

CatalanNumber

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Notations

Traditional name

Catalan number

Traditional notation

C_z

Mathematica StandardForm notation

CatalanNumber[z]

Primary definition

06.41.02.0001.01

$$C_z = \frac{2^{2z} \Gamma\left(z + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma(z+2)}$$

Specific values

Specialized values

06.41.03.0001.01

$$C_{-n} = 0 \text{ ; } n \in \mathbb{Z} \wedge n > 1$$

06.41.03.0002.01

$$C_n = \frac{2^n}{(n+1)!} \prod_{k=1}^n (2k-1) \text{ ; } n \in \mathbb{N}$$

06.41.03.0003.01

$$C_n = \frac{(-1)^n (-2n)_n}{(n+1)!} \text{ ; } n \in \mathbb{N}$$

06.41.03.0004.01

$$C_{n+\frac{1}{2}} = \frac{2^{3n+3} n!}{\pi \prod_{k=1}^{n+2} (2k-1)} \text{ ; } n \in \mathbb{N}$$

06.41.03.0005.01

$$C_{-n-\frac{1}{2}} = \infty \text{ ; } n \in \mathbb{N}$$

Values at fixed points

06.41.03.0006.01
 $C_{-10} = 0$

06.41.03.0007.01
 $C_{-\frac{19}{2}} = \tilde{\infty}$

06.41.03.0008.01
 $C_{-9} = 0$

06.41.03.0009.01
 $C_{-\frac{17}{2}} = \tilde{\infty}$

06.41.03.0010.01
 $C_{-8} = 0$

06.41.03.0011.01
 $C_{-\frac{15}{2}} = \tilde{\infty}$

06.41.03.0012.01
 $C_{-7} = 0$

06.41.03.0013.01
 $C_{-\frac{13}{2}} = \tilde{\infty}$

06.41.03.0014.01
 $C_{-6} = 0$

06.41.03.0015.01
 $C_{-\frac{11}{2}} = \tilde{\infty}$

06.41.03.0016.01
 $C_{-5} = 0$

06.41.03.0017.01
 $C_{-\frac{9}{2}} = \tilde{\infty}$

06.41.03.0018.01
 $C_{-4} = 0$

06.41.03.0019.01
 $C_{-\frac{7}{2}} = \tilde{\infty}$

06.41.03.0020.01
 $C_{-3} = 0$

06.41.03.0021.01
 $C_{-\frac{5}{2}} = \tilde{\infty}$

06.41.03.0022.01
 $C_{-2} = 0$

06.41.03.0023.01
 $C_{-\frac{3}{2}} = \tilde{\infty}$

06.41.03.0024.01

$$C_{-1} = -\frac{1}{2}$$

06.41.03.0025.01

$$C_{-\frac{1}{2}} = \infty$$

06.41.03.0026.01

$$C_0 = 1$$

06.41.03.0027.01

$$C_{\frac{1}{2}} = \frac{8}{3\pi}$$

06.41.03.0028.01

$$C_1 = 1$$

06.41.03.0029.01

$$C_{\frac{3}{2}} = \frac{64}{15\pi}$$

06.41.03.0030.01

$$C_2 = 2$$

06.41.03.0031.01

$$C_{\frac{5}{2}} = \frac{1024}{105\pi}$$

06.41.03.0032.01

$$C_3 = 5$$

06.41.03.0033.01

$$C_{\frac{7}{2}} = \frac{8192}{315\pi}$$

06.41.03.0034.01

$$C_4 = 14$$

06.41.03.0035.01

$$C_{\frac{9}{2}} = \frac{262\,144}{3465\pi}$$

06.41.03.0036.01

$$C_5 = 42$$

06.41.03.0037.01

$$C_{\frac{11}{2}} = \frac{2\,097\,152}{9\,009\pi}$$

06.41.03.0038.01

$$C_6 = 132$$

06.41.03.0039.01

$$C_{\frac{13}{2}} = \frac{33\,554\,432}{45\,045\pi}$$

06.41.03.0040.01

$$C_7 = 429$$

$$C_{\frac{15}{2}} = \frac{268\,435\,456}{109\,395\pi}$$

$$C_8 = 1430$$

$$C_{\frac{17}{2}} = \frac{17\,179\,869\,184}{2\,078\,505\pi}$$

$$C_9 = 4862$$

$$C_{\frac{19}{2}} = \frac{137\,438\,953\,472}{4\,849\,845\pi}$$

$$C_{10} = 16\,796$$

Values at infinities

$$C_{\infty} = \infty$$

$$C_{-\infty} = i$$

$$C_{i\infty} = 0$$

$$C_{-i\infty} = 0$$

$$C_{\tilde{\infty}} = i$$

General characteristics

Domain and analyticity

C_z is an analytical function of z which is defined over the whole complex z -plane with the exception of countably many points $z = -k - 1/2$; $k \in \mathbb{N}$.

