Sin}[b] == \frac{\text{Cos}[a - b] - \text{Cos}[a + b]}{2} \wedge \\ & \text{Cos}[a] \text{Cos}[b] == \frac{\text{Cos}[a - b] + \text{Cos}[a + b]}{2} \wedge \text{Sin}[a] \text{Cos}[b] == \frac{\text{Sin}[a + b] + \text{Sin}[a - b]}{2} \wedge \\ & \text{Sin}\left[\frac{z}{2}\right]^2 == \frac{1 - \text{Cos}[z]}{2} \wedge \text{Cos}\left[\frac{z}{2}\right]^2 == \frac{1 + \text{Cos}[z]}{2} \wedge \\ & \text{Tan}\left[\frac{z}{2}\right] == \frac{1 - \text{Cos}[z]}{\text{Sin}[z]} == \frac{\text{Sin}[z]}{1 + \text{Cos}[z]} \wedge \text{Cot}\left[\frac{z}{2}\right] == \frac{\text{Sin}[z]}{1 - \text{Cos}[z]} == \frac{1 + \text{Cos}[z]}{\text{Sin}[z]} \end{aligned} \right)$$

`True`

The function `Simplify` has the `Assumption` option. For example, *Mathematica* knows that $-1 \leq \sin(x) \leq 1$ for all real x , and uses the periodicity of trigonometric functions for the symbolic integer coefficient k of $k\pi$.

```
Simplify[Abs[Sin[x]] ≤ 1, x ∈ Reals]
```

```
True
```

```
Abs[Sin[x]] ≤ 1 // Simplify[#, x ∈ Reals] &
```

```
True
```

```
Simplify[{Sin[z + 2 k π], Cos[z + 2 k π], Tan[z + k π],  
Cot[z + k π], Csc[z + 2 k π], Sec[z + 2 k π]}, k ∈ Integers]
```

```
{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}
```

```
Simplify[{Sin[z + k π] / Sin[z], Cos[z + k π] / Cos[z], Tan[z + k π] / Tan[z],  
Cot[z + k π] / Cot[z], Csc[z + k π] / Csc[z], Sec[z + k π] / Sec[z]}, k ∈ Integers]
```

```
{(-1)k, (-1)k, 1, 1, (-1)k, (-1)k}
```

Mathematica also knows that the composition of inverse and direct trigonometric functions produces the value of the inner argument under the appropriate restriction. Here are some examples.

```
Simplify[{ArcSin[Sin[z]], ArcTan[Tan[z]], ArcCot[Cot[z]], ArcCsc[Csc[z]]},  
-π / 2 < Re[z] < π / 2]
```

```
{z, z, z, z}
```

```
Simplify[{ArcCos[Cos[z]], ArcSec[Sec[z]]}, 0 < Re[z] < π]
```

```
{z, z}
```

FunctionExpand (and Together)

While the trigonometric functions auto-evaluate for simple fractions of π , for more complicated cases they stay as trigonometric functions to avoid the build up of large expressions. Using the function `FunctionExpand`, such expressions can be transformed into explicit radicals.

$$\text{Cos}\left[\frac{\pi}{32}\right]$$

$$\text{Cos}\left[\frac{\pi}{32}\right]$$

$$\text{FunctionExpand}\left[\text{Cos}\left[\frac{\pi}{32}\right]\right]$$

$$\frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

$$\text{Cot}\left[\frac{\pi}{24}\right] // \text{FunctionExpand}$$

$$\frac{\frac{\sqrt{2-\sqrt{2}}}{4} + \frac{1}{4}\sqrt{3(2+\sqrt{2})}}{-\frac{1}{4}\sqrt{3(2-\sqrt{2})} + \frac{\sqrt{2+\sqrt{2}}}{4}}$$

$$\left\{ \sin\left[\frac{\pi}{16}\right], \cos\left[\frac{\pi}{16}\right], \tan\left[\frac{\pi}{16}\right], \cot\left[\frac{\pi}{16}\right], \csc\left[\frac{\pi}{16}\right], \sec\left[\frac{\pi}{16}\right] \right\}$$

$$\left\{ \sin\left[\frac{\pi}{16}\right], \cos\left[\frac{\pi}{16}\right], \tan\left[\frac{\pi}{16}\right], \cot\left[\frac{\pi}{16}\right], \csc\left[\frac{\pi}{16}\right], \sec\left[\frac{\pi}{16}\right] \right\}$$

FunctionExpand[%]

$$\left\{ \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}, \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}, \sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}, \right.$$

$$\left. \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{2-\sqrt{2+\sqrt{2}}}}, \frac{2}{\sqrt{2-\sqrt{2+\sqrt{2}}}}, \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \right\}$$

$$\left\{ \sin\left[\frac{\pi}{60}\right], \cos\left[\frac{\pi}{60}\right], \tan\left[\frac{\pi}{60}\right], \cot\left[\frac{\pi}{60}\right], \csc\left[\frac{\pi}{60}\right], \sec\left[\frac{\pi}{60}\right] \right\}$$

$$\left\{ \sin\left[\frac{\pi}{60}\right], \cos\left[\frac{\pi}{60}\right], \tan\left[\frac{\pi}{60}\right], \cot\left[\frac{\pi}{60}\right], \csc\left[\frac{\pi}{60}\right], \sec\left[\frac{\pi}{60}\right] \right\}$$

Together[FunctionExpand[%]]

$$\left\{ \frac{1}{16} \left(-\sqrt{2} - \sqrt{6} + \sqrt{10} + \sqrt{30} + 2\sqrt{5+\sqrt{5}} - 2\sqrt{3(5+\sqrt{5})} \right), \right.$$

$$\frac{1}{16} \left(\sqrt{2} - \sqrt{6} - \sqrt{10} + \sqrt{30} + 2\sqrt{5+\sqrt{5}} + 2\sqrt{3(5+\sqrt{5})} \right),$$

$$\frac{-1 - \sqrt{3} + \sqrt{5} + \sqrt{15} + \sqrt{2(5+\sqrt{5})} - \sqrt{6(5+\sqrt{5})}}{1 - \sqrt{3} - \sqrt{5} + \sqrt{15} + \sqrt{2(5+\sqrt{5})} + \sqrt{6(5+\sqrt{5})}},$$

$$\frac{-1 + \sqrt{3} + \sqrt{5} - \sqrt{15} - \sqrt{2(5+\sqrt{5})} - \sqrt{6(5+\sqrt{5})}}{1 + \sqrt{3} - \sqrt{5} - \sqrt{15} - \sqrt{2(5+\sqrt{5})} + \sqrt{6(5+\sqrt{5})}},$$

$$\frac{16}{-\sqrt{2} - \sqrt{6} + \sqrt{10} + \sqrt{30} + 2\sqrt{5+\sqrt{5}} - 2\sqrt{3(5+\sqrt{5})}},$$

$$\left. \frac{16}{\sqrt{2} - \sqrt{6} - \sqrt{10} + \sqrt{30} + 2\sqrt{5+\sqrt{5}} + 2\sqrt{3(5+\sqrt{5})}} \right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$$\left\{ \text{Sin}\left[\frac{\pi}{9}\right], \text{Cos}\left[\frac{\pi}{9}\right], \text{Tan}\left[\frac{\pi}{9}\right], \text{Cot}\left[\frac{\pi}{9}\right], \text{Csc}\left[\frac{\pi}{9}\right], \text{Sec}\left[\frac{\pi}{9}\right] \right\}$$

$$\left\{ \text{Sin}\left[\frac{\pi}{9}\right], \text{Cos}\left[\frac{\pi}{9}\right], \text{Tan}\left[\frac{\pi}{9}\right], \text{Cot}\left[\frac{\pi}{9}\right], \text{Csc}\left[\frac{\pi}{9}\right], \text{Sec}\left[\frac{\pi}{9}\right] \right\}$$

FunctionExpand[%] // Together

$$\left\{ \frac{1}{8} \left(-i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} (-1 + i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right), \right. \\ \frac{1}{8} \left(2^{2/3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} (-1 + i \sqrt{3})^{1/3} - i 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right), \\ \frac{-(-1 - i \sqrt{3})^{1/3} - i \sqrt{3} (-1 - i \sqrt{3})^{1/3} + (-1 + i \sqrt{3})^{1/3} - i \sqrt{3} (-1 + i \sqrt{3})^{1/3}}{-i (-1 - i \sqrt{3})^{1/3} + \sqrt{3} (-1 - i \sqrt{3})^{1/3} - i (-1 + i \sqrt{3})^{1/3} - \sqrt{3} (-1 + i \sqrt{3})^{1/3}}, \\ \frac{(-1 - i \sqrt{3})^{1/3} + i \sqrt{3} (-1 - i \sqrt{3})^{1/3} + (-1 + i \sqrt{3})^{1/3} - i \sqrt{3} (-1 + i \sqrt{3})^{1/3}}{-i (-1 - i \sqrt{3})^{1/3} + \sqrt{3} (-1 - i \sqrt{3})^{1/3} + i (-1 + i \sqrt{3})^{1/3} + \sqrt{3} (-1 + i \sqrt{3})^{1/3}}, \\ 8 \left/ \left(-i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} + i 2^{2/3} (-1 + i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right), \right. \\ \left. - (8 i) \left/ \left(-i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} - i 2^{2/3} (-1 + i \sqrt{3})^{1/3} - 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right) \right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as numbered roots of polynomial equations.

`RootReduce[Simplify[%]]`

$$\left\{ \text{Root}[-3 + 36 \#1^2 - 96 \#1^4 + 64 \#1^6 \&, 4], \text{Root}[-1 - 6 \#1 + 8 \#1^3 \&, 3], \right. \\ \text{Root}[-3 + 27 \#1^2 - 33 \#1^4 + \#1^6 \&, 4], \text{Root}[-1 + 33 \#1^2 - 27 \#1^4 + 3 \#1^6 \&, 6], \\ \left. \text{Root}[-64 + 96 \#1^2 - 36 \#1^4 + 3 \#1^6 \&, 6], \text{Root}[-8 + 6 \#1^2 + \#1^3 \&, 3] \right\}$$

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including hyperbolic functions, to simpler ones. Here are some examples.

`FunctionExpand[Cot[Sqrt[-z^2]]]`

$$\frac{\sqrt{-z} \text{Coth}[z]}{\sqrt{z}}$$

`Tan[Sqrt[i z^2]] // FunctionExpand`

$$\frac{(-1)^{3/4} \sqrt{-(-1)^{3/4} z} \sqrt{(-1)^{3/4} z} \text{Tan}\left[(-1)^{1/4} z\right]}{z}$$

$$\{\text{Sin}[\sqrt{z^2}], \text{Cos}[\sqrt{z^2}], \text{Tan}[\sqrt{z^2}], \text{Cot}[\sqrt{z^2}], \text{Csc}[\sqrt{z^2}], \text{Sec}[\sqrt{z^2}]\} // \text{FunctionExpand}$$

$$\left\{ \frac{\sqrt{-i z} \sqrt{i z} \text{Sin}[z]}{z}, \text{Cos}[z], \frac{\sqrt{-i z} \sqrt{i z} \text{Tan}[z]}{z}, \right.$$

$$\left. \frac{\sqrt{-i z} \sqrt{i z} \text{Cot}[z]}{z}, \frac{\sqrt{-i z} \sqrt{i z} \text{Csc}[z]}{z}, \text{Sec}[z] \right\}$$

Applying Simplify to the last expression gives a more compact result.

Simplify[%]

$$\left\{ \frac{\sqrt{z^2} \text{Sin}[z]}{z}, \text{Cos}[z], \frac{\sqrt{z^2} \text{Tan}[z]}{z}, \frac{\sqrt{z^2} \text{Cot}[z]}{z}, \frac{\sqrt{z^2} \text{Csc}[z]}{z}, \text{Sec}[z] \right\}$$

Here are some similar examples.

