Beta4

Notations

Traditional name

Generalized incomplete beta function

Traditional notation

$\mathrm{B}_{(z_1,z_2)}(a,\, b)$

Mathematica StandardForm notation

`Beta[`z_1`, `z_2`, `a`, `b`]`

Primary definition

06.20.02.0001.01

$$\mathrm{B}_{(z_1,z_2)}(a,\, b) = \int_{z_1}^{z_2} t^{a-1}\, (1-t)^{b-1}\, dt$$

Specific values

Specialized values

For fixed z_1, z_2, a

06.20.03.0001.01

$$\mathrm{B}_{(z_1,z_2)}(a,\, n) = \mathrm{B}(a,\, n) \left(z_2^a \sum_{k=0}^{n-1} \frac{(a)_k\, (1-z_2)^k}{k!} - z_1^a \sum_{k=0}^{n-1} \frac{(a)_k\, (1-z_1)^k}{k!} \right) /;\, n \in \mathbb{N}$$

For fixed z_1, z_2, b

06.20.03.0002.01

$$\mathrm{B}_{(z_1,z_2)}(n,\, b) = \mathrm{B}(n,\, b) \left((1-z_1)^b \sum_{k=0}^{n-1} \frac{(b)_k\, z_1^k}{k!} - (1-z_2)^b \sum_{k=0}^{n-1} \frac{(b)_k\, z_2^k}{k!} \right) /;\, n \in \mathbb{N}$$

For fixed z_1, a, b

06.20.03.0003.01

$$\mathrm{B}_{(z_1,0)}(a,\, b) = -\mathrm{B}_{z_1}(a,\, b) /;\, \mathrm{Re}(a) > 0$$

06.20.03.0004.01

$$\mathrm{B}_{(z_1,0)}(a,\, b) = \tilde{\infty} /;\, \mathrm{Re}(a) < 0$$

06.20.03.0005.01

$B_{(z_1,1)}(a, b) = B(a, b) - B_{z_1}(a, b) /; \mathrm{Re}(b) > 0$

For fixed z_2, a, b

06.20.03.0006.01

$B_{(0,z_2)}(a, b) = B_{z_2}(a, b) /; \mathrm{Re}(a) > 0$

06.20.03.0007.01

$B_{(0,z_2)}(a, b) = \tilde{\infty} /; \mathrm{Re}(a) < 0$

06.20.03.0008.01

$B_{(1,z_2)}(a, b) = B_{z_2}(a, b) - B(a, b) /; \mathrm{Re}(b) > 0$

General characteristics

Domain and analyticity

$B_{(z_1,z_2)}(a, b)$ is an analytical function of z_1, z_2, a, and b which is defined in $\mathbb{C}^4$.

06.20.04.0001.01

$(z_1 * z_2 * a * b) \longrightarrow B_{(z_1,z_2)}(a, b) :: (\mathbb{C}^4) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Mirror symmetry

06.20.04.0002.02

$B_{(\overline{z_1},\overline{z_2})}(\overline{a}, \overline{b}) = \overline{B_{(z_1,z_2)}(a, b)} /; z_1 \notin (-\infty, 0) \bigwedge z_1 \notin (1, \infty) \bigwedge z_2 \notin (-\infty, 0) \bigwedge z_2 \notin (1, \infty)$

Permutation symmetry

06.20.04.0003.01

$B_{(z_1,z_2)}(a, b) = -B_{(z_2,z_1)}(a, b)$

Periodicity

No periodicity

Poles and essential singularities

With respect to b

For fixed z_1, z_2, a, the function $B_{(z_1,z_2)}(a, b)$ has only one singular point at $b = \tilde{\infty}$. It is an essential singular point.

06.20.04.0004.01

$\mathcal{Sing}_b(B_{(z_1,z_2)}(a, b)) = \{\{\tilde{\infty}, \infty\}\}$

With respect to a

For fixed z_1, z_2, b, the function $B_{(z_1,z_2)}(a, b)$ has only one singular point at $a = \tilde{\infty}$. It is an essential singular point.

06.20.04.0005.01

$\mathcal{Sing}_a(B_{(z_1,z_2)}(a, b)) = \{\{\tilde{\infty}, \infty\}\}$

With respect to z_k

For fixed a, b, the function $B_{(z_1,z_2)}(a, b)$ does not have poles and essential singularities.

06.20.04.0006.01

$$\mathcal{Sing}_{z_k}\left(B_{(z_1,z_2)}(a, b)\right) == \{\} \;/;\; k \in \{1, 2\}$$

Branch points

With respect to b

For fixed z_1, z_2, a, the function $B_{(z_1,z_2)}(a, b)$ does not have branch points.

06.20.04.0007.01

$$\mathcal{BP}_b\left(B_{(z_1,z_2)}(a, b)\right) == \{\}$$

With respect to a

For fixed z_1, z_2, b, the function $B_{(z_1,z_2)}(a, b)$ does not have branch points.

06.20.04.0008.01

$$\mathcal{BP}_a\left(B_{(z_1,z_2)}(a, b)\right) == \{\}$$

With respect to z_k

The function $B_{(z_1,z_2)}(a, b)$ has for fixed $z_1$or fixed $z_2$three singular branch points with respect to $z_2$or z_1: $z_k == 0$, $z_k == 1$, $z_k == \tilde{\infty}$, $k = 1, 2$.