$$z \rightarrow C_z :: \mathbb{C} \rightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

$$C_{\bar{z}} = \overline{C_z}$$

Periodicity

No periodicity

Poles and essential singularities

The function C_z has an infinite set of singular points:

a) $z = -k - 1/2$; $k \in \mathbb{N}$ are the simple poles with residues $\frac{(-1)^k 2^{-2k-1}}{\sqrt{\pi} k! \Gamma(3/2-k)}$;

b) $z = \infty$ is the point of convergence of poles, which is an essential singular point.

06.41.04.0003.01

$$\text{Sing}_z(C_z) = \left\{ \left\{ -k - \frac{1}{2}, 1 \right\}; k \in \mathbb{N} \right\}, \{\infty, \infty\}$$

06.41.04.0004.01

$$\text{res}_z(C_z) \left(-k - \frac{1}{2} \right) = \frac{(-1)^k 2^{-2k-1}}{\sqrt{\pi} k! \Gamma\left(\frac{3}{2} - k\right)}; k \in \mathbb{N}$$

Branch points

The function C_z does not have branch points.

06.41.04.0005.01

$$\mathcal{BP}_z(C_z) = \{\}$$

Branch cuts

The function C_z does not have branch cuts.

06.41.04.0006.01

$$\mathcal{BC}_z(C_z) = \{\}$$

Series representations

Generalized power series

Expansions at $z = 0$

06.41.06.0001.01

$$C_z \propto 1 - z + \left(1 + \frac{\pi^2}{6}\right) z^2 - \left(2\zeta(3) + \frac{\pi^2}{6} + 1\right) z^3 + \dots; (z \rightarrow 0)$$

06.41.06.0002.01

$$C_z \propto 1 - z + \left(1 + \frac{\pi^2}{6}\right) z^2 - \left(2\zeta(3) + \frac{\pi^2}{6} + 1\right) z^3 + O(z^4)$$

06.41.06.0003.01

$$C_z = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \sum_{j=0}^k (j+1) d_{k-j} \sum_{r=0}^j \frac{(-1)^r}{r+1} \binom{j}{r} p_{r,j} z^k ; a_k = \frac{\Gamma^{(k)}\left(\frac{1}{2}\right)}{k!} \wedge b_k = \frac{\log^k(4)}{k!} \wedge$$

$$c_k = \frac{\Gamma^{(k)}(2)}{k!} \wedge d_k = \sum_{n=0}^k a_n b_{k-n} \wedge p_{j,0} = 1 \wedge p_{j,k} = \frac{1}{k} \sum_{m=1}^k (-k+jm+m) c_m p_{j,k-m} \wedge k \in \mathbb{N} \wedge |z| < \frac{1}{2}$$

06.41.06.0004.01

$$C_z \propto 1 + O(z)$$

Expansions at $z = z_0$; $z_0 \neq -n - 1/2$

06.41.06.0005.01

$$C_z \propto C_{z_0} \left(1 + \left(\log(4) - \psi(z_0 + 2) + \psi\left(z_0 + \frac{1}{2}\right) \right) (z - z_0) + \right.$$

$$\left. \frac{1}{2} \left(\left(\log(4) - \psi(z_0 + 2) + \psi\left(z_0 + \frac{1}{2}\right) \right)^2 - \psi^{(1)}(z_0 + 2) + \psi^{(1)}\left(z_0 + \frac{1}{2}\right) \right) (z - z_0)^2 + \dots \right) ; (z \rightarrow z_0) \wedge -z_0 - \frac{1}{2} \notin \mathbb{N}$$

06.41.06.0006.01

$$C_z \propto C_{z_0} \left(1 + \left(\log(4) - \psi(z_0 + 2) + \psi\left(z_0 + \frac{1}{2}\right) \right) (z - z_0) + \right.$$

$$\left. \frac{1}{2} \left(\left(\log(4) - \psi(z_0 + 2) + \psi\left(z_0 + \frac{1}{2}\right) \right)^2 - \psi^{(1)}(z_0 + 2) + \psi^{(1)}\left(z_0 + \frac{1}{2}\right) \right) (z - z_0)^2 + O((z - z_0)^3) \right) ; -z_0 - \frac{1}{2} \notin \mathbb{N}$$

