Ceiling

Notations

Traditional name

Ceiling function

Traditional notation

$\lceil z \rceil$

Mathematica StandardForm notation

`Ceiling[z]`

Primary definition

04.02.02.0001.01

$\lceil x \rceil = n \;/;\; x \in \mathbb{R} \land n \in \mathbb{Z} \land n - 1 < x \leq n$

04.02.02.0002.01

$\lceil z \rceil = \lceil \mathrm{Re}(z) \rceil + i \lceil \mathrm{Im}(z) \rceil$

For real z, the function $\lceil z \rceil$ is the smallest integer greater than or equal to z.

Examples: $\lceil 3.2 \rceil = 4$, $\lceil 3 \rceil = 3$, $\lceil -0.2 \rceil = 0$, $\lceil -2.3 \rceil = -2$, $\left\lceil \frac{2}{3} \right\rceil = 1$, $\lceil -\pi \rceil = -3$, $\left\lceil -4 - \frac{5}{3} i \right\rceil = -4 - i$, $\left\lceil \frac{5}{2} \right\rceil = 3$, $\left\lceil \frac{7}{2} \right\rceil = 4$.

Specific values

Specialized values

04.02.03.0001.01

$\lceil x \rceil = x \;/;\; x \in \mathbb{Z}$

04.02.03.0002.01

$\lceil i\, x \rceil = i\, x \;/;\; x \in \mathbb{Z}$

04.02.03.0003.01

$\lceil x + i\, y \rceil = \lceil x \rceil + i \lceil y \rceil \;/;\; x \in \mathbb{R} \bigwedge y \in \mathbb{R}$

Values at fixed points

04.02.03.0004.01

$\lceil 0 \rceil = 0$

04.02.03.0005.01

$\lceil 1 \rceil == 1$

04.02.03.0006.01

$\lceil -1 \rceil == -1$

04.02.03.0007.01

$\lceil i \rceil == i$

04.02.03.0008.01

$\lceil -i \rceil == -i$

04.02.03.0009.01

$\left\lceil \frac{23}{10} \right\rceil == 3$

04.02.03.0010.01

$\lceil -3 \rceil == -3$

04.02.03.0011.01

$\lceil -\pi \rceil == -3$

04.02.03.0012.01

$\left\lceil -\frac{27}{10} \right\rceil == -2$

04.02.03.0013.01

$\lceil -3.4 \rceil == -3$

04.02.03.0014.01

$\left\lceil \frac{23}{10} - i\, e \right\rceil == 3 - 2\, i$

Values at infinities

04.02.03.0015.01

$\lceil \infty \rceil == \infty$

04.02.03.0016.01

$\lceil -\infty \rceil == -\infty$

04.02.03.0017.01

$\lceil i\, \infty \rceil == i\, \infty$

04.02.03.0018.01

$\lceil -i\, \infty \rceil == -i\, \infty$

04.02.03.0019.01

$\lceil \tilde{\infty} \rceil == \tilde{\infty}$

General characteristics

Domain and analyticity

$\lceil z \rceil$ is a nonanalytical function; it is a piecewise constant function which is defined over the whole complex z-plane.

04.02.04.0001.01

$z \longrightarrow \lceil z \rceil :: \mathbb{C} \longrightarrow \mathbb{C}$

Symmetries and periodicities

Mirror symmetry

04.02.04.0002.01

$$\lceil \bar{z} \rceil = \overline{\lceil z \rceil} + i\,(1 - \chi_{\mathbb{Z}}(\operatorname{Im}(z)))$$

Periodicity

No periodicity

Sets of discontinuity

The function $\lceil z \rceil$ is a piecewise constant function with unit jumps on the lines $\operatorname{Re}(z) = k \bigvee \operatorname{Im}(z) = l$ /; $k, l \in \mathbb{Z}$.
The function $\lceil z \rceil$ is continuous from the left on the intervals $(k - i\,\infty, k + i\,\infty)$, $k \in \mathbb{Z}$, and from below on the intervals $(i\,k - \infty, i\,k + \infty)$, $k \in \mathbb{Z}$.