06.20.04.0009.01

$$\mathcal{BP}_{z_k}\left(B_{(z_1,z_2)}(a, b)\right) == \{0, 1, \tilde{\infty}\} \;/;\; k \in \{1, 2\}$$

06.20.04.0010.01

$$\mathcal{R}_{z_k}\left(B_{(z_1,z_2)}(a, b), 0\right) == \log \;/;\; a \notin \mathbb{Z} \land a \notin \mathbb{Q} \land k \in \{1, 2\}$$

06.20.04.0011.01

$$\mathcal{R}_{z_k}\left(B_{(z_1,z_2)}(a, b), 0\right) == q \;/;\; a == \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1 \bigwedge k \in \{1, 2\}$$

06.20.04.0012.01

$$\mathcal{R}_{z_k}\left(B_{(z_1,z_2)}(a, b), 1\right) == \log \;/;\; b \notin \mathbb{Z} \land b \notin \mathbb{Q} \land k \in \{1, 2\}$$

06.20.04.0013.01

$$\mathcal{R}_{z_k}\left(B_{(z_1,z_2)}(a, b), 1\right) == q \;/;\; b == \frac{p}{q} \bigwedge p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) == 1 \bigwedge k \in \{1, 2\}$$

06.20.04.0014.01

$$\mathcal{R}_{z_k}\left(B_{(z_1,z_2)}(a, b), \tilde{\infty}\right) == \log \;/;\; a + b \in \mathbb{Z} \lor a + b \notin \mathbb{Q} \land k \in \{1, 2\}$$

06.20.04.0015.01

$$\mathcal{R}_{z_k}\left(B_{(z_1,z_2)}(a, b), \tilde{\infty}\right) == s \;/;\; a + b == \frac{r}{s} \bigwedge r \in \mathbb{Z} \bigwedge s - 1 \in \mathbb{N}^+ \bigwedge \gcd(r, s) == 1 \bigwedge k \in \{1, 2\}$$

Branch cuts

With respect to b

For fixed z_1, z_2, a, the function $B_{(z_1,z_2)}(a, b)$ does not have branch cuts.

06.20.04.0016.01

$$\mathcal{BC}_b\left(B_{(z_1,z_2)}(a, b)\right) = \{\}$$

With respect to *a*

For fixed z_1, z_2, b, the function $B_{(z_1,z_2)}(a, b)$ does not have branch cuts.

06.20.04.0017.01

$$\mathcal{BC}_a\left(B_{(z_1,z_2)}(a, b)\right) = \{\}$$

With respect to z_1

For fixed a, b, z_2, the function $B_{(z_1,z_2)}(a, b)$ is a single-valued function on the z_1-plane cut along the intervals $(-\infty, 0)$ and $(1, \infty)$.

The function $B_{(z_1,z_2)}(a, b)$ is continuous from above on the interval $(-\infty, 0)$ and from below on the interval $(1, \infty)$.

06.20.04.0018.01

$$\mathcal{BC}_{z_1}\left(B_{(z_1,z_2)}(a, b)\right) = \{\{(-\infty, 0), -i\}, \{(1, \infty), i\}\}$$

06.20.04.0019.01

$$\lim_{\epsilon \to +0} B_{(x_1+i\epsilon,z_2)}(a, b) = B_{(x_1,z_2)}(a, b) /; x_1 < 0$$

06.20.04.0020.01

$$\lim_{\epsilon \to +0} B_{(x_1-i\epsilon,z_2)}(a, b) = B_{(x_1,z_2)}(a, b) + \left(1 - e^{-2 i a \pi}\right) B_{x_1}(a, b) /; x_1 < 0$$

06.20.04.0021.01

$$\lim_{\epsilon \to +0} B_{(x_1-i\epsilon,z_2)}(a, b) = B_{(x_1,z_2)}(a, b) /; x_1 > 1$$

06.20.04.0022.01

$$\lim_{\epsilon \to +0} B_{(x_1+i\epsilon,z_2)}(a, b) = B_{(x_1,z_2)}(a, b) + \left(1 - e^{-2 i b \pi}\right) B_{x_1}(a, b) - 2 i e^{-i b \pi} \sin(b \pi) B(a, b) /; x_1 > 1$$

With respect to z_2

For fixed a, b, z_1, the function $B_{(z_1,z_2)}(a, b)$ is a single-valued function on the z_2-plane cut along the intervals $(-\infty, 0)$ and $(1, \infty)$.

The function $B_{(z_1,z_2)}(a, b)$ is continuous from above on the interval $(-\infty, 0)$ and from below on the interval $(1, \infty)$.