06.41.06.0007.01

$$C_z = \frac{2^{2z_0}}{\sqrt{\pi} \Gamma(z_0 + 2)} \sum_{k=0}^{\infty} \sum_{j=0}^k (j+1) d_{k-j} \sum_{r=0}^j \frac{(-1)^r}{r+1} \binom{j}{r} p_{r,j} (z - z_0)^k ; a_k = \frac{\Gamma^{(k)}\left(z_0 + \frac{1}{2}\right)}{k!} \wedge b_k = \frac{\log^k(4)}{k!} \wedge c_k = \frac{\Gamma^{(k)}(z_0 + 2)}{k!} \wedge$$

$$d_k = \sum_{n=0}^k a_n b_{k-n} \wedge p_{j,0} = 1 \wedge p_{j,k} = \frac{1}{\Gamma(z_0 + 2) k} \sum_{m=1}^k (j m + m - k) c_m p_{j,k-m} \wedge k \in \mathbb{N} \wedge -z_0 - \frac{1}{2} \notin \mathbb{N}$$

06.41.06.0008.01

$$C_z \propto C_{z_0} (1 + O(z - z_0)) ; -z_0 - \frac{1}{2} \notin \mathbb{N}$$

Expansions at $z = -n - 1/2$

06.41.06.0009.01

$$C_z \propto \frac{(-1)^n 2^{-2n-1} (\log(4) - \psi(\frac{3}{2} - n) + \psi(n+1))}{\sqrt{\pi} \Gamma(\frac{3}{2} - n) n!} + \frac{(-1)^n 2^{-2n-1}}{\sqrt{\pi} \Gamma(\frac{3}{2} - n) n! (n+z+\frac{1}{2})} +$$

$$\frac{(-1)^n 4^{-n-1}}{3 \sqrt{\pi} \Gamma(\frac{3}{2} - n) n!} \left(3 \log^2(4) + \pi^2 + 3 \psi\left(\frac{3}{2} - n\right)^2 - 6 \psi\left(\frac{3}{2} - n\right) (\log(4) + \psi(n+1)) + \right.$$

$$\left. 3 \psi(n+1) (\log(16) + \psi(n+1)) - 3 \psi^{(1)}\left(\frac{3}{2} - n\right) - 3 \psi^{(1)}(n+1) \right) \left(n+z+\frac{1}{2} \right) +$$

$$\frac{(-1)^n 4^{-n-1}}{3 \sqrt{\pi} \Gamma(\frac{3}{2} - n) n!} \left(\log^3(4) + \pi^2 \log(4) - 3 \left(\psi^{(1)}\left(\frac{3}{2} - n\right) + \psi^{(1)}(n+1) \right) \log(4) - \psi\left(\frac{3}{2} - n\right)^3 + \psi(n+1)^3 + \right.$$

$$\left. \log(64) \psi(n+1)^2 + 3 \psi\left(\frac{3}{2} - n\right)^2 (\log(4) + \psi(n+1)) + \psi(n+1) \left(3 \log^2(4) + \pi^2 - 3 \psi^{(1)}\left(\frac{3}{2} - n\right) - 3 \psi^{(1)}(n+1) \right) - \right.$$

$$\left. \psi\left(\frac{3}{2} - n\right) \left(3 \log^2(4) + \pi^2 + 3 \psi(n+1) (\log(16) + \psi(n+1)) - 3 \psi^{(1)}\left(\frac{3}{2} - n\right) - 3 \psi^{(1)}(n+1) \right) - \right.$$

$$\left. \psi^{(2)}\left(\frac{3}{2} - n\right) + \psi^{(2)}(n+1) \right) \left(n+z+\frac{1}{2} \right)^2 + \dots /; \left(z \rightarrow -n - \frac{1}{2} \right) \wedge n \in \mathbb{N}$$