Sin[2 ArcTan[z]] // FunctionExpand

$$\frac{2 z}{1 + z^2}$$

Cos[$\frac{\text{ArcCot}[z]}{2}$] // FunctionExpand

$$\frac{\sqrt{1 + \frac{\sqrt{-z} \sqrt{z}}{\sqrt{-1-z^2}}}}{\sqrt{2}}$$

{Sin[2 ArcSin[z]], Cos[2 ArcCos[z]], Tan[2 ArcTan[z]], Cot[2 ArcCot[z]], Csc[2 ArcCsc[z]], Sec[2 ArcSec[z]]} // FunctionExpand

$$\left\{ 2 \sqrt{1-z} z \sqrt{1+z}, -1+2 z^2, -\frac{2 z}{(-1+z)(1+z)}, \right.$$

$$\left. \frac{1}{2} \left(1 + \frac{1}{z^2} \right) z \left(\frac{1}{-1-z^2} - \frac{z^2}{-1-z^2} \right), \frac{\sqrt{-i z} \sqrt{i z} z}{2 \sqrt{(-1+z)(1+z)}}, \frac{z^2}{2-z^2} \right\}$$

{Sin[$\frac{\text{ArcSin}[z]}{2}$], Cos[$\frac{\text{ArcCos}[z]}{2}$], Tan[$\frac{\text{ArcTan}[z]}{2}$], Cot[$\frac{\text{ArcCot}[z]}{2}$], Csc[$\frac{\text{ArcCsc}[z]}{2}$], Sec[$\frac{\text{ArcSec}[z]}{2}$]} // FunctionExpand

$$\left\{ \frac{z \sqrt{1 - \sqrt{1 - z}} \sqrt{1 + z}}{\sqrt{2} \sqrt{-i z} \sqrt{i z}}, \frac{\sqrt{1 + z}}{\sqrt{2}}, \frac{z}{1 + \sqrt{i} (-i + z) \sqrt{-i} (i + z)}, \right. \\ \left. z \left(1 + \frac{\sqrt{-1 - z^2}}{\sqrt{-z} \sqrt{z}} \right), \frac{\sqrt{2} \sqrt{-\frac{i}{z}} \sqrt{\frac{i}{z}} z}{\sqrt{1 - \frac{\sqrt{(-1+z)(1+z)}}{\sqrt{-i z} \sqrt{i z}}}}, \frac{\sqrt{2} \sqrt{-z}}{\sqrt{-1 - z}} \right\}$$

Simplify[%]

$$\left\{ \frac{z \sqrt{1 - \sqrt{1 - z^2}}}{\sqrt{2} \sqrt{z^2}}, \frac{\sqrt{1 + z}}{\sqrt{2}}, \frac{z}{1 + \sqrt{1 + z^2}}, z + \frac{\sqrt{z} \sqrt{-1 - z^2}}{\sqrt{-z}}, \frac{\sqrt{2} \sqrt{\frac{1}{z^2}} z}{\sqrt{1 - \frac{\sqrt{z^2} \sqrt{-1 + z^2}}{z^2}}}, \frac{\sqrt{2}}{\sqrt{1 + \frac{1}{z}}} \right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\text{Cos}\left[\frac{1}{2} i \text{Log}[1 - i z] - \frac{1}{2} i \text{Log}[1 + i z]\right] // \text{Simplify}$$

$$\text{Cosh}\left[\frac{1}{2} (\text{Log}[1 - i z] - \text{Log}[1 + i z])\right]$$

$$\text{Cos}\left[\frac{1}{2} i \text{Log}[1 - i z] - \frac{1}{2} i \text{Log}[1 + i z]\right] // \text{FullSimplify}$$

$$\frac{1}{\sqrt{1 + z^2}}$$

$$\left\{ \text{Sin}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Cos}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \right. \\ \text{Tan}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Cot}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \\ \left. \text{Csc}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Sec}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right] \right\} // \text{Simplify}$$

$$\left\{ z, \frac{1 - z^2 + i z \sqrt{1 - z^2}}{i z + \sqrt{1 - z^2}}, \frac{z (z - i \sqrt{1 - z^2})}{-i + i z^2 + z \sqrt{1 - z^2}}, \frac{1 - z^2 + i z \sqrt{1 - z^2}}{i z^2 + z \sqrt{1 - z^2}}, \frac{1}{z}, \frac{2 (i z + \sqrt{1 - z^2})}{1 + (i z + \sqrt{1 - z^2})^2} \right\}$$

$$\left\{ \text{Sin}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Cos}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \right. \\ \text{Tan}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Cot}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \\ \left. \text{Csc}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right], \text{Sec}\left[-i \text{Log}\left[i z + \sqrt{1 - z^2}\right]\right] \right\} // \text{FullSimplify}$$

$$\left\{ z, \sqrt{1-z^2}, \frac{z}{\sqrt{1-z^2}}, \frac{\sqrt{1-z^2}}{z}, \frac{1}{z}, \frac{1}{\sqrt{1-z^2}} \right\}$$

Operations carried out by specialized *Mathematica* functions

Series expansions

Calculating the series expansion of trigonometric functions to hundreds of terms can be done in seconds. Here are some examples.

Series[Sin[z], {z, 0, 5}]

$$z - \frac{z^3}{6} + \frac{z^5}{120} + O[z]^6$$

Normal[%]

$$z - \frac{z^3}{6} + \frac{z^5}{120}$$

Series[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, 0, 3}]

$$\left\{ z - \frac{z^3}{6} + O[z]^4, 1 - \frac{z^2}{2} + O[z]^4, z + \frac{z^3}{3} + O[z]^4, \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + O[z]^4, \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + O[z]^4, 1 + \frac{z^2}{2} + O[z]^4 \right\}$$

Series[Cot[z], {z, 0, 100}] // Timing

$$\left\{ 1.442 \text{ Second}, \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \frac{z^7}{4725} - \frac{2z^9}{93555} - \frac{1382z^{11}}{638512875} - \frac{4z^{13}}{18243225} - \frac{3617z^{15}}{162820783125} - \frac{87734z^{17}}{38979295480125} - \frac{349222z^{19}}{1531329465290625} - \frac{310732z^{21}}{13447856940643125} - \frac{472728182z^{23}}{201919571963756521875} - \frac{2631724z^{25}}{11094481976030578125} - \frac{13571120588z^{27}}{56465366017076273671875} - \frac{13785346041608z^{29}}{5660878804669082674070015625} - \frac{7709321041217z^{31}}{3124511028551170603633203125} - \frac{303257395102z^{33}}{12130454581433748587292890625} - \frac{52630543106106954746z^{35}}{20777977561866588586487628662044921875} - \frac{61684082396644z^{37}}{2403467618492375776343276883984375} - \frac{522165436992898244102z^{39}}{20080431172289638826798401128390556640625} - \frac{6080390575672283210764z^{41}}{2307789189818960127712594427864667427734375} - \frac{10121188937927645176372z^{43}}{3791367954702577352670690845777679169921875} \right\}$$

$$\begin{aligned}
 & \frac{207\,461\,256\,206\,578\,143\,748\,856\,z^{45}}{7\,670\,102\,214\,448\,301\,053\,033\,358\,480\,610\,212\,529\,462\,890\,625} - \\
 & \frac{11\,218\,806\,737\,995\,635\,372\,498\,255\,094\,z^{47}}{4\,093\,648\,603\,384\,274\,996\,519\,698\,921\,478\,879\,580\,162\,286\,669\,921\,875} - \\
 & \frac{79\,209\,152\,838\,572\,743\,713\,996\,404\,z^{49}}{285\,258\,771\,457\,546\,764\,463\,363\,635\,252\,374\,414\,183\,254\,365\,234\,375} - \\
 & \frac{246\,512\,528\,657\,073\,833\,030\,130\,766\,724\,z^{51}}{8\,761\,982\,491\,474\,419\,367\,550\,817\,114\,626\,909\,562\,924\,278\,968\,505\,859\,375} - \\
 & \frac{233\,199\,709\,079\,078\,899\,371\,344\,990\,501\,528\,z^{53}}{81\,807\,125\,729\,900\,063\,867\,074\,959\,072\,425\,603\,825\,198\,823\,017\,351\,806\,640\,625} - \\
 & \frac{1\,416\,795\,959\,607\,558\,144\,963\,094\,708\,378\,988\,z^{55}}{4\,905\,352\,087\,939\,496\,310\,826\,487\,207\,538\,302\,184\,255\,342\,959\,123\,162\,841\,796\,875} - \\
 & \frac{23\,305\,824\,372\,104\,839\,134\,357\,731\,308\,699\,592\,z^{57}}{796\,392\,368\,980\,577\,121\,745\,974\,726\,570\,063\,253\,238\,310\,542\,073\,919\,837\,646\,484\,375} - \\
 & \frac{9\,721\,865\,123\,870\,044\,576\,322\,439\,952\,638\,561\,968\,331\,928\,z^{59}}{3\,278\,777\,586\,273\,629\,598\,615\,520\,165\,380\,455\,583\,231\,003\,564\,645\,636\,125\,000\,418\,914\,794\,921\,875} - \\
 & \frac{6\,348\,689\,256\,302\,894\,731\,330\,601\,216\,724\,328\,336\,z^{61}}{21\,132\,271\,510\,899\,613\,925\,529\,439\,369\,536\,628\,424\,678\,570\,233\,931\,462\,891\,949\,462\,890\,625} - \\
 & \frac{106\,783\,830\,147\,866\,529\,886\,385\,444\,979\,142\,647\,942\,017\,z^{63}}{3\,508\,062\,732\,166\,890\,409\,707\,514\,582\,539\,928\,001\,638\,766\,051\,683\,792\,497\,378\,070\,587\,158\,203\,125} \\
 & (267\,745\,458\,568\,424\,664\,373\,021\,714\,282\,169\,516\,771\,254\,382\,z^{65}) / \\
 & 86\,812\,790\,293\,146\,213\,360\,651\,966\,604\,262\,937\,105\,495\,141\,563\,588\,806\,888\,204\,273\,501\,373\,291\,015 : \\
 & 625 - (250\,471\,004\,320\,250\,327\,955\,196\,022\,920\,428\,000\,776\,938\,z^{67}) / \\
 & 801\,528\,196\,428\,242\,695\,121\,010\,267\,455\,843\,804\,062\,822\,357\,897\,831\,858\,125\,102\,407\,684\,326\,171\,875 \\
 & - (172\,043\,582\,552\,384\,800\,434\,637\,321\,986\,040\,823\,829\,878\,646\,884\,z^{69}) / \\
 & 5\,433\,748\,964\,547\,053\,581\,149\,916\,185\,708\,338\,218\,048\,392\,402\,830\,337\,634\,114\,958\,370\,880\,742\,156 : \\
 & 982\,421\,875 - (11\,655\,909\,923\,339\,888\,220\,876\,554\,489\,282\,134\,730\,564\,976\,603\,688\,520\,858\,z^{71}) / \\
 & 3\,633\,348\,205\,269\,879\,230\,856\,840\,004\,304\,821\,536\,968\,049\,780\,112\,803\,650\,817\,771\,432\,558\,560\,793 : \\
 & 458\,452\,606\,201\,171\,875 - \\
 & (3\,692\,153\,220\,456\,342\,488\,035\,683\,646\,645\,690\,290\,452\,790\,030\,604\,z^{73}) / \\
 & 11\,359\,005\,221\,796\,317\,918\,049\,302\,062\,760\,294\,302\,183\,889\,391\,189\,419\,445\,133\,951\,612\,582\,060\,536 : \\
 & 346\,435\,546\,875 - (5\,190\,545\,015\,986\,394\,254\,249\,936\,008\,544\,252\,611\,445\,319\,542\,919\,116\,z^{75}) / \\
 & 157\,606\,197\,452\,423\,911\,112\,934\,066\,120\,799\,083\,442\,801\,465\,302\,753\,194\,801\,233\,578\,624\,576\,089 : \\
 & 941\,806\,793\,212\,890\,625 - \\
 & (255\,290\,071\,123\,323\,586\,643\,187\,098\,799\,718\,199\,072\,122\,692\,536\,861\,835\,992\,z^{77}) / \\
 & 76\,505\,736\,228\,426\,953\,173\,738\,238\,352\,183\,101\,801\,688\,392\,812\,244\,485\,181\,277\,127\,930\,109\,049\,138 : \\
 & 257\,655\,704\,498\,291\,015\,625 - \\
 & (9\,207\,568\,598\,958\,915\,293\,871\,149\,938\,038\,093\,699\,588\,515\,745\,502\,577\,839\,313\,734\,z^{79}) / \\
 & 27\,233\,582\,984\,369\,795\,892\,070\,228\,410\,001\,578\,355\,986\,013\,571\,390\,071\,723\,225\,259\,349\,721\,067\,988 : \\
 & 068\,852\,863\,296\,604\,156\,494\,140\,625 - \\
 & (163\,611\,136\,505\,867\,886\,519\,332\,147\,296\,221\,453\,678\,803\,514\,884\,902\,772\,183\,572\,z^{81}) / \\
 & 4\,776\,089\,171\,877\,348\,057\,451\,105\,924\,101\,750\,653\,118\,402\,745\,283\,825\,543\,113\,171\,217\,116\,857\,704 : \\
 & 024\,700\,607\,798\,175\,811\,767\,578\,125 - \\
 & (8\,098\,304\,783\,741\,161\,440\,924\,524\,640\,446\,924\,039\,959\,669\,564\,792\,363\,509\,124\,335\,729\,908\,z^{83}) /
 \end{aligned}$$

