

Introductions to Signature

Introduction to the tensor functions

General

The tensor functions discrete delta and Kronecker delta first appeared in the works L. Kronecker (1866, 1903) and T. Levi-Civita (1896).

Definitions of the tensor functions

For all possible values of their arguments, the discrete delta functions $\delta(n)$ and $\delta(n_1, n_2, \dots)$, Kronecker delta functions δ_n and $\delta_{n_1, n_2, \dots}$, and signature (Levi-Civita symbol) $\varepsilon_{n_1, n_2, \dots, n_d}$ are defined by the formulas:

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{True} \end{cases}$$

$$\delta(n_1, n_2, \dots) = 1 /; n_1 = n_2 = \dots = 0$$

$$\delta(n_1, n_2, \dots) = 0 /; \neg n_1 = n_2 = \dots = 0$$

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{True} \end{cases}$$

$$\delta_{n_1, n_2, \dots} = 1 /; n_1 = n_2 = \dots \wedge n_1 \in \mathbb{Q} \wedge n_2 \in \mathbb{Q} \wedge \dots$$

$$\delta_{n_1, n_2, \dots} = 0 /; \neg n_1 = n_2 = \dots$$

In other words, the Kronecker delta function is equal to 1 if all its arguments are equal.

In the case of one variable, the discrete delta function $\delta(n)$ coincides with the Kronecker delta function δ_n . In the case of several variables, the discrete delta function $\delta(n_1, n_2, \dots, n_m)$ coincides with Kronecker delta function $\delta_{n_1, n_2, \dots, n_m, 0}$:

$$\delta(n) = \delta_n$$

$$\delta(n_1, n_2, \dots, n_m) = \delta_{n_1, n_2, \dots, n_m, 0}$$

$$\varepsilon_{n_1, n_2, \dots, n_d} = (-1)^t$$

$$\varepsilon_{n_1, n_2, \dots, n_d} = 0 /; n_i = n_j \wedge 1 \leq i \leq d \wedge 1 \leq j \leq d,$$

where t is the number of permutations needed to go from the sorted version of $\{n_1, n_2, \dots, n_d\}$ to $\{n_1, n_2, \dots, n_d\}$.

Connections within the group of tensor functions and with other function groups

Representations through equivalent functions

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \dots)$, δ_n , and $\delta_{n_1, n_2, \dots}$ have the following representations through equivalent functions:

$\delta(n)$ and $\delta(n_1, n_2, \dots, n_m)$	δ_n and $\delta_{n_1, n_2, \dots, n_m}$
$\delta(n) = \delta(n_1, n_2, \dots, n_m) /; n_1 = n \wedge m = 1$	$\delta_n = \delta(n)$
$\delta(n) = \delta_{n,0}$	$\delta_{n_1, n_2} = \delta_{n_1 - n_2}$
$\delta(n) = \delta_n$	$\delta_{n_1, n_2, \dots, n_m} = \delta_{n_1 - n_m, n_2 - n_m, \dots, n_{m-1} - n_m}$
$\delta(n_1, n_2, \dots, n_m) = \delta_{n_1, n_2, \dots, n_m, 0}$	$\delta_{n_1, n_2, \dots, n_m, 0} = \delta(n_1, n_2, \dots, n_m)$

The best-known properties and formulas of the tensor functions

Simple values at zero and infinity

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \dots)$, δ_n , and $\delta_{n_1, n_2, \dots}$ can have unit values at infinity:

$\delta(n)$ and $\delta(n_1, n_2)$	δ_n and δ_{n_1, n_2}
$\delta(\infty) = 0$	$\delta_\infty = 0$
$\delta(-\infty) = 0$	$\delta_{-\infty} = 0$
$\delta(\infty, -\infty) = 0$	$\delta_{\infty, -\infty} = 0$
$\delta(-\infty, \infty) = 0$	$\delta_{-\infty, \infty} = 0$

Specific values for specialized variables

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \dots)$, δ_n , $\delta_{n_1, n_2, \dots}$, and $\varepsilon_{n_1, n_2, \dots, n_d}$ have the following values for some specialized variables:

$\delta(n)$	δ_n	ε_n
$\delta(0) = 1$	$\delta_0 = 1$	$\varepsilon_0 = 1$
$\delta(1) = 0$	$\delta_1 = 0$	$\varepsilon_1 = 1$
$\delta(-1) = 0$	$\delta_{-1} = 0$	$\varepsilon_{-1} = 1$
$\delta(2) = 0$	$\delta_2 = 0$	$\varepsilon_2 = 1$
...
$\delta(n) = 0 /; n \neq 0$	$\delta_n = 0 /; n \neq 0$	$\varepsilon_n = 1$

$\delta(n_1, n_2)$	δ_{n_1, n_2}	ε_{n_1, n_2}
$\delta(0, 0) = 1$	$\delta_{1,1} = 1$	$\varepsilon_{1,1} = 0$
$\delta(1, 2) = 0$	$\delta_{1,2} = 0$	$\varepsilon_{1,2} = 1$
$\delta(2, 1) = 0$	$\delta_{2,1} = 0$	$\varepsilon_{2,1} = -1$
$\delta(2, 2) = 0$	$\delta_{2,2} = 1$	$\varepsilon_{2,2} = 0$
...
$\delta(n, 0) = \delta_n$	$\delta_{n,0} = \delta_n$	$\varepsilon_{z,a} = -1$
$\delta(0, n) = \delta_n$	$\delta_{0,n} = \delta_n$	$\varepsilon_{a,z} = 1$

$\delta(n_1, n_2, n_3)$	δ_{n_1, n_2, n_3}	$\varepsilon_{n_1, n_2, n_3}$
$\delta(0, 0, 0) = 1$	$\delta_{1,1,1} = 1$	$\varepsilon_{1,1,2} = 0$
$\delta(0, 0, 1) = 0$	$\delta_{1,1,2} = 0$	$\varepsilon_{1,2,3} = 1$
$\delta(0, 1, 0) = 0$	$\delta_{1,2,1} = 0$	$\varepsilon_{1,3,2} = -1$
$\delta(1, 0, 0) = 0$	$\delta_{2,1,1} = 0$	$\varepsilon_{2,3,1} = 1$
$\delta(1, 1, 1) = 0$	$\delta_{1,1,1} = 1$	$\varepsilon_{2,1,3} = -1$
$\delta(2, 2, 2) = 0$	$\delta_{2,2,2} = 1$	$\varepsilon_{3,1,2} = 1$
$\delta(-1, -1, -1) = 0$	$\delta_{-1,-1,-1} = 1$	$\varepsilon_{3,2,1} = -1$

Analyticity

$\delta(n)$ and δ_n are nonanalytical functions defined over \mathbb{C} . Their possible values are 0 and 1.

$\delta(n_1, n_2, \dots, n_m)$ and $\delta_{n_1, n_2, \dots, n_m}$ are nonanalytical functions defined over \mathbb{C}^m . Their possible values are 0 and 1.

$\varepsilon_{n_1, n_2, \dots, n_d}$ is a nonanalytical function, defined over the set of tuples of complex numbers with possible values 0 and ± 1 .

Periodicity

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \dots)$, δ_n , $\delta_{n_1, n_2, \dots}$, and $\varepsilon_{n_1, n_2, \dots, n_d}$ do not have periodicity.

Parity and symmetry quasi-permutation symmetry

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \dots)$, δ_n , and $\delta_{n_1, n_2, \dots}$ are even functions:

$$\delta(-n) = \delta(n)$$

$$\delta(-n_1, -n_2, \dots, -n_m) = \delta(n_1, n_2, \dots, n_m)$$

$$\delta_{-n} = \delta_n$$

$$\delta_{-n_1, -n_2, \dots, -n_m} = \delta_{n_1, n_2, \dots, n_m}$$

The tensor functions $\delta(n_1, n_2, \dots)$, $\delta_{n_1, n_2, \dots}$, and $\varepsilon_{n_1, n_2, \dots, n_d}$ have permutation symmetry, for example:

$\delta(n_1, n_2, \dots)$	$\delta_{n_1, n_2, \dots}$	$\varepsilon_{n_1, n_2, \dots}$
$\delta(m, n) = \delta(n, m)$	$\delta_{m,n} = \delta_{n,m}$	$\varepsilon_{n_1, n_2, \dots, n_d} = -\varepsilon_{n_2, n_1, \dots, n_d}$
$\delta(n_1, n_2, \dots, n_k, \dots, n_j, \dots, n_m) = \delta(n_1, n_2, \dots, n_j, \dots, n_k, \dots, n_m) / ; n_k \neq n_j \wedge k \neq j$	$\delta_{n_1, n_2, \dots, n_k, \dots, n_j, \dots, n_m} = \delta_{n_1, n_2, \dots, n_j, \dots, n_k, \dots, n_m} / ; n_k \neq n_j \wedge k \neq j$	$\varepsilon_{n_1, n_2, \dots, n_k, \dots, n_j, \dots, n_d} = -\varepsilon_{n_1, n_2, \dots, n_j, \dots, n_k, \dots, n_d}$
$\delta(n_1, n_2, \dots, n_m) = \delta(n_2, n_3, \dots, n_d, n_1)$	$\delta_{n_1, n_2, \dots, n_d} = \delta_{n_2, n_3, \dots, n_d, n_1}$	$\varepsilon_{n_1, n_2, \dots, n_d} = (-1)^{d+1} \varepsilon_{n_2, n_3, \dots, n_d, n_1}$

Integral representations

The discrete delta function $\delta(n)$ and Kronecker delta function $\delta_{n,m}$ have the following integral representations along the interval $(0, 2\pi)$ and unit circle $|z| = 1$:

$$\delta(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{itn} dt$$

$$\delta_{n,m} = \frac{1}{2\pi} \int_0^{2\pi} e^{it(n-m)} dt$$

$$\delta_{n,m} = \frac{1}{2\pi i} \int_{|z|=1} z^{n-m-1} dz.$$

Transformations

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \dots)$, δ_n , $\delta_{n_1, n_2, \dots}$, and $\varepsilon_{n_1, n_2, \dots, n_d}$ satisfy various identities, for example:

$$\delta(-n) = \delta(n)$$

$$\delta(-n_1, -n_2, \dots, -n_m) = \delta(n_1, n_2, \dots, n_m)$$

$$\delta_{-n} = \delta_n$$

$$\delta_{-n_1, -n_2, \dots, -n_m} = \delta_{n_1, n_2, \dots, n_m}$$

$$\delta_{-n,m} = \delta_{n,m} - \theta(|n| - |m|) \operatorname{sgn}(nm) ; n \in \mathbb{Z} \wedge m \in \mathbb{Z}$$

$$\delta(n_1) \delta(n_2) = \delta(n_1, n_2)$$

$$\delta(n_1, n_2, \dots, n_m) \delta(n_{m+1}, n_{m+2}, \dots, n_{m+r}) = \delta(n_1, n_2, \dots, n_m, n_{m+1}, n_{m+2}, \dots, n_{m+r})$$

$$\delta_{n_1, n_2, \dots, n_m} \delta_{n_{m+1}, n_{m+2}, \dots, n_{m+r}} = \delta_{n_1, n_2, \dots, n_m, n_{m+1}, n_{m+2}, \dots, n_{m+r}} ; m \geq 2 \wedge r \geq 2$$

$$\varepsilon_{n_1, n_2, \dots, n_d} \varepsilon_{m_1, m_2, \dots, m_d} = - \sum_{\text{permutations}(m_1, m_2, \dots, m_d)} \varepsilon_{m_1, m_2, \dots, m_d} \prod_{k=1}^d \delta_{n_k, m_k}$$

$$\varepsilon_{n_1, n_2, \dots, n_{r-1}, n_r, n_{r+1}, \dots, n_d} \varepsilon_{n_1, n_2, \dots, n_{r-1}, m_r, m_{r+1}, \dots, m_d} = - \frac{(d-r)! r!}{d!} \sum_{\text{permutations}(m_r, m_{r+1}, \dots, m_d)} \varepsilon_{m_r, m_{r+1}, \dots, m_d} \prod_{k=r}^d \delta_{n_k, m_k}.$$

Complex characteristics

The tensor functions $\delta(n)$, $\delta(n_1, n_2, \dots)$, δ_n , $\delta_{n_1, n_2, \dots}$, and $\varepsilon_{n_1, n_2, \dots, n_d}$ have the following complex characteristics:

	$\delta(n_1, n_2, \dots, n_m)$	$\delta_{n_1, n_2, \dots, n_m}$	$\varepsilon_{n_1, n_2, \dots, n_d}$
Abs	$ \delta(n_1, n_2, \dots, n_m) = \delta(n_1, n_2, \dots, n_m)$	$ \delta_{n_1, n_2, \dots, n_m} = \delta_{n_1, n_2, \dots, n_m}$	$ \varepsilon_{n_1, n_2, \dots, n_d} = \sqrt{\varepsilon_{n_1, n_2, \dots, n_d}}$
Arg	$\operatorname{Arg}(\delta(n_1, n_2, \dots, n_m)) = \tan^{-1}(\delta(n_1, n_2, \dots, n_m), 0)$	$\operatorname{Arg}(\delta_{n_1, n_2, \dots, n_m}) = \tan^{-1}(\delta_{n_1, n_2, \dots, n_m}, 0)$	$\operatorname{Arg}(\varepsilon_{n_1, n_2, \dots, n_d}) = \operatorname{tar}(\varepsilon_{n_1, n_2, \dots, n_d})$
Re	$\operatorname{Re}(\delta(n_1, n_2, \dots, n_m)) = \delta(n_1, n_2, \dots, n_m)$	$\operatorname{Re}(\delta_{n_1, n_2, \dots, n_m}) = \delta_{n_1, n_2, \dots, n_m}$	$\operatorname{Re}(\varepsilon_{n_1, n_2, \dots, n_d}) = \varepsilon_{n_1, n_2, \dots, n_d}$
Im	$\operatorname{Im}(\delta(n_1, n_2, \dots, n_m)) = 0$	$\operatorname{Im}(\delta_{n_1, n_2, \dots, n_m}) = 0$	$\operatorname{Im}(\varepsilon_{n_1, n_2, \dots, n_d}) = 0$
Conjugate	$\overline{\delta(n_1, n_2, \dots, n_m)} = \delta(n_1, n_2, \dots, n_m)$	$\overline{\delta_{n_1, n_2, \dots, n_m}} = \delta_{n_1, n_2, \dots, n_m}$	$\overline{\varepsilon_{n_1, n_2, \dots, n_d}} = \varepsilon_{n_1, n_2, \dots, n_d}$

Differentiation

Differentiation of the tensor functions $\delta(n)$ and δ_n can be provided by the following formulas:

$$\frac{\partial \delta(n)}{\partial n} = 0$$

$$\frac{\partial \delta_n}{\partial n} = 0.$$

Fractional integro-differentiation of the tensor functions $\delta(n)$ and δ_n can be provided by the following formulas:

$$\frac{\partial^\alpha \delta(n)}{\partial n^\alpha} = \frac{n^{-\alpha} \delta(n)}{\Gamma(1 - \alpha)}$$

$$\frac{\partial^\alpha \delta_n}{\partial n^\alpha} = \frac{n^{-\alpha} \delta_n}{\Gamma(1 - \alpha)}.$$

Indefinite integration

Indefinite integration of the tensor functions $\delta(n)$ and δ_n can be provided by the following formulas:

$$\int \delta(z) dz = \delta(z) z$$

$$\int \delta_z dz = \delta_z z.$$

Summation

The following relations represent the sifting properties of the Kronecker and discrete delta functions:

$$\sum_{k=-\infty}^{\infty} \delta_{k,n} a_k = a_n$$

$$\sum_{k=-\infty}^{\infty} \delta(k, n) a_k = a_0.$$

There exist various formulas including finite summation of signature $\varepsilon_{n_1, n_2, \dots, n_d}$, for example:

$$\sum_{\tau_1=1}^n \sum_{\tau_2=1}^n \dots \sum_{\tau_r=1}^n \varepsilon_{\tau_1, \tau_2, \dots, \tau_r, \nu_{r+1}, \dots, \nu_n} \varepsilon_{\tau_1, \tau_2, \dots, \tau_r, \mu_{r+1}, \dots, \mu_n} = r! \sum_{\mu_{r+1}=1}^n \sum_{\mu_{r+2}=1}^n \dots \sum_{\mu_n=1}^n \varepsilon_{\mu_{r+1}, \dots, \mu_n} \prod_{k=r+1}^n \delta_{\nu_k, \mu_k}.$$

Applications of the tensor functions

The tensor functions have numerous applications throughout mathematics, number theory, analysis, and other fields.

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