06.41.06.0010.01

$$C_z = \frac{(-1)^n 2^{-2n-1}}{e_0 \sqrt{\pi} \left(n+z+\frac{1}{2} \right)} \sum_{k=0}^{\infty} (k+1) \sum_{r=0}^k \frac{(-1)^r \binom{k}{r}}{r+1} p_{r,k} \left(n+z+\frac{1}{2} \right)^k /;$$

$$a_k = \frac{\pi^k i^k (1+(-1)^k)}{2(k+1)!} \wedge b_k = \frac{(-1)^k \log^k(4)}{k!} \wedge c_k = \frac{\Gamma^{(k)}(\frac{3}{2} - n)}{k!} \wedge d_k = \frac{(-1)^k \Gamma^{(k)}(n+1)}{k!} \wedge$$

$$e_k = \sum_{u=0}^k \sum_{v=0}^u \sum_{i=0}^v a_{u-v} b_{v-i} c_i d_{k-u} \wedge p_{j,0} = 1 \wedge p_{j,k} = \frac{1}{e_0 k} \sum_{m=1}^k (-k+jm+m) e_m p_{j,k-m} \wedge k \in \mathbb{N} \wedge n \in \left(1 + O\left(n+z+\frac{1}{2} \right) \right)$$

06.41.06.0011.01

$$C_z \propto \frac{(-1)^n 2^{-2n-1}}{\sqrt{\pi} \Gamma(\frac{3}{2} - n) n! \left(n+z+\frac{1}{2} \right)} \left(1 + O\left(n+z+\frac{1}{2} \right) \right) /; n \in \mathbb{N}$$

Asymptotic series expansions

06.41.06.0012.01

$$C_x \propto \frac{4^x e^{-\frac{9}{8x}}}{\sqrt{\pi} x^{3/2}} /; (x \rightarrow \infty)$$

06.41.06.0013.01

$$C_z \propto \frac{2^{2z}}{\sqrt{\pi} z^{3/2}} \left(1 - \frac{9}{8z} + \frac{145}{128z^2} - \frac{1155}{1024z^3} + \frac{36939}{32768z^4} - \frac{295911}{262144z^5} + \frac{4735445}{4194304z^6} - \right.$$

$$\left. \frac{37844235}{33554432z^7} + \frac{2421696563}{2147483648z^8} - \frac{19402289907}{17179869184z^9} + O\left(\frac{1}{z^{10}}\right) \right) /; \left| \arg\left(z + \frac{1}{2}\right) \right| < \pi \wedge (|z| \rightarrow \infty)$$

06.41.06.0014.01

$$C_z \propto \frac{2^{2z}}{\sqrt{\pi} z^{3/2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{3}{2}\right)_k B_k^{(-\frac{1}{2})} \left(\frac{1}{2}\right) z^{-k} /; \left| \arg\left(z + \frac{1}{2}\right) \right| < \pi \wedge (|z| \rightarrow \infty)$$

06.41.06.0015.01

$$C_z \propto \frac{2^{2z} \tan(\pi z)}{\sqrt{\pi} (-z)^{3/2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{3}{2}\right)_k B_k^{(-\frac{1}{2})} \left(\frac{1}{2}\right) z^{-k} /; \arg(z) = \pi \wedge z + \frac{1}{2} \notin \mathbb{Z} \wedge (|z| \rightarrow \infty)$$

06.41.06.0016.01

$$C_z \propto \frac{1}{\sqrt{\pi} z^{3/2}} \left(\frac{z^{3/2} \tan(\pi z)}{(-z)^{3/2}} \right)^{\lfloor \frac{\arg(z)+\pi}{2\pi} \rfloor} 2^{2z} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{3}{2}\right)_k B_k^{(-\frac{1}{2})} \left(\frac{1}{2}\right) z^{-k} /; \frac{1}{2} - z \notin \mathbb{N}^+ \wedge (|z| \rightarrow \infty)$$

06.41.06.0017.01

$$C_z \propto \frac{2^{2z}}{\sqrt{\pi} z^{3/2}} \left(1 + O\left(\frac{1}{z}\right) \right) /; \left| \arg\left(z + \frac{1}{2}\right) \right| < \pi \wedge (|z| \rightarrow \infty)$$

Product representations

06.41.08.0001.01

$$C_z = \frac{2}{(z+1)!} \prod_{k=1}^{z-1} (4k+2)$$

06.41.08.0002.01

$$C_z = \frac{2^{2z+1} e^{\gamma(z+2)} (z+2)}{\sqrt{\pi} (2z+1)} \prod_{k=1}^{\infty} \frac{\left(1 + \frac{1}{k}\right)^{z+\frac{1}{2}} \left(\frac{z+2}{k} + 1\right) e^{-\frac{z+2}{k}}}{\frac{2z+1}{2k} + 1} /; -z - \frac{1}{2} \in \mathbb{N}$$