$$\begin{aligned}
 & 2\ 333\ 207\ 846\ 470\ 426\ 678\ 843\ 707\ 227\ 616\ 712\ 214\ 909\ 162\ 634\ 745\ 895\ 349\ 325\ 948\ 586\ 531\ 533\ 393\ \backslash \\
 & 530\ 725\ 143\ 500\ 144\ 033\ 328\ 342\ 437\ 744\ 140\ 625\ - \\
 & (122\ 923\ 650\ 124\ 219\ 284\ 385\ 832\ 157\ 660\ 699\ 813\ 260\ 991\ 755\ 656\ 444\ 452\ 420\ 836\ 648\ z^{85}) / \\
 & 349\ 538\ 086\ 043\ 843\ 717\ 584\ 559\ 187\ 055\ 386\ 621\ 548\ 470\ 304\ 913\ 596\ 772\ 372\ 737\ 435\ 524\ 697\ 231\ \backslash \\
 & 069\ 047\ 713\ 981\ 709\ 496\ 784\ 210\ 205\ 078\ 125\ - \\
 & (476\ 882\ 359\ 517\ 824\ 548\ 362\ 004\ 154\ 188\ 840\ 670\ 307\ 545\ 554\ 753\ 464\ 961\ 562\ 516\ 323\ 845\ 108\ z^{87}) / \\
 & 13\ 383\ 510\ 964\ 174\ 348\ 021\ 497\ 060\ 628\ 653\ 950\ 829\ 663\ 288\ 548\ 327\ 870\ 152\ 944\ 013\ 988\ 358\ 928\ 114\ \backslash \\
 & 528\ 962\ 242\ 087\ 062\ 453\ 152\ 690\ 410\ 614\ 013\ 671\ 875\ - \\
 & (1\ 886\ 491\ 646\ 433\ 732\ 479\ 814\ 597\ 361\ 998\ 744\ 134\ 040\ 407\ 919\ 471\ 435\ 385\ 970\ 472\ 345\ 164\ 676\ 056 \\
 & z^{89}) / \\
 & 522\ 532\ 651\ 330\ 971\ 490\ 226\ 753\ 590\ 247\ 329\ 744\ 050\ 384\ 290\ 675\ 644\ 135\ 735\ 656\ 667\ 608\ 610\ 471\ \backslash \\
 & 400\ 391\ 047\ 234\ 539\ 824\ 350\ 830\ 981\ 313\ 610\ 076\ 904\ 296\ 875\ - \\
 & (450\ 638\ 590\ 680\ 882\ 618\ 431\ 105\ 331\ 665\ 591\ 912\ 924\ 988\ 342\ 163\ 281\ 788\ 877\ 675\ 244\ 114\ 763\ 912 \\
 & z^{91}) / \\
 & 1\ 231\ 931\ 818\ 039\ 911\ 948\ 327\ 467\ 370\ 123\ 161\ 265\ 684\ 460\ 571\ 086\ 659\ 079\ 080\ 437\ 659\ 781\ 065\ 743\ \backslash \\
 & 269\ 173\ 212\ 919\ 832\ 661\ 978\ 537\ 311\ 246\ 395\ 111\ 083\ 984\ 375\ - \\
 & (415\ 596\ 189\ 473\ 955\ 564\ 121\ 634\ 614\ 268\ 323\ 814\ 113\ 534\ 779\ 643\ 471\ 190\ 276\ 158\ 333\ 713\ 923\ 216 \\
 & z^{93}) / \\
 & 11\ 213\ 200\ 675\ 690\ 943\ 223\ 287\ 032\ 785\ 929\ 540\ 201\ 272\ 600\ 687\ 465\ 377\ 745\ 332\ 153\ 847\ 964\ 679\ 254\ \backslash \\
 & 692\ 602\ 138\ 023\ 498\ 144\ 562\ 090\ 675\ 557\ 613\ 372\ 802\ 734\ 375\ - \\
 & (423\ 200\ 899\ 194\ 533\ 026\ 195\ 195\ 456\ 219\ 648\ 467\ 346\ 087\ 908\ 778\ 120\ 468\ 301\ 277\ 466\ 840\ 101\ 336\ \backslash \\
 & 699\ 974\ 518\ z^{95}) / \\
 & 112\ 694\ 926\ 530\ 960\ 148\ 011\ 367\ 752\ 417\ 874\ 063\ 473\ 378\ 698\ 369\ 880\ 587\ 800\ 838\ 274\ 234\ 349\ 237\ \backslash \\
 & 591\ 647\ 453\ 413\ 782\ 021\ 538\ 312\ 594\ 164\ 677\ 406\ 144\ 702\ 434\ 539\ 794\ 921\ 875\ - \\
 & (5\ 543\ 531\ 483\ 502\ 489\ 438\ 698\ 050\ 411\ 951\ 314\ 743\ 456\ 505\ 773\ 755\ 468\ 368\ 087\ 670\ 306\ 121\ 873\ 229\ \backslash \\
 & 244\ z^{97}) / \\
 & 14\ 569\ 479\ 835\ 935\ 377\ 894\ 165\ 191\ 004\ 250\ 040\ 526\ 616\ 509\ 162\ 234\ 077\ 285\ 176\ 247\ 476\ 968\ 227\ 225\ \backslash \\
 & 810\ 918\ 346\ 966\ 001\ 491\ 701\ 692\ 846\ 112\ 140\ 419\ 483\ 184\ 814\ 453\ 125\ - \\
 & (378\ 392\ 151\ 276\ 488\ 501\ 180\ 909\ 732\ 277\ 974\ 887\ 490\ 811\ 366\ 132\ 267\ 744\ 533\ 542\ 784\ 817\ 245\ 581\ \backslash \\
 & 660\ 788\ 990\ 844\ z^{99}) / \\
 & 9\ 815\ 205\ 420\ 757\ 514\ 710\ 108\ 178\ 059\ 369\ 553\ 458\ 327\ 392\ 260\ 750\ 404\ 049\ 930\ 407\ 987\ 933\ 582\ 359\ \backslash \\
 & 080\ 767\ 225\ 644\ 716\ 670\ 683\ 512\ 153\ 512\ 547\ 802\ 166\ 033\ 089\ 160\ 919\ 189\ 453\ 125 + O[z]^{101} \}
 \end{aligned}$$

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term for odd trigonometric functions can be evaluated.

```

<< DiscreteMath`RSolve`
SeriesTerm[{Sin[z], Tan[z], Cot[z], Csc[z], Cos[z], Sec[z]}, {z, 0, n}]

```

$$\left\{ \frac{i^{-1+n} \text{KroneckerDelta}[\text{Mod}[-1+n, 2]] \text{UnitStep}[-1+n]}{\text{Gamma}[1+n]}, \right. \\ \text{If}\left[\text{Odd}[n], \frac{i^{-1+n} 2^{1+n} (-1+2^{1+n}) \text{BernoulliB}[1+n]}{(1+n)!}, 0\right], \frac{i i^n 2^{1+n} \text{BernoulliB}[1+n]}{(1+n)!}, \\ \left. \frac{i i^n 2^{1+n} \text{BernoulliB}\left[1+n, \frac{1}{2}\right]}{(1+n)!}, \frac{i^n \text{KroneckerDelta}[\text{Mod}[n, 2]]}{\text{Gamma}[1+n]}, \frac{i^n \text{EulerE}[n]}{n!} \right\}$$

Differentiation

Mathematica can evaluate derivatives of trigonometric functions of an arbitrary positive integer order.

D[Sin[z], z]

Cos[z]

Sin[z] // D[#, z] &

Cos[z]

∂_z{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}

{Cos[z], -Sin[z], Sec[z]², -Csc[z]², -Cot[z] Csc[z], Sec[z] Tan[z]}

∂_{z,2}{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}

{-Sin[z], -Cos[z], 2 Sec[z]² Tan[z], 2 Cot[z] Csc[z]²,
Cot[z]² Csc[z] + Csc[z]³, Sec[z]³ + Sec[z] Tan[z]²}

Table[D[{Sin[z], Cos[z], Tan[z], Cot[z], Csc[z], Sec[z]}, {z, n}], {n, 4}]

{ {Cos[z], -Sin[z], Sec[z]², -Csc[z]², -Cot[z] Csc[z], Sec[z] Tan[z]}, {-Sin[z], -Cos[z],
2 Sec[z]² Tan[z], 2 Cot[z] Csc[z]², Cot[z]² Csc[z] + Csc[z]³, Sec[z]³ + Sec[z] Tan[z]²},
{-Cos[z], Sin[z], 2 Sec[z]⁴ + 4 Sec[z]² Tan[z]², -4 Cot[z]² Csc[z]² - 2 Csc[z]⁴,
-Cot[z]³ Csc[z] - 5 Cot[z] Csc[z]³, 5 Sec[z]³ Tan[z] + Sec[z] Tan[z]³},
{Sin[z], Cos[z], 16 Sec[z]⁴ Tan[z] + 8 Sec[z]² Tan[z]³,
8 Cot[z]³ Csc[z]² + 16 Cot[z] Csc[z]⁴, Cot[z]⁴ Csc[z] + 18 Cot[z]² Csc[z]³ + 5 Csc[z]⁵,
5 Sec[z]⁵ + 18 Sec[z]³ Tan[z]² + Sec[z] Tan[z]⁴}}

Finite summation

Mathematica can calculate finite sums that contain trigonometric functions. Here are two examples.

Sum[Sin[a k], {k, 0, n}]

$$\frac{1}{2} \left(\cos\left[\frac{a}{2}\right] - \cos\left[\frac{a}{2} + a n\right] \right) \text{Csc}\left[\frac{a}{2}\right]$$

$$\sum_{k=0}^n (-1)^k \text{Sin}[a k]$$

$$\frac{1}{2} \operatorname{Sec}\left[\frac{a}{2}\right] \left(-\operatorname{Sin}\left[\frac{a}{2}\right] + \operatorname{Sin}\left[\frac{a}{2} + a n + n \pi\right]\right)$$

Infinite summation

Mathematica can calculate infinite sums that contain trigonometric functions. Here are some examples.

$$\sum_{k=1}^{\infty} z^k \operatorname{Sin}[k x]$$

$$\frac{i (-1 + e^{2 i x}) z}{2 (e^{i x} - z) (-1 + e^{i x} z)}$$

$$\sum_{k=1}^{\infty} \frac{\operatorname{Sin}[k x]}{k!}$$

$$\frac{1}{2} i (e^{-i x} - e^{i x})$$

$$\sum_{k=1}^{\infty} \frac{\operatorname{Cos}[k x]}{k}$$

$$\frac{1}{2} (-\operatorname{Log}[1 - e^{-i x}] - \operatorname{Log}[1 - e^{i x}])$$

Finite products

Mathematica can calculate some finite symbolic products that contain the trigonometric functions. Here are two examples.

$$\operatorname{Product}\left[\operatorname{Sin}\left[\frac{\pi k}{n}\right], \{k, 1, n-1\}\right]$$

$$2^{1-n} n$$

$$\prod_{k=1}^{n-1} \operatorname{Cos}\left[z + \frac{\pi k}{n}\right]$$

$$-(-1)^n 2^{1-n} \operatorname{Sec}[z] \operatorname{Sin}\left[\frac{1}{2} n (\pi - 2 z)\right]$$

Infinite products

Mathematica can calculate infinite products that contain trigonometric functions. Here are some examples.

$$\text{In[2]:= } \prod_{k=1}^{\infty} \operatorname{Exp}\left[z^k \operatorname{Sin}[k x]\right]$$

$$\text{Out[2]= } e^{\frac{i (-1 + e^{2 i x}) z}{2 (z + e^{2 i x} z - e^{i x} (1 + z^2))}}$$

$$\text{In[3]:= } \prod_{k=1}^{\infty} \text{Exp}\left[\frac{\text{Cos}[k x]}{k!}\right]$$

$$\text{Out[3]= } e^{\frac{1}{2}(-2 + e^{-1} + e^1)}$$

Indefinite integration

Mathematica can calculate a huge number of doable indefinite integrals that contain trigonometric functions. Here are some examples.

$$\int \text{Sin}[7 z] \, dz$$

$$-\frac{1}{7} \text{Cos}[7 z]$$

$$\int \left\{ \left\{ \text{Sin}[z], \text{Sin}[z]^a \right\}, \left\{ \text{Cos}[z], \text{Cos}[z]^a \right\}, \left\{ \text{Tan}[z], \text{Tan}[z]^a \right\}, \right. \\ \left. \left\{ \text{Cot}[z], \text{Cot}[z]^a \right\}, \left\{ \text{Csc}[z], \text{Csc}[z]^a \right\}, \left\{ \text{Sec}[z], \text{Sec}[z]^a \right\} \right\} dz$$

$$\left\{ \left\{ -\text{Cos}[z], -\text{Cos}[z] \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1-a}{2}, \frac{3}{2}, \text{Cos}[z]^2\right] \text{Sin}[z]^{1+a} (\text{Sin}[z]^2)^{\frac{1}{2}(-1-a)} \right\}, \right.$$

$$\left\{ \text{Sin}[z], -\frac{\text{Cos}[z]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, \frac{1}{2}, \frac{3+a}{2}, \text{Cos}[z]^2\right] \text{Sin}[z]}{(1+a) \sqrt{\text{Sin}[z]^2}} \right\},$$

$$\left\{ -\text{Log}[\text{Cos}[z]], \frac{\text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, -\text{Tan}[z]^2\right] \text{Tan}[z]^{1+a}}{1+a} \right\},$$

$$\left\{ \text{Log}[\text{Sin}[z]], -\frac{\text{Cot}[z]^{1+a} \text{Hypergeometric2F1}\left[\frac{1+a}{2}, 1, 1 + \frac{1+a}{2}, -\text{Cot}[z]^2\right]}{1+a} \right\},$$

$$\left\{ -\text{Log}\left[\text{Cos}\left[\frac{z}{2}\right]\right] + \text{Log}\left[\text{Sin}\left[\frac{z}{2}\right]\right], \right.$$

$$\left. -\text{Cos}[z] \text{Csc}[z]^{-1+a} \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{1+a}{2}, \frac{3}{2}, \text{Cos}[z]^2\right] (\text{Sin}[z]^2)^{\frac{1}{2}(-1+a)} \right\},$$

$$\left\{ -\text{Log}\left[\text{Cos}\left[\frac{z}{2}\right] - \text{Sin}\left[\frac{z}{2}\right]\right] + \text{Log}\left[\text{Cos}\left[\frac{z}{2}\right] + \text{Sin}\left[\frac{z}{2}\right]\right], \right.$$

$$\left. -\frac{\text{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \text{Cos}[z]^2\right] \text{Sec}[z]^{-1+a} \text{Sin}[z]}{(1-a) \sqrt{\text{Sin}[z]^2}} \right\}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain trigonometric functions. Here are some examples.