04.02.04.0003.01

$$\mathcal{DS}_z(\lfloor z \rfloor) = \{\{\{(k - i\,\infty, k + i\,\infty), 1\} /; k \in \mathbb{Z}\}, \{\{(i\,k - \infty, i\,k + \infty), i\} /; k \in \mathbb{Z}\}\}$$

04.02.04.0004.01

$$\lim_{\epsilon \to +0} \lceil z - \epsilon \rceil = \lceil z \rceil \;/;\; \operatorname{Re}(z) \in \mathbb{Z}$$

04.02.04.0005.01

$$\lim_{\epsilon \to +0} \lceil z + \epsilon \rceil = \lceil z \rceil + 1 \;/;\; \operatorname{Re}(z) \in \mathbb{Z}$$

04.02.04.0006.01

$$\lim_{\epsilon \to +0} \lceil z - i\,\epsilon \rceil = \lceil z \rceil \;/;\; \operatorname{Im}(z) \in \mathbb{Z}$$

04.02.04.0007.01

$$\lim_{\epsilon \to +0} \lceil z + i\,\epsilon \rceil = \lceil z \rceil + i \;/;\; \operatorname{Im}(z) \in \mathbb{Z}$$

Series representations

Exponential Fourier series

04.02.06.0001.01

$$\lceil x \rceil = x + \frac{1}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\,\pi\,k\,x)}{k} \;/;\; x \in \mathbb{R} \bigwedge x \notin \mathbb{Z}$$

Other series representations

04.02.06.0002.01

$$\left\lceil \frac{m}{n} \right\rceil = \frac{m}{n} + \frac{1}{2\,n} \sum_{k=1}^{n-1} \sin\left(\frac{2\,\pi\,k\,m}{n}\right) \cot\left(\frac{\pi\,k}{n}\right) + \frac{1}{2} \;/;\; m \in \mathbb{Z} \bigwedge n \in \mathbb{Z} \bigwedge \frac{m}{n} \notin \mathbb{Z} \bigwedge n > 1$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

04.02.16.0001.01

$\lceil -z \rceil = -\lceil z \rceil$ /; $\operatorname{Re}(z) \in \mathbb{Z} \land \operatorname{Im}(z) \in \mathbb{Z}$

04.02.16.0002.01

$\lceil -z \rceil = -\lceil z \rceil + i \operatorname{sgn}(|\operatorname{Im}(z)|) + \operatorname{sgn}(|\operatorname{Re}(z)|)$ /; $\operatorname{Re}(z) \notin \mathbb{Z} \bigwedge \operatorname{Im}(z) \notin \mathbb{Z}$

04.02.16.0003.01

$\lceil -z \rceil = -\lceil z \rceil + i (1 - \chi_{\mathbb{Z}}(\operatorname{Im}(z))) \operatorname{sgn}(|\operatorname{Im}(z)|) + (1 - \chi_{\mathbb{Z}}(\operatorname{Re}(z))) \operatorname{sgn}(|\operatorname{Re}(z)|)$

04.02.16.0004.01

$\lceil i z \rceil = i \lceil z \rceil - \chi_{\mathbb{Z}}(\operatorname{Im}(z)) + 1$

04.02.16.0005.01

$\lceil i z \rceil = -\lfloor \operatorname{Im}(z) \rfloor + i \lceil \operatorname{Re}(z) \rceil$

04.02.16.0006.01

$\lceil -i z \rceil = -i \lceil z \rceil + i (1 - \chi_{\mathbb{Z}}(\operatorname{Re}(z)))$

04.02.16.0007.01

$\lceil -i z \rceil = \lceil \operatorname{Im}(z) \rceil - i \lfloor \operatorname{Re}(z) \rfloor$

04.02.16.0008.01

$\lceil z + n \rceil = \lceil z \rceil + n$ /; $n \in \mathbb{Z}$

04.02.16.0020.01

$\left\lceil \frac{\lceil n x \rceil}{n} \right\rceil = \lceil x \rceil$ /; $x \in \mathbb{R} \land n \in \mathbb{Z}$