06.20.04.0023.01

$$\mathcal{BC}_{z_2}\left(B_{(z_1,z_2)}(a, b)\right) = \{\{(-\infty, 0), -i\}, \{(1, \infty), i\}\}$$

06.20.04.0024.01

$$\lim_{\epsilon \to +0} B_{(z_1, x_2+i\epsilon)}(a, b) = B_{(z_1,x_2)}(a, b) /; x_2 < 0$$

06.20.04.0025.01

$$\lim_{\epsilon \to +0} B_{(z_1, x_2-i\epsilon)}(a, b) = B_{(z_1,x_2)}(a, b) - \left(1 - e^{-2 i a \pi}\right) B_{x_2}(a, b) /; x_2 < 0$$

06.20.04.0026.01

$$\lim_{\epsilon \to +0} B_{(z_1, x_2-i\epsilon,z_2)}(a, b) = B_{(z_1,x_2)}(a, b) /; x_2 > 1$$

06.20.04.0027.01

$$\lim_{\epsilon \to +0} B_{(z_1, x_2 + i\epsilon, z_2)}(a, b) = B_{(z_1, x_2)}(a, b) - \left(1 - e^{-2 i b \pi}\right) B_{x_2}(a, b) + 2 i e^{-i b \pi} \sin(b \pi) B(a, b) /; x_2 > 1$$

Series representations

Generalized power series

Expansions at $\{z_1, z_2\} = \{0, 0\}$

06.20.06.0001.02

$$B_{(z_1, z_2)}(a, b) \propto z_2^a \left(\frac{1}{a} + \frac{(1-b) z_2}{1+a} + \frac{(1-b)(2-b) z_2^2}{2(2+a)} + \ldots \right) - z_1^a \left(\frac{1}{a} + \frac{(1-b) z_1}{1+a} + \frac{(1-b)(2-b) z_1^2}{2(2+a)} + \ldots \right) /;$$
$$(z_1 \to 0) \land (z_2 \to 0) \land -a \notin \mathbb{N}$$

06.20.06.0002.01

$$B_{(z_1, z_2)}(a, b) = \sum_{k=0}^{\infty} \frac{(1-b)_k \left(z_2^{a+k} - z_1^{a+k}\right)}{(a+k)\, k!} /; |z_1| < 1 \land |z_2| < 1 \land -a \notin \mathbb{N}$$

06.20.06.0003.01

$$B_{(z_1, z_2)}(a, b) = \frac{z_2^a}{a} {}_2F_1(a, 1-b; a+1; z_2) - \frac{z_1^a}{a} {}_2F_1(a, 1-b; a+1; z_1)$$

06.20.06.0004.01

$$B_{(z_1, z_2)}(a, b) \propto \frac{z_2^a}{a} (1 + O(z_2)) - \frac{z_1^a}{a} (1 + O(z_1)) /; (z_1 \to 0) \land (z_2 \to 0) \land -a \notin \mathbb{N}$$

Integral representations

On the real axis

Of the direct function

06.20.07.0001.01

$$B_{(z_1, z_2)}(a, b) = \int_{z_1}^{z_2} t^{a-1} (1-t)^{b-1} \, dt$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

06.20.13.0001.01

$$(1 - z_1) z_1 w''(z_1) + (1 - a + (a + b - 2) z_1) w'(z_1) = 0 /; w(z_1) = c_1 B_{(z_1, z_2)}(a, b) + c_2$$

06.20.13.0003.01

$$W_{z_1}(1, B(z_1, z_2, a, b)) = -(1 - z_1)^{b-1} z_1^{a-1}$$

06.20.13.0002.01

$$(1 - z_2) z_2 w''(z_2) + (1 - a + (a + b - 2) z_2) w'(z_2) = 0 /; w(z_2) = c_1 B_{(z_1, z_2)}(a, b) + c_2$$

06.20.13.0004.01

$$W_{z_2}(1, \mathrm{B}(z_1, z_2, a, b)) = (1 - z_2)^{-1+b} z_2^{-1+a}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.20.16.0001.01

$$\mathrm{B}_{(1-z_1\,1-,z_2)}(a, b) = -\mathrm{B}_{(z_1,z_2)}(b, a)$$

06.20.16.0002.01

$$\mathrm{B}_{(z_1,z_2)}(a+1, b) = \frac{a}{a+b} \mathrm{B}_{(z_1,z_2)}(a, b) + \frac{1}{a+b} \left((1-z_1)^b z_1^a - (1-z_2)^b z_2^a\right)$$

06.20.16.0003.01

$$\mathrm{B}_{(z_1,z_2)}(a-1, b) = \frac{a+b-1}{a-1} \mathrm{B}_{(z_1,z_2)}(a, b) + \frac{1}{a-1} \left((1-z_2)^b z_2^{a-1} - (1-z_1)^b z_1^{a-1}\right)$$

06.20.16.0004.01

$$\mathrm{B}_{(z_1,z_2)}(a+n, b) = \frac{(a)_n}{(a+b)_n} \mathrm{B}_{(z_1,z_2)}(a, b) + \frac{1}{a+b+n-1} \sum_{k=0}^{n-1} \frac{(1-a-n)_k}{(2-a-b-n)_k} \left((1-z_1)^b z_1^{a+n-k-1} - (1-z_2)^b z_2^{a+n-k-1}\right) /; n \in \mathbb{N}$$

06.20.16.0005.01

$$\mathrm{B}_{(z_1,z_2)}(a-n, b) = \frac{(1-a-b)_n}{(1-a)_n} \mathrm{B}_{(z_1,z_2)}(a, b) - \frac{(2-a-b)_{n-1}}{(1-a)_n} \sum_{k=0}^{n-1} \frac{(1-a)_k}{(2-a-b)_k} \left((1-z_2)^b z_2^{a-k-1} - (1-z_1)^b z_1^{a-k-1}\right) /; n \in \mathbb{N}$$