$$\int_0^{\pi/2} \sqrt[3]{\text{Sin}[z]} \, dz$$

$$\frac{\sqrt{\pi} \text{Gamma}\left[\frac{2}{3}\right]}{2 \text{Gamma}\left[\frac{7}{6}\right]}$$

$$\int_0^{\pi/2} \left\{ \sqrt{\text{Sin}[z]}, \sqrt{\text{Cos}[z]}, \sqrt{\text{Tan}[z]}, \sqrt{\text{Cot}[z]}, \sqrt{\text{Csc}[z]}, \sqrt{\text{Sec}[z]} \right\} dz$$

$$\left\{ 2 \text{EllipticE}\left[\frac{\pi}{4}, 2\right], 2 \text{EllipticE}\left[\frac{\pi}{4}, 2\right], \frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}, \frac{2 \sqrt{\pi} \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]}, \frac{2 \sqrt{\pi} \text{Gamma}\left[\frac{5}{4}\right]}{\text{Gamma}\left[\frac{3}{4}\right]} \right\}$$

$$\int_0^{\frac{\pi}{2}} \left\{ \{\text{Sin}[z], \text{Sin}[z]^a\}, \{\text{Cos}[z], \text{Cos}[z]^a\}, \{\text{Tan}[z], \text{Tan}[z]^a\}, \right.$$

$$\left. \{\text{Cot}[z], \text{Cot}[z]^a\}, \{\text{Csc}[z], \text{Csc}[z]^a\}, \{\text{Sec}[z], \text{Sec}[z]^a\} \right\} dz$$

$$\left\{ \left\{ 1, \frac{\sqrt{\pi} \text{Gamma}\left[\frac{1+a}{2}\right]}{a \text{Gamma}\left[\frac{a}{2}\right]} \right\}, \left\{ 1, \frac{\sqrt{\pi} \text{Gamma}\left[\frac{1+a}{2}\right]}{a \text{Gamma}\left[\frac{a}{2}\right]} \right\}, \right.$$

$$\left. \left\{ \int_0^{\frac{\pi}{2}} \text{Tan}[z] dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2} \pi \text{Sec}\left[\frac{a \pi}{2}\right], \int_0^{\frac{\pi}{2}} \text{Tan}[z]^a dz\right] \right\}, \right.$$

$$\left. \left\{ \int_0^{\frac{\pi}{2}} \text{Cot}[z] dz, \text{If}\left[\text{Re}[a] < 1, \frac{1}{2} \pi \text{Sec}\left[\frac{a \pi}{2}\right], \int_0^{\frac{\pi}{2}} \text{Cot}[z]^a dz\right] \right\}, \right.$$

$$\left. \left\{ \int_0^{\frac{\pi}{2}} \text{Csc}[z] dz, \frac{\sqrt{\pi} \text{Gamma}\left[\frac{1}{2} - \frac{a}{2}\right]}{2 \text{Gamma}\left[1 - \frac{a}{2}\right]} \right\}, \left\{ \int_0^{\frac{\pi}{2}} \text{Sec}[z] dz, \frac{\sqrt{\pi} \text{Gamma}\left[\frac{1}{2} - \frac{a}{2}\right]}{2 \text{Gamma}\left[1 - \frac{a}{2}\right]} \right\} \right\}$$

Limit operation

Mathematica can calculate limits that contain trigonometric functions.

$$\text{Limit}\left[\frac{\text{Sin}[z]}{z} + \text{Cos}[z]^3, z \rightarrow 0\right]$$

2

$$\text{Limit}\left[\left(\frac{\text{Tan}[x]}{x}\right)^{\frac{1}{x^2}}, x \rightarrow 0\right]$$

e^{1/3}

Solving equations

The next input solves equations that contain trigonometric functions. The message indicates that the multivalued functions are used to express the result and that some solutions might be absent.

$$\text{Solve}\left[\text{Tan}[z]^2 + 3 \text{Sin}\left[z + \text{Pi}/6\right] == 4, z\right]$$

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

$$\left\{ \left\{ z \rightarrow \text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 1\right]\right]\right\}, \right. \\ \left. \left\{ z \rightarrow -\text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 2\right]\right]\right\}, \right. \\ \left. \left\{ z \rightarrow -\text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 3\right]\right]\right\}, \right. \\ \left. \left\{ z \rightarrow \text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 4\right]\right]\right\}, \right. \\ \left. \left\{ z \rightarrow \text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 5\right]\right]\right\}, \right. \\ \left. \left\{ z \rightarrow \text{ArcCos}\left[\text{Root}\left[4 - 40 \#1^2 + 12 \#1^3 + 73 \#1^4 - 60 \#1^5 + 36 \#1^6 \&, 6\right]\right]\right\} \right\}$$

Complete solutions can be obtained by using the function `Reduce`.

Reduce[Sin[x] == a, x] // TraditionalForm

```
// InputForm =
C[1] ∈ Integers && (x == Pi - ArcSin[a] + 2 * Pi * C[1] || x == ArcSin[a] + 2 * Pi * C[1])
```

Reduce[Cos[x] == a, x] // TraditionalForm

```
// InputForm = C[1] ∈ Integers && (x == -ArcCos[a] + 2 * Pi * C[1] || x == ArcCos[a] + 2 * Pi * C[1])
```

Reduce[Tan[x] == a, x] // TraditionalForm

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcTan[a] + Pi * C[1]
```

Reduce[Cot[x] == a, x] // TraditionalForm

```
// InputForm = C[1] ∈ Integers && 1 + a^2 ≠ 0 && x == ArcCot[a] + Pi * C[1]
```

Reduce[Csc[x] == a, x] // TraditionalForm

$$c_1 \in \mathbb{Z} \wedge a \neq 0 \wedge \left(x = -\sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 + \pi \vee x = \sin^{-1}\left(\frac{1}{a}\right) + 2\pi c_1 \right)$$

Reduce[Sec[x] == a, x] // TraditionalForm

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
(x == -ArcCos[a^(-1)] + 2 * Pi * C[1] || x == ArcCos[a^(-1)] + 2 * Pi * C[1])
```

Solving differential equations

Here are differential equations whose linear-independent solutions are trigonometric functions. The solutions of the simplest second-order linear ordinary differential equation with constant coefficients can be represented through $\sin(z)$ and $\cos(z)$.

DSolve[w''[z] + w[z] == 0, w[z], z]

```
{{w[z] → C[1] Cos[z] + C[2] Sin[z]}}
```

dsol1 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w[z], z]

```
{{{w[z] → C[3] Cos[z] + C[1] Cos[√2 z] + C[4] Sin[z] + C[2] Sin[√2 z]}}}
```

In the last input, the differential equation was solved for $w(z)$. If the argument is suppressed, the result is returned as a pure function (in the sense of the λ -calculus).

```
dsol2 = DSolve[2 w[z] + 3 w''[z] + w^(4)[z] == 0, w, z]
{{w -> Function[{z}, C[3] Cos[z] + C[1] Cos[Sqrt[2] z] + C[4] Sin[z] + C[2] Sin[Sqrt[2] z]]}}
```

The advantage of such a pure function is that it can be used for different arguments, derivatives, and more.

```
w'[ξ] /. dsol1
```

```
{w'[ξ]}
```

```
w'[ξ] /. dsol2
```

```
{C[4] Cos[ξ] + Sqrt[2] C[2] Cos[Sqrt[2] ξ] - C[3] Sin[ξ] - Sqrt[2] C[1] Sin[Sqrt[2] ξ]}
```

All trigonometric functions satisfy first-order nonlinear differential equations. In carrying out the algorithm to solve the nonlinear differential equation, *Mathematica* has to solve a transcendental equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

```
DSolve[{w'[z] == Sqrt[1 - w[z]^2], w[0] == 0}, w[z], z]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] -> Sin[z]}}
```

```
DSolve[{w'[z] == Sqrt[1 - w[z]^2], w[0] == 1}, w[z], z]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] -> Cos[z]}}
```

```
DSolve[{w'[z] - w[z]^2 - 1 == 0, w[0] == 0}, w[z], z]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] -> Tan[z]}}
```

```
DSolve[{w'[z] + w[z]^2 + 1 == 0, w[Pi/2] == 0}, w[z], z]
```

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] -> Cot[z]}}
```

```
DSolve[{w'[z] == Sqrt[w[z]^4 - w[z]^2], 1/w[0] == 0}, w[z], z] // Simplify[#, 0 < z < Pi/2] &
```

Solve::verif: Potential solution {C[1] -> Indeterminate} (possibly discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

Solve::verif: Potential solution {C[1] -> Indeterminate} (possibly discarded by verifier) should be checked by hand. May require use of limits.

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{{w[z] -> -Csc[z]}, {w[z] -> Csc[z]}}
```

```
DSolve[{w'[z] == Sqrt[w[z]^4 - w[z]^2], 1/w[z] == 0}, w[z], z] // Simplify[#, 0 < z < Pi/2] &
```

```
Solve::verif: Potential solution {C[1] -> Indeterminate} (possibly
discarded by verifier) should be checked by hand. May require use of limits.
```

```
Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.
```

```
Solve::verif: Potential solution {C[1] -> Indeterminate} (possibly
discarded by verifier) should be checked by hand. May require use of limits.
```

```
Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.
```

```
{{w[z] -> -Sec[z]}, {w[z] -> Sec[z]}}
```

Integral transforms

Mathematica supports the main integral transforms like direct and inverse Fourier, Laplace, and Z transforms that can give results that contain classical or generalized functions. Here are some transforms of trigonometric functions.

```
LaplaceTransform[Sin[t], t, s]
```

$$\frac{1}{1 + s^2}$$

```
FourierTransform[Sin[t], t, s]
```

$$i \sqrt{\frac{\pi}{2}} \text{DiracDelta}[-1 + s] - i \sqrt{\frac{\pi}{2}} \text{DiracDelta}[1 + s]$$

```
FourierSinTransform[Sin[t], t, s]
```

$$\sqrt{\frac{\pi}{2}} \text{DiracDelta}[-1 + s] - \sqrt{\frac{\pi}{2}} \text{DiracDelta}[1 + s]$$

```
FourierCosTransform[Sin[t], t, s]
```

$$-\frac{1}{\sqrt{2\pi}(-1 + s)} + \frac{1}{\sqrt{2\pi}(1 + s)}$$

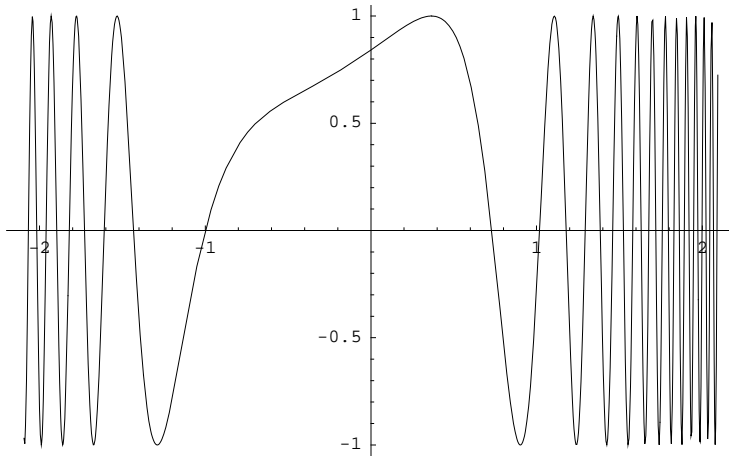
```
ZTransform[Sin[pi t], t, s]
```

```
0
```

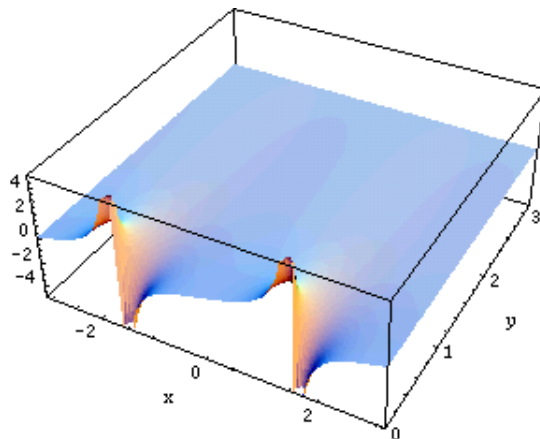
Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

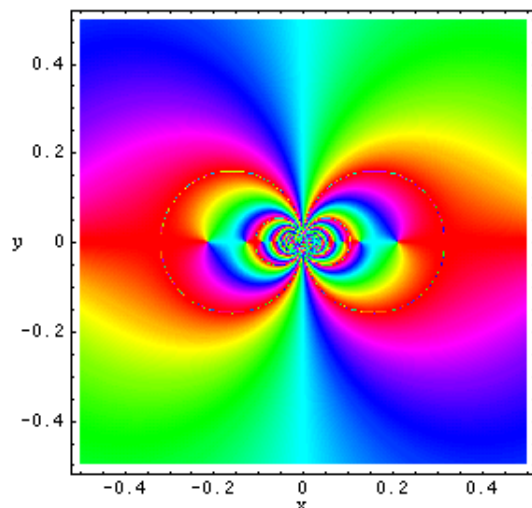
```
Plot[Sin[Sum[z^k, {k, 0, 5}], {z, -2/3, 2/3}];
```



```
Plot3D[Re[Tan[x + i y]], {x, -π, π}, {y, 0, π},
  PlotPoints → 240, PlotRange → {-5, 5},
  ClipFill → None, Mesh → False, AxesLabel → {"x", "y", None}];
```



```
ContourPlot[Arg[Sec[1/(x + i y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},
  PlotPoints → 400, PlotRange → {-π, π}, FrameLabel → {"x", "y", None, None},
  ColorFunction → Hue, ContourLines → False, Contours → 200];
```



Introduction to the Secant Function in *Mathematica*

Overview

The following shows how the secant function is realized in *Mathematica*. Examples of evaluating *Mathematica* functions applied to various numeric and exact expressions that involve the secant function or return it are shown. These involve numeric and symbolic calculations and plots.

