ArithmeticGeometricMean

Notations

Traditional name

Arithmetic-geometric mean

Traditional notation

agm(a, b)

Mathematica StandardForm notation

ArithmeticGeometricMean[a, b]

Primary definition

\[ \text{agm}(a, b) = \frac{\pi (a + b)}{4 K\left(\frac{a-b}{a+b}\right)^2} \]

Specific values

Specialized values

For fixed \(a\)

\[ \text{agm}(a, a) = a \]

For fixed \(b\)

\[ \text{agm}(0, b) = 0 \]

\[ \text{agm}(1, b) = \frac{\pi}{2 K(1 - b^2)} \]

\[ \text{agm}(a, \sqrt{2} a) = a \sqrt{\frac{2}{\pi} \left(1 - \frac{3}{4}\right)^2} \]
Values at fixed points

\[ \text{agm}(0, 1) = 0 \]

Values at infinities

\[ \text{agm}(1, \infty) = \infty \]

General characteristics

Domain and analyticity

\text{agm}(a, b) \text{ is an analytical function of } a \text{ and } b \text{ which is defined over } \mathbb{C}^2.

\[ (a \times b) \rightarrow \text{agm}(a, b) : (\mathbb{C} \otimes \mathbb{C}) \rightarrow \mathbb{C} \]

Symmetries and periodicities

Parity

\text{agm}(a, b) \text{ is an odd function.}

\[ \text{agm}(-a, -b) = -\text{agm}(a, b) \; ; \; a \notin \mathbb{R} / \; b \notin \mathbb{R} \]

Mirror symmetry

\[ \text{agm}(\bar{a}, \bar{b}) = \text{agm}(a, b) \; ; \; -\frac{a}{b} \notin (-\infty, 0) \]

Permutation symmetry

\[ \text{agm}(b, a) = \text{agm}(a, b) \]

Periodicity

No periodicity

Homogeneity

\[ \text{agm}(ca, cb) = c \text{agm}(a, b) \; ; \; c > 0 \]

Poles and essential singularities
With respect to $a$

The function $\text{agm}(a, b)$ does not have poles and essential singularities with respect to $a$.

\[ \text{Sing}_a(\text{agm}(a, b)) = \{ \} \]

With respect to $b$

The function $\text{agm}(a, b)$ does not have poles and essential singularities with respect to $b$.

\[ \text{Sing}_b(\text{agm}(a, b)) = \{ \} \]

Branch points

The function $\text{agm}(a, b)$ on the $\frac{a}{b}$-plane has two branch points: $\frac{a}{b} = 0$, $\frac{a}{b} = \infty$.

\[ \text{BP}_a(\text{agm}(a, b)) = \{0, \infty\} \]

\[ \mathcal{R}_a(\text{agm}(a, b), 0) = \log \]

\[ \mathcal{R}_b(\text{agm}(a, b), \infty) = \log \]

Branch cuts

The function $\text{agm}(a, b)$ is a single-valued function on the $\frac{a}{b}$-plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

\[ \text{BC}_a(\text{agm}(a, b)) = \{(-\infty, 0), -i\} \]

\[ \lim_{\epsilon\to0} \text{agm}(a + i \epsilon, 1) = \text{agm}(a, 1) / \epsilon; a < 0 \]

\[ \lim_{\epsilon\to0} \text{agm}(a - i \epsilon, 1) = a \text{agm} \left( 1, \frac{1}{a} \right) / \epsilon; a < 0 \]

Series representations

Generalized power series

Expansions at $b = 0$ for $a = 1$
Expansions at \( b = 1 \) for \( a = 1 \)

\[
\text{agm}(1, b) \propto \frac{\pi}{2 (\log(4) - \log(b))} + O\left(\frac{b^2}{\log(b)}\right); (b \to 0)
\]

\[
\frac{b - 1}{2} - \frac{(b - 1)^2}{16} - \frac{(b - 1)^3}{32} - \frac{21 (b - 1)^4}{1024} - \frac{31 (b - 1)^5}{2048} - \frac{195 (b - 1)^6}{16384} + \\
\frac{319 (b - 1)^7}{32768} - \frac{34325 (b - 1)^8}{4194304} - \frac{58899 (b - 1)^9}{8388608} - \frac{410771 (b - 1)^{10}}{67108864} + O((b - 1)^{11}); (b \to 1)
\]

Expansions at \( b = \infty \) for \( a = 1 \)

\[
\text{agm}(1, b) \propto 1 + O(b - 1); (b \to 1)
\]
\[
\text{Integral representations}
\]

\section*{On the real axis}

\textbf{Of the direct function}

\[
\text{agm}(a, b) = \frac{\pi b}{2 \log(4b)} + O\left(\frac{1}{b \log(b)}\right); (|b| \to \infty)
\]