Identities

Recurrence identities

Consecutive neighbors

06.20.17.0001.01

$$\mathrm{B}_{(z_1,z_2)}(a, b) = \frac{a+b}{a} \mathrm{B}_{(z_1,z_2)}(a+1, b) + \frac{1}{a} \left((1-z_2)^b z_2^a - (1-z_1)^b z_1^a\right)$$

06.20.17.0002.01

$$\mathrm{B}_{(z_1,z_2)}(a, b) = \frac{a-1}{a+b-1} \mathrm{B}_{(z_1,z_2)}(a-1, b) + \frac{1}{a+b-1} \left((1-z_1)^b z_1^{a-1} - (1-z_2)^b z_2^{a-1}\right)$$

Distant neighbors

06.20.17.0003.02

$$\mathrm{B}_{(z_1,z_2)}(a, b) = \frac{(a+b)_n}{(a)_n} \mathrm{B}_{(z_1,z_2)}(a+n, b) + \frac{1}{a+b-1} \sum_{k=1}^{n} \frac{(a+b-1)_k}{(a)_k} \left((1-z_2)^b z_2^{a+k-1} - (1-z_1)^b z_1^{a+k-1}\right) /; n \in \mathbb{N}$$

06.20.17.0004.01

$$\mathrm{B}_{(z_1,z_2)}(a, b) = \frac{(1-a)_n}{(1-a-b)_n} \mathrm{B}_{(z_1,z_2)}(a-n, b) + \frac{1}{a+b-1} \sum_{k=0}^{n-1} \frac{(1-a)_k}{(2-a-b)_k} \left((1-z_1)^b z_1^{a-k-1} - (1-z_2)^b z_2^{a-k-1}\right) /; n \in \mathbb{N}$$

Functional identities

Relations between contiguous functions

06.20.17.0005.01

$$B_{(z_1,z_2)}(a,b) = B_{(z_1,z_2)}(a+1,b) + B_{(z_1,z_2)}(a,b+1)$$

Major general cases

06.20.17.0006.01

$$B_{(z_1,z_2)}(a,b) = -B_{(1-z_1,1-z_2)}(b,a)$$

06.20.17.0007.01

$$B_{(1-z_1,z_2)}(a,b) = B_{(1-z_2,z_1)}(b,a)$$

Differentiation

Low-order differentiation

With respect to z_1

06.20.20.0001.01

$$\frac{\partial B_{(z_1,z_2)}(a,b)}{\partial z_1} = -(1-z_1)^{b-1} z_1^{a-1}$$

06.20.20.0002.01

$$\frac{\partial^2 B(z_1,z_2,a,b)}{\partial z_1^2} = (1-z_1)^{b-2} z_1^{a-2} (-a+(a+b-2) z_1 + 1)$$

With respect to z_2

06.20.20.0003.01

$$\frac{\partial B_{(z_1,z_2)}(a,b)}{\partial z_2} = (1-z_2)^{b-1} z_2^{a-1}$$

06.20.20.0004.01

$$\frac{\partial^2 B(z_1,z_2,a,b)}{\partial z_2^2} = (1-z_2)^{b-2} z_2^{a-2} (a-(a+b-2) z_2 - 1)$$

With respect to a

06.20.20.0005.01

$$\frac{\partial B_{(z_1,z_2)}(a,b)}{\partial a} = \Gamma(a)^2 \left(z_1^a\, {}_3\tilde{F}_2(a,a,1-b;a+1,a+1;z_1) - z_2^a\, {}_3\tilde{F}_2(a,a,1-b;a+1,a+1;z_2)\right) - \log(z_1) B_{z_1}(a,b) + \log(z_2) B_{z_2}(a,b)$$

06.20.20.0006.01

$$\frac{\partial^2 B(z_1,z_2,a,b)}{\partial a^2} = 2\,\Gamma(a)^2 z_2^a\left(\Gamma(a)\, {}_4\tilde{F}_3(a,a,a,1-b;a+1,a+1,a+1;z_2) - \log(z_2)\, {}_3\tilde{F}_2(a,a,1-b;a+1,a+1;z_2)\right) - 2\,\Gamma(a)^2 z_1^a\left(\Gamma(a)\, {}_4\tilde{F}_3(a,a,a,1-b;a+1,a+1,a+1;z_1) - \log(z_1)\, {}_3\tilde{F}_2(a,a,1-b;a+1,a+1;z_1)\right) - \log^2(z_1) B_{z_1}(a,b) + \log^2(z_2) B_{z_2}(a,b)$$

With respect to b

06.20.20.0007.01

$$\frac{\partial B_{(z_1,z_2)}(a,b)}{\partial b} = -\Gamma(b)^2 \left((1-z_1)^b \, {}_3\tilde{F}_2(b,b,1-a;b+1,b+1;1-z_1) - (1-z_2)^b \, {}_3\tilde{F}_2(b,b,1-a;b+1,b+1;1-z_2)\right) + \log(1-z_1) \, B_{1-z_1}(b,a) - \log(1-z_2) \, B_{1-z_2}(b,a)$$