Notations

Mathematica forms of notations

Following *Mathematica*'s general naming convention, function names in `StandardForm` are just the capitalized versions of their traditional mathematics names. This shows the secant function in `StandardForm`.

```
Sec[z]
```

```
Sec[z]
```

This shows the secant function in `TraditionalForm`.

```
% // TraditionalForm
```

```
sec(z)
```

Additional forms of notations

Mathematica has other popular forms of notations that are used for print and electronic publications. In this particular instance the task is not difficult. However, it must be made to work in *Mathematica*'s `CForm`, `TeXForm`, and `FortranForm`.

```
{CForm[Sec[2 Pi z]], FortranForm[Sec[2 Pi z]], TeXForm[Sec[2 Pi z]]}
```

```
{Sec(2 * Pi * z), Sec(2 * Pi * z), \sec(2 \, \pi \, , z)}
```

Automatic evaluations and transformations

Evaluation for exact and machine-number values of arguments

For the exact argument $z = \pi/4$, *Mathematica* returns an exact result.

$$\text{Sec}\left[\frac{\pi}{4}\right]$$

$$\sqrt{2}$$

$$\text{Sec}[z] /. z \rightarrow \frac{\pi}{4}$$

$$\sqrt{2}$$

For a machine-number argument (numerical argument with a decimal point), a machine number is also returned.

$$\text{Sec}[4.]$$

$$-1.52989$$

$$\text{Sec}[z] /. z \rightarrow 2.$$

$$-2.403$$

The next inputs calculate 100-digit approximations at $z = 1$ and $z = 2$.

$$\text{N}[\text{Sec}[z] /. z \rightarrow 1, 100]$$

$$1.8508157176809256179117532413986501934703966550940092988351582778588154112615967059\backslash$$

$$21841413287306671$$

$$\text{N}[\text{Sec}[2], 100]$$

$$-2.402997961722380989754600401420066226245121093154526013044400970212074081045498386\backslash$$

$$923810087998256180$$

$$\text{Sec}[2] // \text{N}[\#, 100] \&$$

$$-2.402997961722380989754600401420066226245121093154526013044400970212074081045498386\backslash$$

$$923810087998256180$$

Within a second, it is possible to calculate thousands of digits for the secant function. The next input calculates 10000 digits for $\text{sec}(1)$ and analyzes the frequency of the digit k in the resulting decimal number.

$$\text{Map}[\text{Function}[w, \{\text{First}[\#], \text{Length}[\#]\} \& /@ \text{Split}[\text{Sort}[\text{First}[\text{RealDigits}[w]]]]],$$

$$\text{N}[\{\text{Sec}[z]\} /. z \rightarrow 1, 10000]$$

$$\{\{0, 978\}, \{1, 1034\}, \{2, 1016\}, \{3, 974\},$$

$$\{4, 987\}, \{5, 1067\}, \{6, 943\}, \{7, 1006\}, \{8, 1027\}, \{9, 968\}\}$$

Here is a 50-digit approximation to the secant function at the complex argument $z = 3 - 2i$.

$$\text{N}[\text{Sec}[3 - 2i], 50]$$

```
-0.26351297515838930964360417460542626393326353695118 -
0.036211636558768520871456897789535761206336177766579 i
```

```
{N[Sec[z] /. z -> 3 - 2 i, 50], Sec[3 - 2 i] // N[#, 50] &}
```

```
{-0.26351297515838930964360417460542626393326353695118 -
0.036211636558768520871456897789535761206336177766579 i,
-0.26351297515838930964360417460542626393326353695118 -
0.036211636558768520871456897789535761206336177766579 i}
```

Mathematica automatically evaluates mathematical functions with machine precision, if the arguments of the function are numerical values and include machine-number elements. In this case, only six digits after the decimal point are shown in the results. The remaining digits are suppressed, but can be displayed using the function `InputForm`.

```
{Sec[3.], N[Sec[3]], N[Sec[3], 16], N[Sec[3], 5], N[Sec[3], 20]}
```

```
{-1.01011, -1.01011, -1.01011, -1.01011, -1.0101086659079937513}
```

```
% // InputForm
```

```
{-1.0101086659079939, -1.0101086659079939, -1.0101086659079939, -1.0101086659079939,
-1.01010866590799375130303648146318085228`20}
```

Simplification of the argument

Mathematica knows the symmetry and periodicity of the secant function. Here are some examples.

```
Sec[-3]
```

```
Sec[3]
```

```
{Sec[-z], Sec[z + π], Sec[z + 2 π], Sec[-z + 21 π]}
```

```
{Sec[z], -Sec[z], Sec[z], -Sec[z]}
```

Mathematica automatically simplifies the composition of the direct and the inverse secant functions into its argument.

```
Sec[ArcSec[z]]
```

```
z
```

Mathematica also automatically simplifies the composition of the direct and any of the inverse trigonometric functions into algebraic functions of the argument.

```
{Sec[ArcSin[z]], Sec[ArcCos[z]], Sec[ArcTan[z]],
Sec[ArcCot[z]], Sec[ArcCsc[z]], Sec[ArcSec[z]]}
```

$$\left\{ \frac{1}{\sqrt{1-z^2}}, \frac{1}{z}, \sqrt{1+z^2}, \sqrt{1+\frac{1}{z^2}}, \frac{1}{\sqrt{1-\frac{1}{z^2}}}, z \right\}$$

In cases where the argument has the structure $\pi k/2 + z$ or $\pi k/2 - z$, and $\pi k/2 + iz$ or $\pi k/2 - iz$ with integer k , the secant function can be automatically transformed into trigonometric or hyperbolic secant or cosecant functions.

$$\text{Sec}\left[\frac{\pi}{2} - z\right]$$

$$\text{Csc}[z]$$

$$\left\{ \text{Sec}\left[\frac{\pi}{2} - z\right], \text{Sec}\left[\frac{\pi}{2} + z\right], \text{Sec}\left[-\frac{\pi}{2} - z\right], \text{Sec}\left[-\frac{\pi}{2} + z\right], \text{Sec}[\pi - z], \text{Sec}[\pi + z] \right\}$$

$$\{\text{Csc}[z], -\text{Csc}[z], -\text{Csc}[z], \text{Csc}[z], -\text{Sec}[z], -\text{Sec}[z]\}$$

$$\text{Sec}[iz]$$

$$\text{Sech}[z]$$

$$\left\{ \text{Sec}[iz], \text{Sec}\left[\frac{\pi}{2} - iz\right], \text{Sec}\left[\frac{\pi}{2} + iz\right], \text{Sec}[\pi - iz], \text{Sec}[\pi + iz] \right\}$$

$$\{\text{Sech}[z], -i \text{Csch}[z], i \text{Csch}[z], -\text{Sech}[z], -\text{Sech}[z]\}$$

Simplification of combinations of secant functions

Sometimes simple arithmetic operations containing the secant function can automatically generate other equal trigonometric functions.

$$1 / \text{Sec}[z]$$

$$\text{Cos}[z]$$

$$\{1 / \text{Sec}[z], 1 / \text{Sec}[\pi/2 - z], \text{Sec}[\pi/2 - z] / \text{Sec}[z], \text{Sec}[z] / \text{Sec}[\pi/2 - z], 1 / \text{Sec}[\pi/2 - z], \text{Sec}[\pi/2 - z] / \text{Sec}[z]^2\}$$

$$\{\text{Cos}[z], \text{Sin}[z], \text{Cot}[z], \text{Tan}[z], \text{Sin}[z], \text{Cos}[z] \text{Cot}[z]\}$$

The secant function arising as special cases from more general functions

The secant function can be treated as a particular case of some more general special functions. For example, $\sec(z)$ appears automatically from Bessel, Struve, Mathieu, Jacobi, hypergeometric, and Meijer functions for appropriate values of their parameters.

$$\text{In[440]:= } \left\{ \sqrt{\frac{2}{\pi}} / \left(\sqrt{z} \text{BesselJ}\left[-\frac{1}{2}, z\right] \right), \frac{1}{1 - \sqrt{\frac{\pi z}{2}} \text{StruveH}\left[\frac{1}{2}, z\right]}, \frac{1}{\text{MathieuC}[1, 0, z]}, \right. \\ \text{JacobiDC}[z, 0], \text{JacobiNC}[z, 0], \text{JacobiDS}\left[\frac{\pi}{2} - z, 0\right], \text{JacobiNS}\left[\frac{\pi}{2} - z, 0\right], \\ \text{JacobiCN}[iz, 1], \text{JacobiDN}[iz, 1], i \text{JacobiCS}\left[\frac{\pi i}{2} - iz, 1\right], i \text{JacobiDS}\left[\frac{\pi i}{2} - iz, 1\right], \\ \left. 1 / \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{1}{2}\right\}, -\frac{z^2}{4}\right], 1 / \text{MeijerG}\left[\{\{\}, \{\}\}, \left\{-\frac{1}{2}\right\}, \{0\}\right], \frac{z^2}{4}\right\}$$

$$\text{Out[440]= } \left\{ \text{Sec}[z], \frac{1}{1 - \sqrt{\frac{\pi}{2}} \sqrt{z} \left(\frac{\sqrt{\frac{2}{\pi}}}{\sqrt{z}} - \frac{\sqrt{\frac{2}{\pi}} \text{Cos}[z]}{\sqrt{z}} \right)}, \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \right. \\ \left. \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \text{Sec}[\sqrt{z^2}], \frac{1}{2} \sqrt{\pi} \sqrt{z^2} \text{Sec}[z] \right\}$$

In[441]= % // Simplify

$$\text{Out[441]= } \left\{ \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \right. \\ \left. \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \text{Sec}[z], \text{Sec}[\sqrt{z^2}], \frac{1}{2} \sqrt{\pi} \sqrt{z^2} \text{Sec}[z] \right\}$$

Equivalence transformations using specialized *Mathematica* functions

General remarks

Almost everybody prefers using $1 - \sec(z)$ instead of $\sec(\pi - z) + \sec(0)$. *Mathematica* automatically transforms the second expression into the first one. The automatic application of transformation rules to mathematical expressions can give overly complicated results. Compact expressions like $\sec(\pi/16)$ should not be automatically expanded into the more complicated expression $2/(2 + (2 + 2^{1/2}))$. *Mathematica* has special functions that produce such expansions. Some are demonstrated in the next section.

TrigExpand

The function `TrigExpand` expands out trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then expands out the products of trigonometric and hyperbolic functions into sums of powers, using trigonometric and hyperbolic identities where possible. Here are some examples.

`TrigExpand[Sec[x - y]]`

$$\frac{1}{\text{Cos}[x] \text{Cos}[y] + \text{Sin}[x] \text{Sin}[y]}$$

`Sec[4 z] // TrigExpand`

$$\frac{1}{\text{Cos}[z]^4 - 6 \text{Cos}[z]^2 \text{Sin}[z]^2 + \text{Sin}[z]^4}$$

`Sec[2 z]^2 // TrigExpand`

$$\frac{1}{\text{Cos}[z]^4 - 2 \text{Cos}[z]^2 \text{Sin}[z]^2 + \text{Sin}[z]^4}$$

`TrigExpand[{Sec[x + y + z], Sec[3 z]}]`

$$\left\{ 1 / (\cos[x] \cos[y] \cos[z] - \cos[z] \sin[x] \sin[y] - \cos[y] \sin[x] \sin[z] - \cos[x] \sin[y] \sin[z]), \frac{1}{\cos[z]^3 - 3 \cos[z] \sin[z]^2} \right\}$$

TrigFactor

The function TrigFactor factors trigonometric and hyperbolic functions. In more detail, it splits up sums and integer multiples that appear in the arguments of trigonometric and hyperbolic functions, and then factors the resulting polynomials into trigonometric and hyperbolic functions, using trigonometric and hyperbolic identities where possible. Here are some examples.

TrigFactor[Sec[x] + Sec[y]]

$$\left(2 \cos\left[\frac{x}{2} - \frac{y}{2}\right] \cos\left[\frac{x}{2} + \frac{y}{2}\right] \right) / \left(\left(\cos\left[\frac{x}{2}\right] - \sin\left[\frac{x}{2}\right] \right) \left(\cos\left[\frac{x}{2}\right] + \sin\left[\frac{x}{2}\right] \right) \left(\cos\left[\frac{y}{2}\right] - \sin\left[\frac{y}{2}\right] \right) \left(\cos\left[\frac{y}{2}\right] + \sin\left[\frac{y}{2}\right] \right) \right)$$

Sec[x] - Csc[y] // TrigFactor

$$\frac{\csc\left[\frac{y}{2}\right] \sec\left[\frac{y}{2}\right] \left(\cos\left[\frac{x}{2} - \frac{y}{2}\right] + \sin\left[\frac{x}{2} - \frac{y}{2}\right] \right) \left(\cos\left[\frac{x}{2} + \frac{y}{2}\right] - \sin\left[\frac{x}{2} + \frac{y}{2}\right] \right)}{2 \left(\cos\left[\frac{x}{2}\right] - \sin\left[\frac{x}{2}\right] \right) \left(\cos\left[\frac{x}{2}\right] + \sin\left[\frac{x}{2}\right] \right)}$$

TrigReduce

The function TrigReduce rewrites the products and powers of trigonometric and hyperbolic functions in terms of trigonometric and hyperbolic functions with combined arguments. In more detail, it typically yields a linear expression involving trigonometric and hyperbolic functions with more complicated arguments. TrigReduce is approximately opposite to TrigExpand and TrigFactor. Here are some examples.

TrigReduce[Sec[x] Sec[y]]

$$\frac{2}{\cos[x - y] + \cos[x + y]}$$

Sec[x] Csc[y] // TrigReduce

$$-\frac{2}{\sin[x - y] - \sin[x + y]}$$

Table[TrigReduce[Sec[z]^n], {n, 2, 5}]

$$\left\{ \frac{2}{1 + \cos[2z]}, \frac{4}{3 \cos[z] + \cos[3z]}, \frac{8}{3 + 4 \cos[2z] + \cos[4z]}, \frac{16}{10 \cos[z] + 5 \cos[3z] + \cos[5z]} \right\}$$

TrigReduce[TrigExpand[{Sec[x + y + z], Sec[3z], Sec[x] Sec[y]}]]

$$\left\{ \sec[x + y + z], \sec[3z], \frac{2}{\cos[x - y] + \cos[x + y]} \right\}$$

TrigFactor[Sec[x] + Sec[y]] // TrigReduce

$$\frac{2 (\cos [x] + \cos [y])}{\cos [x - y] + \cos [x + y]}$$

TrigToExp

The function `TrigToExp` converts trigonometric and hyperbolic functions to exponentials. It tries, where possible, to give results that do not involve explicit complex numbers. Here are some examples.