Generating functions

06.41.11.0001.01

$$C_n = \left([t^n] \frac{1 - \sqrt{1-4t}}{2t} \right) /; n \in \mathbb{N}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.41.16.0001.01

$$C_{-z} = \frac{\tan(\pi z)}{\pi z (1-z^2)} C_z$$

06.41.16.0002.01

$$C_{z+1} = \frac{2(2z+1)}{z+2} C_z$$

06.41.16.0003.01

$$C_{n+z} = \frac{2^{2n} \left(z + \frac{1}{2}\right)_n}{(z+2)_n} C_z /; n \in \mathbb{Z}$$

06.41.16.0004.01

$$C_{z-1} = \frac{z+1}{2(2z-1)} C_z$$

06.41.16.0005.01

$$C_{z-n} = \frac{2^{-2n} (-z-1)_n}{\left(\frac{1}{2}-z\right)_n} C_z /; n \in \mathbb{Z}$$

Products, sums, and powers of the direct function

Products of the direct function

06.41.16.0006.01

$$C_z C_{-z} = \frac{\tan(\pi z)}{\pi z (1-z^2)}$$

06.41.16.0007.01

$$C_z C_{1-z} = \frac{2(1-2z)\tan(\pi z)}{\pi(z-2)(z-1)z(z+1)}$$

06.41.16.0008.01

$$C_z C_{-n-z} = -\frac{4^{-n} (z+1)_{n-2} \tan(\pi z)}{\pi(z+1)\left(z + \frac{1}{2}\right)_n} /; n \in \mathbb{Z}$$

06.41.16.0009.01

$$\frac{C_{n+z}}{C_z} = 2^n \prod_{k=0}^{n-1} \frac{2k+2z+1}{k+z+2} /; n \in \mathbb{Z}$$

06.41.16.0010.01

$$\frac{C_{z-n}}{C_z} = 2^{-n} \prod_{k=0}^{n-1} \frac{k-z-1}{2k-2z+1} /; n \in \mathbb{N}^+$$

Identities

Recurrence identities

Consecutive neighbors

06.41.17.0001.01

$$C_z = \frac{2(2z-1)}{z+1} C_{z-1}$$

06.41.17.0002.01

$$C_z = \frac{z+2}{2(2z+1)} C_{z+1}$$

Distant neighbors

$$C_z = \frac{(z+2)_n}{2^{2n} \left(z + \frac{1}{2}\right)_n} C_{n+z} /; n \in \mathbb{Z}$$

$$C_z = \frac{2^{2n} \left(\frac{1}{2} - z\right)_n}{(-z-1)_n} C_{z-n} /; n \in \mathbb{Z}$$

Differentiation

Low-order differentiation

$$\frac{\partial C_z}{\partial z} = C_z \left(\log(4) - \psi(z+2) + \psi\left(z + \frac{1}{2}\right) \right)$$

$$\frac{\partial^2 C_z}{\partial z^2} = C_z \left(\left(\log(4) - \psi(z+2) + \psi\left(z + \frac{1}{2}\right) \right)^2 - \psi^{(1)}(z+2) + \psi^{(1)}\left(z + \frac{1}{2}\right) \right)$$

Summation

Infinite summation

$$\sum_{n=0}^{\infty} C_n z^n = \frac{1 - \sqrt{1-4z}}{2z} /; |z| < \frac{1}{4}$$

Representations through equivalent functions

With related functions

$$C_z = \frac{1}{z+1} \binom{2z}{z}$$

$$C_z = \frac{(2z)!}{(z+1)! z!}$$

$$C_z = \frac{1}{(z+1)!} 2^{z-\frac{1}{4}} \cos(2\pi z) + \frac{1}{4} \pi^{-\frac{1}{2}} \sin^2(\pi z) (2z-1)!!$$

$$C_z = \frac{\Gamma(2z+1)}{\Gamma(z+1)\Gamma(z+2)}$$

$$C_z = \frac{(z+1)_z}{\Gamma(z+2)}$$

$$C_z = \frac{1}{z(z+1)B(z, z+1)}$$

Zeros

$$C_z = 0 /; z \in \mathbb{Z} \wedge z \leq -2$$

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