06.20.20.0008.01

$$\frac{\partial^2 B(z_1,z_2,a,b)}{\partial b^2} = 2\,\Gamma(b)^2 (1-z_1)^b \left(\Gamma(b) \, {}_4\tilde{F}_3(b,b,b,1-a;b+1,b+1,b+1;1-z_1) - \log(1-z_1) \, {}_3\tilde{F}_2(b,b,1-a;b+1,b+1;1-z_1)\right) + \log^2(1-z_1) \, B_{1-z_1}(b,a) - \log^2(1-z_2) \, B_{1-z_2}(b,a) - 2\,\Gamma(b)^2 (1-z_2)^b \left(\Gamma(b) \, {}_4\tilde{F}_3(b,b,b,1-a;b+1,b+1,b+1;1-z_2) - \log(1-z_2) \, {}_3\tilde{F}_2(b,b,1-a;b+1,b+1;1-z_2)\right)$$

Symbolic differentiation

With respect to z_1

06.20.20.0022.01

$$\frac{\partial^n B(z_1,z_2,a,b)}{\partial z_1^n} = \delta_n \, B(z_1,z_2,a,b) + (1-z_1)^{b-1} z_1^{a-n} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} (1-b)_k (1-a)_{n-k-1} \left(\frac{z_1}{1-z_1}\right)^k /; n \in \mathbb{N}$$

06.20.20.0009.02

$$\frac{\partial^n B_{(z_1,z_2)}(a,b)}{\partial z_1^n} = (-1)^n (1-z_1)^{b-n} z_1^{a-1} \Gamma(b) \, {}_2\tilde{F}_1\left(1-a, 1-n; b-n+1; 1-\frac{1}{z_1}\right) /; n \in \mathbb{N}$$

With respect to z_2

06.20.20.0023.01

$$\frac{\partial^n B(z_1,z_2,a,b)}{\partial z_2^n} = \delta_n \, B(z_1,z_2,a,b) - (1-z_2)^{b-1} z_2^{a-n} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n-1}{k} (1-b)_k (1-a)_{n-k-1} \left(\frac{z_2}{1-z_2}\right)^k /; n \in \mathbb{N}$$

06.20.20.0010.02

$$\frac{\partial^n B_{(z_1,z_2)}(a,b)}{\partial z_2^n} = (-1)^{n-1} (1-z_2)^{b-n} z_2^{a-1} \Gamma(b) \, {}_2\tilde{F}_1\left(1-a, 1-n; b-n+1; 1-\frac{1}{z_2}\right) /; n \in \mathbb{N}$$

With respect to a

06.20.20.0011.02

$$\frac{\partial^n B_{(z_1,z_2)}(a,b)}{\partial a^n} = \Gamma(a) \left(z_2^a \log^n(z_2) \sum_{j=0}^{n} \binom{n}{j} j! \, {}_{j+2}\tilde{F}_{j+1}(a_1, a_2, \ldots, a_{j+1}, 1-b; a_1+1, a_2+1, \ldots, a_{j+1}+1; z_2) \left(-\frac{\Gamma(a)}{\log(z_2)}\right)^j - z_1^a \log^n(z_1) \sum_{j=0}^{n} \binom{n}{j} j! \, {}_{j+2}\tilde{F}_{j+1}(a_1, a_2, \ldots, a_{j+1}, 1-b; a_1+1, a_2+1, \ldots, a_{j+1}+1; z_1) \left(-\frac{\Gamma(a)}{\log(z_1)}\right)^j\right) /; a_1 = a_2 = \ldots = a_{n+1} = a \land n \in \mathbb{N}$$

06.20.20.0012.02

$$\frac{\partial^n B_{(z_1,z_2)}(a,b)}{\partial a^n} = (-1)^n \sum_{k=0}^{\infty} \frac{(1-b)_k}{(a+k)^{n+1} \, k!} \Gamma(n+1, -(a+k)\log(z_2), -(a+k)\log(z_1)) /; n \in \mathbb{N}$$

With respect to b

06.20.20.0013.02

$$\frac{\partial^n \mathrm{B}_{(z_1,z_2)}(a,\,b)}{\partial b^n} == \Gamma(b)\left((1-z_1)^b \log^n(1-z_1) \sum_{j=0}^{n} \binom{n}{j}\, j!\; {}_{j+2}\tilde{F}_{j+1}\big(a_1,\,a_2,\,\ldots,\,a_{j+1},\,1-a;\,a_1+1,\,a_2+1,\,\ldots,\,a_{j+1}+1;\,1-z_1\big)\left(-\frac{\Gamma(b)}{\log(1-z_1)}\right)^j - (1-z_2)^b \log^n(1-z_2) \sum_{j=0}^{n} \binom{n}{j}\, j!\; {}_{j+2}\tilde{F}_{j+1}\big(a_1,\,a_2,\,\ldots,\,a_{j+1},\,1-a;\,a_1+1,\,a_2+1,\,\ldots,\,a_{j+1}+1;\,1-z_2\big)\left(-\frac{\Gamma(b)}{\log(1-z_2)}\right)^j\right) /;\; a_1 == a_2 == \ldots == a_{n+1} == b \bigwedge n \in \mathbb{N}$$

06.20.20.0024.01

$$\frac{\partial^n \mathrm{B}_{(z_1,z_2)}(a,\,b)}{\partial b^n} == (-1)^{n-1} \sum_{k=0}^{\infty} \frac{(1-a)_k}{(b+k)^{n+1}\, k!}\, \Gamma(n+1,\,-(b+k)\log(1-z_2),\,-(b+k)\log(1-z_1)) /;\; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z_1