`TrigToExp[Sec[z]]`

$$\frac{2}{e^{-i z} + e^{i z}}$$

`Sec[a z] + Sec[b z] // TrigToExp`

$$\frac{2}{e^{-i a z} + e^{i a z}} + \frac{2}{e^{-i b z} + e^{i b z}}$$

ExpToTrig

The function `ExpToTrig` converts exponentials to trigonometric and hyperbolic functions. It is approximately opposite to `TrigToExp`. Here are some examples.

`ExpToTrig[TrigToExp[Sec[z]]]`

`Sec[z]`

`{α e-i x β + α ei x β / (α e-i x β + γ ei x β)}` // `ExpToTrig`

$$\left\{ \alpha \cos [x \beta] - i \alpha \sin [x \beta] + \frac{\alpha (\cos [x \beta] + i \sin [x \beta])}{\alpha \cos [x \beta] + \gamma \cos [x \beta] - i \alpha \sin [x \beta] + i \gamma \sin [x \beta]} \right\}$$

ComplexExpand

The function `ComplexExpand` expands expressions assuming that all the variables are real. The option `TargetFunctions` can be given as a list of functions from the set {`Re`, `Im`, `Abs`, `Arg`, `Conjugate`, `Sign`}. `ComplexExpand` will try to give results in terms of the specified functions. Here are some examples.

`ComplexExpand[Sec[x + i y]]`

$$\frac{2 \cos [x] \cosh [y]}{\cos [2 x] + \cosh [2 y]} + \frac{2 i \sin [x] \sinh [y]}{\cos [2 x] + \cosh [2 y]}$$

`Sec[x + i y] + Sec[x - i y] // ComplexExpand`

$$\frac{4 \cos [x] \cosh [y]}{\cos [2 x] + \cosh [2 y]}$$

`ComplexExpand[Re[Sec[x + i y]], TargetFunctions -> {Re, Im}]`

$$\frac{2 \cos [x] \cosh [y]}{\cos [2 x] + \cosh [2 y]}$$

`ComplexExpand[Im[Sec[x + i y]], TargetFunctions -> {Re, Im}]`

$$\frac{2 \sin[x] \sinh[y]}{\cos[2x] + \cosh[2y]}$$

`ComplexExpand[Abs[Sec[x + i y]], TargetFunctions -> {Re, Im}]`

$$\sqrt{\frac{4 \cos[x]^2 \cosh[y]^2}{(\cos[2x] + \cosh[2y])^2} + \frac{4 \sin[x]^2 \sinh[y]^2}{(\cos[2x] + \cosh[2y])^2}}$$

`ComplexExpand[Abs[Sec[x + i y]], TargetFunctions -> {Re, Im}] // Simplify[#, {x, y} ∈ Reals] &`

$$\frac{\sqrt{2}}{\sqrt{\cos[2x] + \cosh[2y]}}$$

`ComplexExpand[Re[Sec[x + i y]] + Im[Sec[x + i y]], TargetFunctions -> {Re, Im}]`

$$\frac{2 \cos[x] \cosh[y]}{\cos[2x] + \cosh[2y]} + \frac{2 \sin[x] \sinh[y]}{\cos[2x] + \cosh[2y]}$$

`ComplexExpand[Arg[Sec[x + i y]], TargetFunctions -> {Re, Im}]`

$$\text{ArcTan}\left[\frac{2 \cos[x] \cosh[y]}{\cos[2x] + \cosh[2y]}, \frac{2 \sin[x] \sinh[y]}{\cos[2x] + \cosh[2y]}\right]$$

`ComplexExpand[Arg[Sec[x + i y]], TargetFunctions -> {Re, Im}] // Simplify[#, {x, y} ∈ Reals] &`

$$\text{ArcTan}[\cos[x] \cosh[y], \sin[x] \sinh[y]]$$

`ComplexExpand[Conjugate[Sec[x + i y]], TargetFunctions -> {Re, Im}] // Simplify`

$$\frac{1}{\cos[x] \cosh[y] + i \sin[x] \sinh[y]}$$

Simplify

The function `Simplify` performs a sequence of algebraic transformations on the expression, and returns the simplest form it finds. Here are some examples.

$$\left(\sec[z_1]^2 \sec[z_2]^2 - \sec[z_1]^2 - \sec[z_2]^2\right) \sec[z_1 + z_2]^2 + 2 \sec[z_1] \sec[z_2] \sec[z_1 + z_2] // \text{Simplify}$$

$$\sec[z_1]^2 \sec[z_2]^2$$

$$\text{Simplify}\left[\sec\left[z - \frac{\pi}{3}\right] \sec\left[\frac{\pi}{3} + z\right] + \sec\left[z - \frac{\pi}{3}\right] \sec[z] + \sec[z] \sec\left[\frac{\pi}{3} + z\right]\right]$$

$$\frac{8}{-1 + 2 \cos[2z]}$$

Here is a collection of trigonometric identities. Each is written as a logical conjunction.

`Simplify[#] & /@`

$$\left\{ \sec[2z] (2 - \sec[z]^2) == \sec[z]^2 \wedge \sec[4z] (\cos[z]^4 - 6 \cos[z]^2 \sin[z]^2 + \sin[z]^4) == 1 \wedge \right.$$

$$\sec[a + i b] == \frac{2 \cosh[b] \cos[a] + 2 i \sin[a] \sinh[b]}{\cos[2 a] + \cosh[2 b]} \wedge$$

$$\sec[a]^2 - \sec[b]^2 == \sec[a]^2 \sec[b]^2 \sin[a - b] \sin[a + b] \wedge \sec[\text{ArcCsc}[z]] == \frac{\sqrt{z^2}}{\sqrt{z^2 - 1}} \wedge$$

$$\sec\left[\frac{z}{2}\right]^2 == \frac{2}{1 + \cos[z]} \wedge \sec[x] \sec[y] == \frac{2}{\cos[x - y] + \cos[x + y]} \wedge$$

$$\left. \tan[a]^2 - \tan[b]^2 == \sec[a]^2 \sec[b]^2 \sin[a - b] \sin[a + b] \right\}$$

True

The function `Simplify` has the `Assumption` option. For example, *Mathematica* treats the periodicity of trigonometric functions for the symbolic integer coefficient k of $k\pi$.

`Simplify[{Sec[z + 2 k π], Sec[z + k π] / Sec[z]}, k ∈ Integers]`

`{Sec[z], (-1)k}`

Mathematica also knows that the composition of the inverse and direct trigonometric functions produces the value of the internal argument under the corresponding restriction.

`ArcSec[Sec[z]]`

`ArcSec[Sec[z]]`

`Simplify[ArcSec[Sec[z]], 0 < Re[z] < π]`

`z`

FunctionExpand (and Together)

While the secant function auto-evaluates for simple fractions of π , for more complicated cases it stays as a secant function to avoid the build up of large expressions. Using the function `FunctionExpand`, the secant function can sometimes be transformed into explicit radicals. Here are some examples.

`{Sec[π/16], Sec[π/60]}`

`{Sec[π/16], Sec[π/60]}`

`FunctionExpand[%]`

$$\left\{ \frac{2}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}, \frac{1}{-\frac{\frac{1}{8}\sqrt{3}(-1+\sqrt{5})-\frac{1}{4}\sqrt{\frac{1}{2}(5+\sqrt{5})}}{\sqrt{2}} - \frac{\frac{1}{8}(-1+\sqrt{5})-\frac{1}{4}\sqrt{\frac{3}{2}(5+\sqrt{5})}}{\sqrt{2}}} \right\}$$

Together[%]

$$\left\{ \frac{2}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}, \frac{16}{\sqrt{2} - \sqrt{6} - \sqrt{10} + \sqrt{30} + 2\sqrt{5 + \sqrt{5}} + 2\sqrt{3(5 + \sqrt{5})}} \right\}$$

If the denominator contains squares of integers other than 2, the results always contain complex numbers (meaning that the imaginary number $i = \sqrt{-1}$ appears unavoidably).

$$\left\{ \text{Sec}\left[\frac{\pi}{9}\right] \right\}$$

$$\left\{ \text{Sec}\left[\frac{\pi}{9}\right] \right\}$$

FunctionExpand[%]//Together

$$\left\{ - (8 i) / \left(-i 2^{2/3} (-1 - i \sqrt{3})^{1/3} + 2^{2/3} \sqrt{3} (-1 - i \sqrt{3})^{1/3} - i 2^{2/3} (-1 + i \sqrt{3})^{1/3} - 2^{2/3} \sqrt{3} (-1 + i \sqrt{3})^{1/3} \right) \right\}$$

Here the function `RootReduce` is used to express the previous algebraic numbers as roots of polynomial equations.

RootReduce[Simplify[%]]

$$\left\{ \text{Root}[-8 + 6 \#1^2 + \#1^3 \&, 3] \right\}$$

The function `FunctionExpand` also reduces trigonometric expressions with compound arguments or compositions, including trigonometric functions, to simpler ones. Here are some examples.

$$\left\{ \text{Sec}\left[\sqrt{z^2}\right], \text{Sec}\left[\frac{\text{ArcSec}[z]}{2}\right], \text{Sec}[2 \text{ArcSec}[z]], \text{Sec}[3 \text{ArcSin}[z]] \right\} // \text{FunctionExpand}$$

$$\left\{ \text{Sec}[z], \frac{\sqrt{2} \sqrt{-z}}{\sqrt{-1-z}}, \frac{z^2}{2-z^2}, \frac{1}{-3\sqrt{1-z} z^2 \sqrt{1+z} + (1-z)^{3/2} (1+z)^{3/2}} \right\}$$

Applying `Simplify` to the last expression gives a more compact result.

Simplify[%]

$$\left\{ \text{Sec}[z], \frac{\sqrt{2}}{\sqrt{1 + \frac{1}{z}}}, \frac{z^2}{2-z^2}, \frac{1}{(1-4z^2)\sqrt{1-z^2}} \right\}$$

FullSimplify

The function `FullSimplify` tries a wider range of transformations than `Simplify` and returns the simplest form it finds. Here are some examples that contrast the results of applying these functions to the same expressions.

$$\begin{aligned} \text{set1} = & \left\{ \text{Sec} \left[-i \text{Log} \left[i z + \sqrt{1 - z^2} \right] \right], \text{Sec} \left[\frac{\pi}{2} + i \text{Log} \left[i z + \sqrt{1 - z^2} \right] \right], \right. \\ & \text{Sec} \left[\frac{1}{2} i \text{Log} [1 - i z] - \frac{1}{2} i \text{Log} [1 + i z] \right], \text{Sec} \left[\frac{1}{2} i \text{Log} \left[1 - \frac{i}{z} \right] - \frac{1}{2} i \text{Log} \left[1 + \frac{i}{z} \right] \right], \\ & \left. \text{Sec} \left[-i \text{Log} \left[\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z} \right] \right], \text{Sec} \left[\frac{\pi}{2} + i \text{Log} \left[\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z} \right] \right] \right\} \\ & \left\{ \frac{2 \left(i z + \sqrt{1 - z^2} \right)}{1 + \left(i z + \sqrt{1 - z^2} \right)^2}, \frac{2 i \left(i z + \sqrt{1 - z^2} \right)}{-1 + \left(i z + \sqrt{1 - z^2} \right)^2}, \text{Sech} \left[\frac{1}{2} \text{Log} [1 - i z] - \frac{1}{2} \text{Log} [1 + i z] \right], \right. \\ & \left. \text{Sech} \left[\frac{1}{2} \text{Log} \left[1 - \frac{i}{z} \right] - \frac{1}{2} \text{Log} \left[1 + \frac{i}{z} \right] \right], \frac{2 \left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z} \right)}{1 + \left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z} \right)^2}, \frac{2 i \left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z} \right)}{-1 + \left(\sqrt{1 - \frac{1}{z^2}} + \frac{i}{z} \right)^2} \right\} \end{aligned}$$

`set1 // Simplify`

$$\begin{aligned} & \left\{ \frac{2 \left(i z + \sqrt{1 - z^2} \right)}{1 + \left(i z + \sqrt{1 - z^2} \right)^2}, \frac{1}{z}, \text{Sech} \left[\frac{1}{2} \left(\text{Log} [1 - i z] - \text{Log} [1 + i z] \right) \right], \right. \\ & \left. \text{Sech} \left[\frac{1}{2} \left(\text{Log} \left[\frac{-i + z}{z} \right] - \text{Log} \left[\frac{i + z}{z} \right] \right) \right], \frac{z \left(i + \sqrt{1 - \frac{1}{z^2}} z \right)}{-1 + i \sqrt{1 - \frac{1}{z^2}} z + z^2}, z \right\} \end{aligned}$$

`set1 // FullSimplify`

$$\left\{ \frac{1}{\sqrt{1 - z^2}}, \frac{1}{z}, \sqrt{1 + z^2}, \sqrt{1 + \frac{1}{z^2}}, \frac{1}{\sqrt{1 - \frac{1}{z^2}}}, z \right\}$$

Operations under special *Mathematica* functions

Series expansions

Calculating the series expansion of a secant function to hundreds of terms can be done in seconds.