\section*{Product representations}

\[
\text{agm}(1, b) = \prod_{k=0}^{\infty} \frac{1}{2} (q_k + 1); q_0 = b \wedge q_{k+1} = \frac{2 \sqrt{q_k}}{q_k + 1}
\]

\section*{Limit representations}

\[
\text{agm}(a, b) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n; a_0 = a > b_0 = b > 0
\]

\[
a_{n+1} = \frac{1}{2} (a_n + b_n) = \text{agm}(a_n, b_n) \sqrt{0, z^{2n+1}} \wedge b_{n+1} = \sqrt{a_n b_n} = \text{agm}(a_n, b_n) \sqrt{0, z^{2n+1}} \wedge z = q \left(1 - \frac{b_0}{a_0}\right)^2
\]
Differential equations

Ordinary nonlinear differential equations

\[ 2 a (b^2 - a^2) \left( \frac{\partial w(a)}{\partial a} \right)^2 - a w(a)^2 + \left( 3 a^2 - b^2 \right) \frac{\partial w(a)}{\partial a} + a (a^2 - b^2) \frac{\partial^2 w(a)}{\partial a^2} \right) w(a) = 0 ; \ w(a) = \text{agm}(a, b) \]

Partial differential equations

\[ \text{agm}(a, b) - a \frac{\partial \text{agm}(a, b)}{\partial a} - b \frac{\partial \text{agm}(a, b)}{\partial b} = 0 \]

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

\[ \text{agm}(-a, -b) = -\text{agm}(a, b) ; \ a \notin \mathbb{R} \land b \notin \mathbb{R} \]

\[ \text{agm}(c a, c b) = c \text{agm}(a, b) ; \ c > 0 \]

\[ \text{agm}(1, z) = \frac{1}{a} \text{agm}(a, a z) ; \ a > 0 \]

\[ \text{agm} \left( \frac{a + b}{2}, \sqrt{ab} \right) = \text{agm}(a, b) \]

\[ \text{agm} \left( 1, \sqrt{1 - z^2} \right) = \text{agm}(z + 1, 1 - z) \]

\[ \text{agm} \left( 1, \frac{2 \sqrt{b}}{b + 1} \right) = \frac{2}{b + 1} \text{agm}(1, b) \]

Identities

Functional identities

\[ \text{agm}(c a, c b) = c \text{agm}(a, b) ; \ c > 0 \]
\[\text{agm}(a, b) = a \text{agm}\left(1, \frac{b}{a}\right) ; a > 0\]

\[\text{agm}(a, b) = \text{agm}\left(\frac{a + b}{2}, \sqrt{a b}\right)\]

\[\text{agm}(a, 2 - a) = \text{agm}\left(1, \sqrt{a (2 - a)}\right)\]

\[\text{agm}(1, b) = \frac{b + 1}{2} \text{agm}\left(1, \frac{2 \sqrt{b}}{b + 1}\right)\]

## Differentiation

### Low-order differentiation

**With respect to** \(a\)

\[
\frac{\partial \text{agm}(a, b)}{\partial a} = \frac{\text{agm}(a, b)}{a (a - b) \pi} \left(a \pi - 2 \text{agm}(a, b) E \left(\frac{(a - b)^2}{(a + b)^2}\right)\right)
\]

\[
\frac{\partial^2 \text{agm}(a, b)}{\partial a^2} = \frac{2 \text{agm}(a, b)^2}{a^2 (a - b)^2 (a + b)^2 \pi^2} \left\{4 (a + b) \text{agm}(a, b) E \left(\frac{(a - b)^2}{(a + b)^2}\right)^2 - \pi \left(\frac{a^2 + 4 a b + b^2}{(a + b)^2}\right) E \left(\frac{(a - b)^2}{(a + b)^2}\right) - 2 a b K \left(\frac{(a - b)^2}{(a + b)^2}\right)\right\}
\]

**With respect to** \(b\)

\[
\frac{\partial \text{agm}(a, b)}{\partial b} = \frac{\text{agm}(a, b)}{(a - b) b \pi} \left(2 \text{agm}(a, b) E \left(\frac{(a - b)^2}{(a + b)^2}\right) - b \pi\right)
\]

\[
\frac{\partial^2 \text{agm}(a, b)}{\partial b^2} = \frac{2 \text{agm}(a, b)^2}{(a - b)^2 b^2 (a + b)^2 \pi^2} \left\{4 (a + b) \text{agm}(a, b) E \left(\frac{(a - b)^2}{(a + b)^2}\right)^2 - \pi \left(\frac{a^2 + 4 a b + b^2}{(a + b)^2}\right) E \left(\frac{(a - b)^2}{(a + b)^2}\right) - 2 a b K \left(\frac{(a - b)^2}{(a + b)^2}\right)\right\}
\]