06.20.20.0014.01

$$\frac{\partial^\alpha \mathrm{B}_{(z_1,z_2)}(a,\,b)}{\partial z_1^\alpha} == \frac{z_1^{-\alpha}}{\Gamma(1-\alpha)}\, \mathrm{B}_{z_2}(a,\,b) - z_1^{a-\alpha}\, \Gamma(a)\, {}_2\tilde{F}_1(a,\,1-b;\,a-\alpha+1;\,z_1) /;\; -a \notin \mathbb{N}^+$$

06.20.20.0015.01

$$\frac{\partial^\alpha \mathrm{B}_{(z_1,z_2)}(a,\,b)}{\partial z_1^\alpha} == \frac{z_1^{-\alpha}}{\Gamma(1-\alpha)}\, \mathrm{B}_{z_2}(a,\,b) - \sum_{k=0}^{\infty} \frac{(1-b)_k\, \mathcal{FC}_{\exp}^{(\alpha)}(z_1,\,a+k)\, z_1^{a+k-\alpha}}{(a+k)\, k!} /;\; |z_1| < 1$$

With respect to z_2

06.20.20.0016.01

$$\frac{\partial^\alpha \mathrm{B}_{(z_1,z_2)}(a,\,b)}{\partial z_2^\alpha} == z_2^{a-\alpha}\, \Gamma(a)\, {}_2\tilde{F}_1(a,\,1-b;\,a-\alpha+1;\,z_2) - \frac{z_2^{-\alpha}}{\Gamma(1-\alpha)}\, \mathrm{B}_{z_1}(a,\,b) /;\; -a \notin \mathbb{N}^+$$

06.20.20.0017.01

$$\frac{\partial^\alpha \mathrm{B}_{(z_1,z_2)}(a,\,b)}{\partial z_2^\alpha} == \sum_{k=0}^{\infty} \frac{(1-b)_k\, \mathcal{FC}_{\exp}^{(\alpha)}(z_2,\,a+k)\, z_2^{a+k-\alpha}}{(a+k)\, k!} - \frac{z_2^{-\alpha}}{\Gamma(1-\alpha)}\, \mathrm{B}_{z_1}(a,\,b) /;\; |z_2| < 1$$

With respect to a

06.20.20.0018.01

$$\frac{\partial^\alpha \mathrm{B}_{(z_1,z_2)}(a,\,b)}{\partial a^\alpha} == a^{-\alpha}\big(\log(z_2)\, {}_2\tilde{F}_2(1,\,1;\,2,\,1-\alpha;\,a\log(z_2)) - \log(z_1)\, {}_2\tilde{F}_2(1,\,1;\,2,\,1-\alpha;\,a\log(z_1))\big) + (1-b)\, a^{-\alpha} \sum_{k=0}^{\infty} \frac{(2-b)_k}{(k+1)^2\, k!}\, {}_2\tilde{F}_1\left(1,\,1;\,1-\alpha;\,-\frac{a}{k+1}\right)\left(z_2^{a+k+1} - z_1^{a+k+1}\right) /;\; |z_1| < 1 \bigwedge |z_2| < 1 \bigwedge -a \notin \mathbb{N}$$

06.20.20.0019.01

$$\frac{\partial^\alpha \mathrm{B}_{(z_1,z_2)}(a,\,b)}{\partial a^\alpha} == a^{-\alpha} \int_{z_1}^{z_2} t^{a-1}\, (1-t)^{b-1}\, (a\log(t))^\alpha\, Q(-\alpha,\,0,\,a\log(t))\, dt$$

With respect to b

06.20.20.0020.01

$$\frac{\partial^\alpha \mathrm{B}_{(z_1,z_2)}(a,b)}{\partial b^\alpha} = \sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{(-1)^k\, k!\, b^{k-\alpha}\, S_j^{(k)}}{(a+j)\,\Gamma(k-\alpha+1)\, j!} \left(z_2^{a+j}\, {}_2F_1(j+1, a+j; a+j+1; z_2) - z_1^{a+j}\, {}_2F_1(j+1, a+j; a+j+1; z_1)\right) /;$$

$|z_1| < 1 \bigwedge |z_2| < 1 \bigwedge -a \notin \mathbb{N}$

06.20.20.0021.01

$$\frac{\partial^\alpha \mathrm{B}_{(z_1,z_2)}(a,b)}{\partial b^\alpha} = b^{-\alpha} \int_{z_1}^{z_2} t^{a-1} (1-t)^{b-1} (b \log(1-t))^\alpha\, Q(-\alpha, 0, b\log(1-t))\, dt$$

Integration

Indefinite integration

Involving only one direct function with respect to z_1

06.20.21.0001.01

$$\int \mathrm{B}(a z_1, z_2, a, b)\, dz_1 = \frac{1}{a} \mathrm{B}_{a z_1}(a+1, b) + \mathrm{B}(a z_1, z_2, a, b)\, z_1$$

06.20.21.0002.01

$$\int \mathrm{B}_{(z_1,z_2)}(a,b)\, dz_1 = \mathrm{B}_{z_1}(a+1,b) + z_1\, \mathrm{B}_{(z_1,z_2)}(a,b)$$