`Series[Sec[z], {z, 0, 3}]`

$$1 + \frac{z^2}{2} + O[z]^4$$

Normal[%]

$$1 + \frac{z^2}{2}$$

Series[Sec[z], {z, 0, 100}] // Timing

$$\left\{ 0.4 \text{ Second}, 1 + \frac{z^2}{2} + \frac{5 z^4}{24} + \frac{61 z^6}{720} + \frac{277 z^8}{8064} + \frac{50 521 z^{10}}{3 628 800} + \frac{540 553 z^{12}}{95 800 320} + \frac{199 360 981 z^{14}}{87 178 291 200} + \frac{3 878 302 429 z^{16}}{4 184 557 977 600} + \frac{2 404 879 675 441 z^{18}}{6 402 373 705 728 000} + \frac{14 814 847 529 501 z^{20}}{97 316 080 327 065 600} + \frac{69 348 874 393 137 901 z^{22}}{1 124 000 727 777 607 680 000} + \frac{238 685 140 977 801 337 z^{24}}{9 545 360 026 665 222 144 000} + \frac{4 087 072 509 293 123 892 361 z^{26}}{403 291 461 126 605 635 584 000 000} + \frac{13 181 680 435 827 682 794 403 z^{28}}{3 209 350 995 912 777 478 963 200 000} + \frac{441 543 893 249 023 104 553 682 821 z^{30}}{265 252 859 812 191 058 636 308 480 000 000} + \frac{2 088 463 430 347 521 052 196 056 349 z^{32}}{3 095 656 905 102 276 825 496 682 496 000 000} + \frac{80 723 299 235 887 898 062 168 247 453 281 z^{34}}{295 232 799 039 604 140 847 618 609 643 520 000 000} + \frac{634 185 543 684 887 724 959 185 841 096 293 z^{36}}{5 722 974 258 306 172 576 430 760 740 782 080 000 000} + \frac{23 489 580 527 043 108 252 017 828 576 198 947 741 z^{38}}{523 022 617 466 601 111 760 007 224 100 074 291 200 000 000} + \frac{594 046 028 724 599 200 715 086 271 256 233 067 377 z^{40}}{32 636 611 329 915 909 373 824 450 783 844 635 770 880 000 000} + \frac{10 364 622 733 519 612 119 397 957 304 745 185 976 310 201 z^{42}}{1 405 006 117 752 879 898 543 142 606 244 511 569 936 384 000 000 000} + \frac{1 589 515 884 519 518 540 721 608 102 017 614 123 903 854 761 z^{44}}{531 654 314 957 689 753 608 725 162 202 923 178 063 927 705 600 000 000} + \frac{350 923 027 193 976 051 443 948 867 093 355 168 290 742 509 719 z^{46}}{289 611 692 621 688 892 097 384 496 252 644 994 366 402 723 840 000 000 000} + \frac{5 516 994 249 383 296 071 214 195 242 422 482 492 286 460 673 697 z^{48}}{11 234 312 753 426 310 109 377 637 146 944 230 804 091 842 854 912 000 000 000} + \frac{6 053 285 248 188 621 896 314 383 785 111 649 088 103 498 225 146 815 121 z^{50}}{30 414 093 201 713 378 043 612 608 166 064 768 844 377 641 568 960 512 000 000 000 000} + \frac{41 975 241 849 578 121 598 166 907 316 650 470 861 184 801 527 518 616 623 z^{52}}{520 375 323 683 508 894 010 713 786 170 346 883 711 545 196 392 795 340 800 000 000 000} + \frac{(175 503 719 511 831 141 815 382 152 695 110 156 842 145 116 853 760 461 828 967 z^{54})}{5 368 458 077 656 776 290 048 668 434 489 013 513 564 617 780 739 719 561 216 000 000 000 000} + \frac{(64 967 737 216 691 318 649 670 939 570 279 884 325 581 992 086 894 110 409 620 041 z^{56})}{4 903 438 536 585 265 185 200 314 810 094 654 825 768 951 578 490 816 956 465 152 000 000 000 000} +$$

$$\begin{aligned}
& (12\ 622\ 019\ 251\ 806\ 218\ 719\ 903\ 409\ 237\ 287\ 489\ 255\ 482\ 341\ 061\ 191\ 825\ 594\ 069\ 964\ 920\ 041\ z^{58}) / \\
& 2\ 350\ 561\ 331\ 282\ 878\ 571\ 829\ 474\ 910\ 515\ 074\ 683\ 828\ 862\ 318\ 181\ 142\ 924\ 420\ 699\ 914\ 240\ 000\ 000\ 000\ 000\ 000 + \\
& (1\ 185\ 526\ 120\ 888\ 983\ 993\ 227\ 221\ 456\ 081\ 974\ 382\ 241\ 144\ 834\ 647\ 221\ 872\ 733\ 754\ 203\ 223\ z^{60}) / \\
& 544\ 745\ 473\ 829\ 223\ 577\ 366\ 699\ 913\ 795\ 310\ 270\ 425\ 805\ 080\ 612\ 847\ 525\ 528\ 594\ 284\ 544\ 000\ 000\ 000\ 000 + \\
& (27\ 757\ 101\ 702\ 071\ 580\ 597\ 366\ 980\ 908\ 371\ 527\ 449\ 233\ 019\ 594\ 800\ 917\ 578\ 033\ 782\ 766\ 889\ 782\ 501 \\
& \quad z^{62}) / \\
& 31\ 469\ 973\ 260\ 387\ 937\ 525\ 653\ 122\ 354\ 950\ 764\ 088\ 012\ 280\ 797\ 258\ 232\ 192\ 163\ 168\ 247\ 821\ 107\ 200\ 000\ 000\ 000\ 000 + \\
& (28\ 085\ 512\ 897\ 843\ 894\ 225\ 849\ 466\ 174\ 345\ 363\ 694\ 335\ 517\ 968\ 712\ 302\ 712\ 983\ 804\ 234\ 006\ 003\ 567 \\
& \quad z^{64}) / \\
& 78\ 567\ 759\ 867\ 420\ 535\ 048\ 565\ 566\ 151\ 802\ 774\ 490\ 938\ 400\ 108\ 077\ 518\ 389\ 351\ 018\ 188\ 987\ 432\ 960\ 000\ 000\ 000\ 000 + \\
& (1\ 292\ 833\ 476\ 501\ 932\ 691\ 491\ 921\ 433\ 154\ 096\ 790\ 811\ 117\ 425\ 770\ 553\ 589\ 060\ 815\ 696\ 637\ 002\ 753\ 202\ 901\ z^{66}) / \\
& 8\ 923\ 687\ 525\ 859\ 722\ 360\ 716\ 872\ 790\ 948\ 241\ 846\ 627\ 755\ 153\ 914\ 735\ 648\ 079\ 674\ 210\ 978\ 214\ 445\ 056\ 000\ 000\ 000\ 000\ 000 + \\
& (29\ 123\ 688\ 760\ 279\ 263\ 001\ 430\ 094\ 018\ 988\ 465\ 332\ 372\ 162\ 571\ 662\ 986\ 597\ 289\ 539\ 553\ 612\ 919\ 097\ 772\ 580\ 417\ z^{68}) / \\
& 496\ 007\ 108\ 487\ 366\ 119\ 920\ 198\ 083\ 713\ 834\ 316\ 209\ 479\ 840\ 271\ 073\ 534\ 474\ 342\ 147\ 603\ 644\ 289\ 142\ 436\ 659\ 200\ 000\ 000\ 000\ 000 + \\
& (285\ 051\ 783\ 223\ 697\ 718\ 732\ 198\ 729\ 556\ 739\ 339\ 504\ 255\ 241\ 778\ 255\ 239\ 879\ 353\ 211\ 106\ 980\ 427\ 546\ 235\ 397\ 447\ 421\ z^{70}) / \\
& 11\ 978\ 571\ 669\ 969\ 891\ 796\ 072\ 783\ 721\ 689\ 098\ 736\ 458\ 938\ 142\ 546\ 425\ 857\ 555\ 362\ 864\ 628\ 009\ 582\ 789\ 845\ 319\ 680\ 000\ 000\ 000\ 000\ 000 + \\
& (245\ 561\ 214\ 460\ 517\ 407\ 673\ 652\ 820\ 081\ 679\ 655\ 456\ 146\ 784\ 275\ 186\ 259\ 846\ 056\ 537\ 278\ 029\ 024\ 968\ 924\ 948\ 811\ 253\ z^{72}) / \\
& 25\ 461\ 313\ 254\ 422\ 489\ 339\ 511\ 047\ 977\ 245\ 186\ 170\ 801\ 701\ 365\ 778\ 515\ 169\ 988\ 779\ 610\ 801\ 823\ 279\ 510\ 057\ 910\ 272\ 000\ 000\ 000\ 000\ 000 + \\
& (1\ 292\ 973\ 664\ 187\ 864\ 170\ 497\ 603\ 235\ 938\ 698\ 754\ 076\ 170\ 519\ 123\ 672\ 606\ 411\ 370\ 597\ 343\ 787\ 035\ 331\ 808\ 195\ 731\ 850\ 937\ 881\ z^{74}) / \\
& 330\ 788\ 544\ 151\ 938\ 641\ 225\ 953\ 028\ 221\ 253\ 782\ 145\ 683\ 251\ 820\ 934\ 971\ 170\ 611\ 926\ 835\ 411\ 235\ 700\ 971\ 565\ 459\ 250\ 872\ 320\ 000\ 000\ 000\ 000\ 000 + \\
& (597\ 385\ 636\ 656\ 915\ 390\ 186\ 148\ 730\ 443\ 428\ 121\ 138\ 584\ 473\ 874\ 136\ 140\ 562\ 762\ 566\ 693\ 379\ 607\ 634\ 403\ 131\ 161\ 792\ 057\ 690\ 569\ z^{76}) / \\
& 377\ 098\ 940\ 333\ 210\ 050\ 997\ 586\ 452\ 172\ 229\ 311\ 646\ 078\ 907\ 075\ 865\ 867\ 134\ 497\ 596\ 592\ 368\ 808\ 699\ 107\ 584\ 623\ 545\ 994\ 444\ 800\ 000\ 000\ 000\ 000 + \\
& (7\ 270\ 601\ 714\ 016\ 864\ 143\ 803\ 280\ 651\ 699\ 281\ 851\ 647\ 234\ 288\ 049\ 207\ 905\ 108\ 309\ 583\ 687\ 335\ 688\ 017\ 641\ 546\ 191\ 095\ 009\ 395\ 592\ 341\ z^{78}) / \\
& 11\ 324\ 281\ 178\ 206\ 297\ 831\ 457\ 521\ 158\ 732\ 046\ 228\ 731\ 749\ 579\ 488\ 251\ 990\ 048\ 962\ 825\ 668\ 835\ 325\ 234\ 200\ 766\ 245\ 086\ 213\ 177\ 344\ 000\ 000\ 000\ 000\ 000 + \\
& (1\ 068\ 746\ 958\ 875\ 908\ 008\ 291\ 678\ 180\ 949\ 390\ 301\ 975\ 323\ 397\ 102\ 930\ 046\ 690\ 221\ 009\ 795\ 234\ 210\ 820\ 441\ 976\ 633\ 130\ 205\ 853\ 968\ 257\ z^{80}) / \\
& 4\ 107\ 285\ 913\ 702\ 370\ 289\ 515\ 726\ 469\ 049\ 442\ 305\ 055\ 073\ 592\ 101\ 334\ 437\ 710\ 728\ 554\ 274\ 148\ 594\ 288\ 674\ 252\ 090\ 039\ 877\ 605\ 785\ 600\ 000\ 000\ 000\ 000 + \\
& (85\ 112\ 138\ 214\ 108\ 313\ 434\ 479\ 955\ 312\ 191\ 866\ 587\ 577\ 821\ 260\ 136\ 739\ 840\ 362\ 642\ 167\ 337\ 133\ 108\ 017\ 641\ 546\ 191\ 095\ 009\ 395\ 592\ 341\ z^{82}) / \\
& 11\ 324\ 281\ 178\ 206\ 297\ 831\ 457\ 521\ 158\ 732\ 046\ 228\ 731\ 749\ 579\ 488\ 251\ 990\ 048\ 962\ 825\ 668\ 835\ 325\ 234\ 200\ 766\ 245\ 086\ 213\ 177\ 344\ 000\ 000\ 000\ 000\ 000 +
\end{aligned}$$