### Symbolic differentiation

**With respect to** \(a\)
\[
\frac{\partial^n \text{agm}(a, b)}{\partial a^n} = \text{agm}(a, b) \delta_a^n + \frac{\pi}{4b^n} \left( b \delta_{a-1} + b n! \sum_{q=1}^{n} \frac{(-1)^q}{(q+1)! (n-q-1)!} K\left( \frac{a-b}{a+b} \right)^{2^{-q-1}} \right)
\]

\[
\sum_{k_1=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \sum_{k_2=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \ldots \sum_{k_q=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \left( \prod_{j=1}^{q-1} A(k_j, a, b) \right) \left( n - \sum_{j=1}^{q-1} k_j \right) \right) \left( n - \sum_{j=1}^{q} k_j - 1, b, a \right) + \right.

\]

\[
(a + b)(n + 1)! \sum_{q=1}^{n} \frac{(-1)^q}{(q+1)! (n-q)!} K\left( \frac{a-b}{a+b} \right)^{2^{-q-1}} \sum_{k_1=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \sum_{k_2=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \ldots \sum_{k_q=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \left( \prod_{j=1}^{q-1} A(k_j, a, b) \right) \left( n - \sum_{j=1}^{q-1} k_j \right) \left( n - \sum_{j=1}^{q} k_j - 1, b, a \right) + \right.

\]

With respect to \( b \)

\[
\frac{\partial^n \text{agm}(a, b)}{\partial b^n} = \text{agm}(a, b) \delta_b^n + \frac{\pi}{4a^n} \left( a \delta_{b-1} + a n! \sum_{q=1}^{n} \frac{(-1)^q}{(q+1)! (n-q-1)!} K\left( \frac{a-b}{a+b} \right)^{2^{-q-1}} \right)
\]

\[
\sum_{k_1=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \sum_{k_2=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \ldots \sum_{k_q=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \left( \prod_{j=1}^{q-1} A(k_j, b, a) \right) \left( n - \sum_{j=1}^{q-1} k_j \right) \left( n - \sum_{j=1}^{q} k_j - 1, b, a \right) + \right.

\]

\[
(a + b)(n + 1)! \sum_{q=1}^{n} \frac{(-1)^q}{(q+1)! (n-q)!} K\left( \frac{a-b}{a+b} \right)^{2^{-q-1}} \sum_{k_1=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \sum_{k_2=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \ldots \sum_{k_q=0}^{n-\Sigma_{j=1}^{n-k_1} k_j} \left( \prod_{j=1}^{q-1} A(k_j, b, a) \right) \left( n - \sum_{j=1}^{q-1} k_j \right) \left( n - \sum_{j=1}^{q} k_j - 1, b, a \right) + \right.

\]

Representations through more general functions
Through hypergeometric functions

Involving $_2F_1$

\[ agm(a, b) = \frac{a + b}{2} \frac{\binom{1}{\frac{1}{2}} \frac{1}{2}; 1; \left(\frac{a-b}{a+b}\right)^2}{_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{a-b}{a+b}\right)^2\right)} \]

Through Meijer G

Classical cases for the direct function itself

\[ agm(a, b) = \frac{\pi (a + b)}{\sqrt{2}} G_{2,2}^{1,2}\left(1 - \left(\frac{a-b}{a+b}\right)^2, \frac{1}{2}; \frac{1}{2}; 0, 0\right) \]

Through other functions

Involving some hypergeometric-type functions

\[ agm(a, b) = \frac{\pi (a + b)}{4 \sqrt{2}} \frac{\binom{1}{\frac{1}{2}} \frac{1}{2}; 1; \left(\frac{a-b}{a+b}\right)^2}{_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{a-b}{a+b}\right)^2\right)} \]

Inequalities

\[ \sqrt{ab} \leq agm(a, b) \leq \frac{a + b}{2} \]

Theorems

Representation of $\pi$

\[ \pi = \lim_{n \to \infty} 2 \frac{a_{n+1}^2}{\sqrt{1 - \sum_{k=0}^{n} 2^k c_k^2}} /; c_{n+1} = \frac{a_n - b_n}{2} \wedge c_0 = \sqrt{a_0^2 - b_0^2} \]

History

– J. Landen (1771, 1775)
– J.-L. Lagrange (1784-85)
– C. F. Gauss (1791–1799, 1800, 1876); Gauss (1800) derived the relation to $F_1(a, b; c; z)$

Applications include fast high-precision computation of $\pi$, $\log(z)$, $e^z$, $\sin(z)$, $\cos(z)$, etc.
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