Involving one direct function and elementary functions with respect to z_1

Involving power function

06.20.21.0003.01

$$\int z_1^{\alpha-1}\, \mathrm{B}(a z_1, z_2, a, b)\, dz_1 = \frac{z_1^\alpha}{\alpha} \left(\mathrm{B}_{a z_1}(a+\alpha, b)\, (a z_1)^{-\alpha} + \mathrm{B}(a z_1, z_2, a, b)\right)$$

06.20.21.0004.01

$$\int z_1^{\alpha-1}\, \mathrm{B}_{(z_1,z_2)}(a,b)\, dz_1 = \frac{1}{\alpha} \left(\mathrm{B}_{z_1}(a+\alpha, b) + z_1^\alpha\, \mathrm{B}_{(z_1,z_2)}(a,b)\right)$$

Involving only one direct function with respect to z_2

06.20.21.0005.01

$$\int \mathrm{B}(z_1, a z_2, a, b)\, dz_2 = z_2\, \mathrm{B}(z_1, a z_2, a, b) - \frac{1}{a} \mathrm{B}_{a z_2}(a+1, b)$$

06.20.21.0006.01

$$\int \mathrm{B}_{(z_1,z_2)}(a,b)\, dz_2 = z_2\, \mathrm{B}_{(z_1,z_2)}(a,b) - \mathrm{B}_{z_2}(a+1,b)$$

Involving one direct function and elementary functions with respect to z_1

Involving power function

06.20.21.0007.01

$$\int z_2^{\alpha-1}\, \mathrm{B}(z_1, a z_2, a, b)\, dz_2 = \frac{z_2^\alpha}{\alpha} \left(\mathrm{B}(z_1, a z_2, a, b) - \mathrm{B}_{a z_2}(a+\alpha, b)\, (a z_2)^{-\alpha}\right)$$

06.20.21.0008.01

$$\int z_2^{\alpha-1}\,\mathrm{B}_{(z_1,z_2)}(a,b)\,dz_2 = \frac{1}{\alpha}\left(z_2^{\alpha}\,\mathrm{B}_{(z_1,z_2)}(a,b) - \mathrm{B}_{z_2}(a+\alpha,b)\right)$$

Involving only one direct function with respect to *a*

06.20.21.0009.01

$$\int \mathrm{B}_{(z_1,z_2)}(a,b)\,da = \Gamma(0,-a\log(z_1),-a\log(z_2)) + \log(-a\log(z_1)) - \log(-a\log(z_2)) + (1-b)\sum_{k=0}^{\infty}\frac{(2-b)_k\left(z_2^{a+k+1}-z_1^{a+k+1}\right)}{(k+1)!}\log\left(1+\frac{a}{k+1}\right) /;\ |z_1|<1\bigwedge|z_2|<1\bigwedge -a\notin\mathbb{N}$$

Involving one direct function and elementary functions with respect to *a*

Involving power function

06.20.21.0010.01

$$\int a^{\alpha-1}\,\mathrm{B}_{(z_1,z_2)}(a,b)\,da = \frac{(z_2^a-z_1^a)\,a^{\alpha-1}}{\alpha-1} + \frac{(1-b)\,a^{\alpha}}{\alpha}\sum_{k=0}^{\infty}\frac{(2-b)_k\left(z_2^{a+k+1}-z_1^{a+k+1}\right)}{(k+1)^2\,k!}\,{}_2F_1\left(\alpha,1;\alpha+1;-\frac{a}{k+1}\right) + \frac{a^{\alpha}}{\alpha-1}\left((-a\log(z_1))^{-\alpha}\log(z_1)\,\Gamma(\alpha,0,-a\log(z_1)) - (-a\log(z_2))^{-\alpha}\log(z_2)\,\Gamma(\alpha,0,-a\log(z_2))\right) /;\ |z_1|<1\bigwedge|z_2|<1\bigwedge -a\notin\mathbb{N}$$

Involving only one direct function with respect to *b*

06.20.21.0011.01

$$\int \mathrm{B}_{(z_1,z_2)}(a,b)\,db = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k\,k!\,b^{k+1}\,S_j^{(k)}}{(a+j)\,(k+1)!\,j!}\left(z_2^{a+j}\,{}_2F_1(j+1,a+j;a+j+1;z_2) - z_1^{a+j}\,{}_2F_1(j+1,a+j;a+j+1;z_1)\right) /;\ |z_1|<1\bigwedge|z_2|<1\bigwedge -a\notin\mathbb{N}$$

Involving one direct function and elementary functions with respect to *b*

Involving power function

06.20.21.0012.01

$$\int b^{\alpha-1}\,\mathrm{B}_{(z_1,z_2)}(a,b)\,db = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{(-1)^k\,k!\,b^{k+\alpha}\,S_j^{(k)}}{(a+j)\,(k+\alpha)\,k!\,j!}\left(z_2^{a+j}\,{}_2F_1(j+1,a+j;a+j+1;z_2) - z_1^{a+j}\,{}_2F_1(j+1,a+j;a+j+1;z_1)\right) /;\ |z_1|<1\bigwedge|z_2|<1\bigwedge -a\notin\mathbb{N}$$