$$\begin{aligned}
 & \left(921\,391\,785\,317\,917\,129\,176\,247\,795\,509\,z^{82} \right) / \\
 & 807\,070\,176\,063\,300\,806\,183\,596\,276\,722\,249\,484\,964\,029\,361\,745\,319\,742\,983\,295\,304\,035\,219\,006 \cdot \\
 & 341\,085\,141\,957\,567\,286\,601\,832\,202\,240\,000\,000\,000\,000\,000\,000\,000 + \\
 & \left(75\,147\,245\,410\,003\,320\,910\,727\,888\,281\,178\,604\,989\,642\,358\,351\,638\,149\,809\,389\,998\,326\,458\,656\,760 \cdot \right. \\
 & \quad \left. 950\,050\,385\,743\,229\,535\,819\,860\,148\,866\,073\,z^{84} \right) / \\
 & 1\,758\,217\,578\,018\,755\,048\,806\,041\,156\,037\,204\,927\,321\,308\,374\,664\,217\,234\,545\,806\,913\,731\,065\,852 \cdot \\
 & 144\,146\,877\,522\,485\,794\,678\,364\,325\,281\,792\,000\,000\,000\,000\,000\,000\,000 + \\
 & \left(419\,664\,316\,404\,024\,471\,322\,573\,414\,069\,418\,891\,818\,962\,628\,391\,683\,907\,039\,212\,228\,549\,032\,921 \cdot \right. \\
 & \quad \left. 853\,217\,838\,146\,608\,053\,808\,786\,365\,440\,570\,254\,969\,261\,z^{86} \right) / \\
 & 24\,227\,095\,383\,672\,732\,381\,765\,523\,203\,441\,259\,715\,284\,870\,552\,429\,381\,750\,838\,764\,496\,720\,162\,249 \cdot \\
 & 742\,450\,276\,789\,464\,634\,901\,319\,465\,571\,660\,595\,200\,000\,000\,000\,000\,000\,000 + \\
 & \left(260\,431\,918\,104\,809\,279\,625\,171\,738\,266\,163\,736\,271\,351\,522\,797\,322\,006\,135\,619\,151\,648\,480\,857 \cdot \right. \\
 & \quad \left. 326\,745\,852\,459\,424\,735\,439\,948\,718\,349\,601\,240\,929\,373\,797\,z^{88} \right) / \\
 & 37\,096\,528\,451\,479\,687\,822\,959\,369\,129\,109\,256\,876\,044\,193\,789\,879\,869\,336\,884\,316\,197\,377\,912\,436 \cdot \\
 & 805\,639\,863\,820\,028\,248\,960\,900\,365\,683\,326\,703\,370\,240\,000\,000\,000\,000\,000\,000 + \\
 & \left(4\,227\,240\,686\,139\,909\,064\,705\,589\,929\,214\,593\,102\,933\,845\,388\,672\,369\,082\,676\,644\,542\,650\,248\,228 \cdot \right. \\
 & \quad \left. 369\,590\,525\,634\,078\,984\,302\,153\,217\,507\,945\,782\,396\,923\,579\,721\,z^{90} \right) / \\
 & 1\,485\,715\,964\,481\,761\,497\,309\,522\,733\,620\,825\,737\,885\,569\,961\,284\,688\,766\,942\,216\,863\,704\,985\,393 \cdot \\
 & 094\,065\,876\,545\,992\,131\,370\,884\,059\,645\,617\,234\,469\,978\,112\,000\,000\,000\,000\,000\,000\,000 + \\
 & \left(42\,815\,560\,573\,064\,428\,078\,010\,992\,956\,950\,942\,861\,432\,484\,026\,397\,489\,883\,900\,223\,747\,790\,418\,497 \cdot \right. \\
 & \quad \left. 948\,114\,193\,578\,680\,162\,571\,793\,178\,004\,310\,552\,004\,531\,066\,419\,z^{92} \right) / \\
 & 37\,129\,594\,192\,959\,126\,135\,747\,236\,793\,652\,397\,246\,501\,467\,808\,583\,326\,438\,329\,073\,381\,904\,888\,689 \cdot \\
 & 503\,043\,338\,635\,958\,578\,618\,033\,872\,695\,843\,244\,724\,348\,518\,400\,000\,000\,000\,000\,000\,000 + \\
 & \left(50\,817\,990\,724\,580\,425\,164\,559\,757\,643\,090\,736\,003\,482\,435\,671\,513\,413\,926\,813\,239\,886\,828\,210\,876 \cdot \right. \\
 & \quad \left. 247\,074\,897\,752\,122\,164\,140\,484\,881\,907\,534\,297\,068\,189\,565\,042\,330\,181\,z^{94} \right) / \\
 & 108\,736\,615\,665\,674\,308\,027\,365\,285\,256\,786\,601\,004\,186\,803\,580\,182\,872\,307\,497\,374\,434\,045\,199 \cdot \\
 & 869\,417\,927\,630\,229\,109\,214\,583\,415\,458\,560\,865\,651\,202\,385\,340\,530\,688\,000\,000\,000\,000\,000\,000 + \\
 & 000 + \\
 & \left(50\,039\,705\,954\,360\,322\,297\,158\,013\,840\,847\,032\,326\,634\,079\,577\,003\,760\,741\,000\,075\,956\,513\,037\,893 \cdot \right. \\
 & \quad \left. 631\,352\,039\,017\,573\,706\,673\,862\,825\,679\,894\,245\,530\,824\,535\,763\,677\,z^{96} \right) / \\
 & 264\,187\,840\,714\,111\,516\,871\,972\,848\,425\,452\,269\,212\,303\,016\,542\,748\,737\,692\,261\,352\,467\,906\,130 \cdot \\
 & 325\,025\,008\,754\,780\,828\,981\,920\,632\,387\,127\,607\,867\,726\,182\,219\,776\,000\,000\,000\,000\,000\,000\,000 + \\
 & \left(10\,192\,301\,945\,118\,109\,544\,467\,431\,855\,447\,637\,929\,383\,937\,537\,761\,510\,320\,573\,414\,521\,901\,629\,141 \cdot \right. \\
 & \quad \left. 453\,868\,275\,940\,735\,720\,302\,151\,703\,532\,547\,073\,126\,053\,721\,100\,491\,517\,027\,671\,z^{98} \right) / \\
 & 132\,773\,104\,913\,848\,559\,797\,551\,911\,874\,045\,668\,645\,206\,954\,423\,647\,206\,528\,087\,870\,102\,742\,353 \cdot \\
 & 098\,918\,645\,054\,689\,805\,059\,263\,790\,420\,050\,786\,628\,008\,569\,865\,202\,995\,036\,160\,000\,000\,000\,000 \cdot \\
 & 000\,000\,000 + \\
 & \left(1\,222\,538\,251\,225\,725\,261\,917\,727\,930\,427\,130\,894\,769\,496\,002\,798\,382\,713\,718\,102\,029\,114\,525\,323 \cdot \right. \\
 & \quad \left. 524\,473\,694\,269\,201\,451\,423\,144\,149\,126\,167\,703\,052\,004\,443\,591\,028\,403\,213\,886\,579\,z^{100} \right) / \\
 & 39\,295\,248\,607\,976\,485\,339\,662\,837\,413\,164\,926\,522\,406\,723\,479\,739\,630\,092\,039\,142\,692\,723\,199\,997 \cdot \\
 & 149\,438\,151\,133\,247\,989\,960\,639\,278\,422\,609\,650\,874\,620\,529\,789\,972\,720\,386\,048\,000\,000\,000\,000 \cdot \\
 & 000\,000\,000 + O[z]^{101} \}
 \end{aligned}$$

Mathematica comes with the add-on package `DiscreteMath`RSolve`` that allows finding the general terms of series for many functions. After loading this package, and using the package function `SeriesTerm`, the following n^{th} term of $\sec(z)$ can be evaluated.

$$\prod_{k=1}^{n-1} \operatorname{Sec}\left[\frac{k \pi}{n}\right]$$

$$- (-1)^{-n} 2^{-1+n} \operatorname{Csc}\left[\frac{n \pi}{2}\right]$$

$$\prod_{k=1}^{n-1} \operatorname{Sec}\left[z + \frac{k \pi}{n}\right]$$

$$- (-1)^{-n} 2^{-1+n} \operatorname{Cos}[z] \operatorname{Csc}\left[\frac{1}{2} n (\pi - 2 z)\right]$$

Indefinite integration

Mathematica can calculate a huge number of doable indefinite integrals that contain the secant function. The results can contain special functions. Here are some examples.

$$\int \operatorname{Sec}[z] \, dz$$

$$- \operatorname{Log}\left[\operatorname{Cos}\left[\frac{z}{2}\right] - \operatorname{Sin}\left[\frac{z}{2}\right]\right] + \operatorname{Log}\left[\operatorname{Cos}\left[\frac{z}{2}\right] + \operatorname{Sin}\left[\frac{z}{2}\right]\right]$$

$$\int \frac{1}{\sqrt{\operatorname{Sec}[z]}} \, dz$$

$$\frac{2 \operatorname{EllipticE}\left[\frac{z}{2}, 2\right]}{\sqrt{\operatorname{Cos}[z]} \sqrt{\operatorname{Sec}[z]}}$$

$$\int \operatorname{Sec}[z]^a \, dz$$

$$\frac{\operatorname{Hypergeometric2F1}\left[\frac{1-a}{2}, \frac{1}{2}, \frac{3-a}{2}, \operatorname{Cos}[z]^2\right] \operatorname{Sec}[z]^{-1+a} \operatorname{Sin}[z]}{(1-a) \sqrt{\operatorname{Sin}[z]^2}}$$

$$\int \frac{1}{a + b \operatorname{Sec}[z]} \, dz \text{ // Simplify}$$

$$z + \frac{2 b \operatorname{ArcTanh}\left[\frac{(-a+b) \operatorname{Tan}\left[\frac{z}{2}\right]}{\sqrt{a^2-b^2}}\right]}{\sqrt{a^2-b^2}}$$

Definite integration

Mathematica can calculate wide classes of definite integrals that contain the secant function. Here are some examples.

$$\int_0^{\pi/2} \sqrt{\operatorname{Sec}[z]} \, dz$$

$$\frac{2 \sqrt{\pi} \Gamma\left[\frac{5}{4}\right]}{\Gamma\left[\frac{3}{4}\right]}$$

$$\frac{\int_0^{\frac{\pi}{2}} z \sqrt{\sec[z]} dz}{1}$$

$$\frac{1}{8 \Gamma\left[\frac{1}{4}\right] \Gamma\left[\frac{3}{4}\right]} \left(\sqrt{\pi} \left(\pi \Gamma\left[\frac{1}{4}\right]^2 + 8 \Gamma\left[\frac{3}{4}\right]^2 \left(2 \text{HypergeometricPFQ}\left[\left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}, \left\{\frac{5}{4}, \frac{5}{4}\right\}, 1\right] - \text{HypergeometricPFQ}\left[\left\{\frac{1}{2}, \frac{1}{2}, \frac{3}{4}\right\}, \left\{\frac{5}{4}, \frac{3}{2}\right\}, 1\right] \right) \right) \right)$$

Limit operation

Mathematica can calculate limits that contain the secant function. Here are some examples.

`Limit[Sec[3 z] - 1 / z^2, z -> 0]`

$$\frac{9}{2}$$

`Limit[z Sec[pi/2 - 2 sqrt[z^2]], z -> 0, Direction -> 1]`

$$-\frac{1}{2}$$

`Limit[z Sec[pi/2 - 2 sqrt[z^2]], z -> 0, Direction -> -1]`

$$\frac{1}{2}$$

Solving equations

The next inputs solve two equations that contain the secant function. Because of the multivalued nature of the inverse secant function, a printed message indicates that only some of the possible solutions are returned.

`Solve[Sec[z]^2 + 2 Sec[z + Pi/6] == 4, z]`

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{ { z → ArcCos[Root[1 - 24 #1^2 + 208 #1^4 - 1088 #1^6 + 4096 #1^8 - 7168 #1^10 + 4096 #1^12 &, 3]] },
  { z → ArcCos[Root[1 - 24 #1^2 + 208 #1^4 - 1088 #1^6 + 4096 #1^8 - 7168 #1^10 + 4096 #1^12 &, 5]] },
  { z → -ArcCos[Root[1 - 24 #1^2 + 208 #1^4 - 1088 #1^6 + 4096 #1^8 - 7168 #1^10 + 4096 #1^12 &, 7]] },
  { z → ArcCos[Root[1 - 24 #1^2 + 208 #1^4 - 1088 #1^6 + 4096 #1^8 - 7168 #1^10 + 4096 #1^12 &, 8]] },
  { z → -ArcCos[Root[1 - 24 #1^2 + 208 #1^4 - 1088 #1^6 + 4096 #1^8 - 7168 #1^10 + 4096 #1^12 &, 9]] },
  { z → -ArcCos[Root[1 - 24 #1^2 + 208 #1^4 - 1088 #1^6 + 4096 #1^8 - 7168 #1^10 + 4096 #1^12 &, 10]] } }
```

Solve[Sec[x] == a, x]

Solve::ifun: Inverse functions are being used by Solve, so some solutions may not be found.

```
{ { x → -ArcSec[a] }, { x → ArcSec[a] } }
```

A complete solution of the previous equation can be obtained using the function Reduce.

Reduce[Sec[x] == a, x] // InputForm

```
// InputForm = C[1] ∈ Integers && a ≠ 0 &&
(x == -ArcCos[a^(-1)] + 2 * Pi * C[1] || x == ArcCos[a^(-1)] + 2 * Pi * C[1])
```

Solving differential equations

Here is a nonlinear first-order differential equation that is obeyed by the secant function.

```
In[72]:= w'[z]^2 - w[z]^4 + w[z]^2 == 0 /. w → Sec // Simplify
```

```
Out[72]= True
```

Mathematica can find the general solution of this differential equation. In doing so, the generically multivariate inverse of a function is encountered, and a message is issued that a solution branch is potentially missed.

```
In[73]:= DSolve[{w'[z]^2 - w[z]^4 + w[z]^2 == 0}, w[z], z]
```

Solve::tdep: The equations appear to involve the variables to be solved for in an essentially non-algebraic way. More...

Solve::tdep: The equations appear to involve the variables to be solved for in an essentially non-algebraic way. More...

```
Out[73]= { { w[z] → InverseFunction[ - (ArcTan[ 1 / sqrt(-1 + #1^2) ] #1 sqrt(-1 + #1^2) / sqrt(-#1^2 + #1^4) & ] [-z + C[1]] },
  { w[z] → InverseFunction[ - (ArcTan[ 1 / sqrt(-1 + #1^2) ] #1 sqrt(-1 + #1^2) / sqrt(-#1^2 + #1^4) & ] [z + C[1]] } }
```

Plotting

Mathematica has built-in functions for 2D and 3D graphics. Here are some examples.

```
In[74]:= Plot[Sec[Sum[z^k, {k, 0, 3}], {z, -2 Pi/3, 2 Pi/3}];
```

```
In[75]:= Plot3D[Re[Sec[x + i y]], {x, -Pi, Pi}, {y, 0, 2},  
PlotPoints -> 240, PlotRange -> {-5, 5},  
ClipFill -> None, Mesh -> False, AxesLabel -> {"x", "y", None}];
```

```
In[76]:= ContourPlot[Arg[Sec[1/(x + i y)]], {x, -1/2, 1/2}, {y, -1/2, 1/2},  
PlotPoints -> 400, PlotRange -> {-Pi, Pi}, FrameLabel -> {"x", "y", None, None},  
ColorFunction -> Hue, ContourLines -> False, Contours -> 200];
```

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