Integral transforms

Laplace transforms

06.20.22.0001.01

$$\mathcal{L}_t\left[\mathrm{B}_{(t,z_2)}(a,b)\right](z) = \frac{1}{z}\left(\mathrm{B}_{z_2}(a,b) - (-1)^{-a}\,\Gamma(a)\,U(a,a+b,-z)\right) /;\ \mathrm{Re}(z)>0\land\mathrm{Re}(a)>-1$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_2\tilde{F}_1$

06.20.26.0001.01

$$B_{(z_1,z_2)}(a,\,b) = \Gamma(a)\left(z_2^a\,{}_2\tilde{F}_1(a,\,1-b;\,a+1;\,z_2) - z_1^a\,{}_2\tilde{F}_1(a,\,1-b;\,a+1;\,z_1)\right) /;\; -a \notin \mathbb{N}$$

06.20.26.0002.01

$$B_{(z_1,z_2)}(a,\,b) = \Gamma(b)\left((1-z_1)^b\,z_1^a\,{}_2\tilde{F}_1(1,\,a+b;\,b+1;\,1-z_1) - (1-z_2)^b\,z_2^a\,{}_2\tilde{F}_1(1,\,a+b;\,b+1;\,1-z_2)\right) /;\; -b \notin \mathbb{N}$$

06.20.26.0003.01

$$B_{(z_1,z_2)}(a,\,b) = B(1-a-b,\,a)\left((-z_2)^{-a}\,z_2^a - (-z_1)^{-a}\,z_1^a\right) - \Gamma(1-a-b)$$
$$\left((-z_2)^{b-1}\,z_2^a\,{}_2\tilde{F}_1\!\left(1-b,\,1-a-b;\,2-a-b;\,\frac{1}{z_2}\right) - (-z_1)^{b-1}\,z_1^a\,{}_2\tilde{F}_1\!\left(1-b,\,1-a-b;\,2-a-b;\,\frac{1}{z_1}\right)\right) /;\; a+b \notin \mathbb{N}^+$$

Involving ${}_2F_1$

06.20.26.0004.01

$$B_{(z_1,z_2)}(a,\,b) = \frac{1}{a}\left(z_2^a\,{}_2F_1(a,\,1-b;\,a+1;\,z_2) - z_1^a\,{}_2F_1(a,\,1-b;\,a+1;\,z_1)\right) /;\; -a \notin \mathbb{N}$$

06.20.26.0005.01

$$B_{(z_1,z_2)}(a,\,b) = \frac{1}{b}\left((1-z_1)^b\,z_1^a\,{}_2F_1(1,\,a+b;\,b+1;\,1-z_1) - (1-z_2)^b\,z_2^a\,{}_2F_1(1,\,a+b;\,b+1;\,1-z_2)\right) /;\; -b \notin \mathbb{N}$$

06.20.26.0006.01

$$B_{(z_1,z_2)}(a,\,b) = \frac{1}{a+b-1}\left((-z_2)^{b-1}\,z_2^a\,{}_2F_1\!\left(1-b,\,1-a-b;\,2-a-b;\,\frac{1}{z_2}\right) - (-z_1)^{b-1}\,z_1^a\,{}_2F_1\!\left(1-b,\,1-a-b;\,2-a-b;\,\frac{1}{z_1}\right)\right) +$$
$$B(1-a-b,\,a)\left((-z_2)^{-a}\,z_2^a - (-z_1)^{-a}\,z_1^a\right) /;\; a+b \notin \mathbb{N}^+$$

Through Meijer *G*

Classical cases for the direct function itself

06.20.26.0007.01

$$B_{(z_1,z_2)}(a,\,b) = \frac{1}{\Gamma(1-b)}\left(z_2^a\,G_{2,2}^{1,2}\!\left(-z_2\,\middle|\,\begin{matrix}1-a,\,b\\0,\,-a\end{matrix}\right) - z_1^a\,G_{2,2}^{1,2}\!\left(-z_1\,\middle|\,\begin{matrix}1-a,\,b\\0,\,-a\end{matrix}\right)\right)$$

Classical cases involving algebraic functions in the arguments

06.20.26.0008.01

$$B\!\left(\frac{1}{z+1},\,1,\,a,\,b\right) = \frac{1}{\Gamma(a+b)}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}1,\,1-a\\b,\,0\end{matrix}\right) /;\; z \notin (-\infty,\,-1)$$

06.20.26.0009.01

$$(z+1)^{a+b-1}\,B\!\left(\frac{1}{z+1},\,1,\,a,\,b\right) = \frac{\Gamma(b)}{\Gamma(1-a)}\,G_{2,2}^{1,2}\!\left(z\,\middle|\,\begin{matrix}b,\,a+b\\b,\,0\end{matrix}\right) /;\; z \notin (-\infty,\,-1)$$

Representations through equivalent functions

With inverse function

06.20.27.0001.01

$$B_{\left(z_1,\,I^{-1}_{(z_1,z_2)}(a,b)\right)}(a,\,b) = B(a,\,b)\,z_2$$

With related functions

06.20.27.0002.01

$$\mathrm{B}_{(z_1,z_2)}(a,\, b) == \mathrm{B}_{z_2}(a,\, b) - \mathrm{B}_{z_1}(a,\, b)$$

06.20.27.0003.01

$$\mathrm{B}_{(z_1,z_2)}(a,\, b) == \mathrm{B}(a,\, b)\, I_{(z_1,z_2)}(a,\, b)$$

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