Argument involving related functions

04.02.16.0009.01

$\lceil \lceil z \rceil \rceil = \lceil z \rceil$

04.02.16.0010.01

$\lceil z - \lceil z \rceil \rceil = 0$

04.02.16.0011.01

$\lceil \lfloor z \rceil \rceil = \lfloor z \rceil$

04.02.16.0012.01

$\lceil \lfloor z \rfloor \rceil = \lfloor z \rfloor$

04.02.16.0013.01

$\lceil \operatorname{int}(z) \rceil = \operatorname{int}(z)$

04.02.16.0021.01

$\lceil \operatorname{frac}(z) \rceil = i (1 - \chi_{\mathbb{Z}}(\operatorname{Im}(-z))) \theta(\operatorname{Im}(z)) + (1 - \chi_{\mathbb{Z}}(\operatorname{Re}(-z))) \theta(\operatorname{Re}(z))$

04.02.16.0022.01

$\lceil \operatorname{quotient}(m, n) \rceil = \left\lfloor \frac{m}{n} \right\rfloor$

04.02.16.0014.01

$\operatorname{Nest}[f, x, n] = \left\lceil \left(x - \frac{a b}{b-1}\right) b^{-n} + \frac{a b}{b-1} \right\rceil$ /; $f(x) = a + \left\lceil \frac{x}{b} \right\rceil \bigwedge a \in \mathbb{Z} \bigwedge b \in \mathbb{N}^+$

Addition formulas

04.02.16.0015.01

$\lceil z+n \rceil == \lceil z \rceil + n \;/;\; n \in \mathbb{Z}$

04.02.16.0016.01

$\lceil z_1 + z_2 \rceil == \lceil z_1 \rceil + \lceil z_2 \rceil + \lceil z_1 + z_2 - \lceil z_1 \rceil - \lceil z_2 \rceil \rceil$

Multiple arguments

04.02.16.0017.01

$$\lceil n\, z \rceil == n \lceil z \rceil - \sum_{k=0}^{n-1} k\, \theta\left(-\frac{k}{n} - z \bmod 1 + 1\right)\left(1 - \theta\left(-\frac{k+1}{n} - z \bmod 1 + 1\right)\right) /;\; n \in \mathbb{N} \land z \in \mathbb{R}$$

Products, sums, and powers of the direct function

Sums of the direct function

04.02.16.0018.01

$\lceil z_1 \rceil + \lceil z_2 \rceil == \lceil z_1 + z_2 \rceil - \lceil z_1 + z_2 - \lceil z_1 \rceil - \lceil z_2 \rceil \rceil$

04.02.16.0019.01

$$\sum_{k=0}^{n-1} \left\lceil \frac{x - k\, m}{n} \right\rceil == \sum_{k=0}^{m-1} \left\lceil \frac{x - k\, n}{m} \right\rceil /;\; x \in \mathbb{R} \bigwedge n \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+$$

Complex characteristics

Real part

04.02.19.0001.01

$\mathrm{Re}(\lceil x + i\, y \rceil) == \lceil x \rceil$

04.02.19.0006.01

$\mathrm{Re}(\lceil z \rceil) == \lceil \mathrm{Re}(z) \rceil$

Imaginary part

04.02.19.0002.01

$\mathrm{Im}(\lceil x + i\, y \rceil) == \lceil y \rceil$

04.02.19.0007.01

$\mathrm{Im}(\lceil z \rceil) == \lceil \mathrm{Im}(z) \rceil$

Absolute value

04.02.19.0003.01

$|\lceil x + i\, y \rceil| == \sqrt{\lceil x \rceil^2 + \lceil y \rceil^2}$

04.02.19.0008.01

$|\lceil z \rceil| == \sqrt{\lceil \mathrm{Im}(z) \rceil^2 + \lceil \mathrm{Re}(z) \rceil^2}$

Argument

04.02.19.0004.01

$\arg(\lceil x + i\, y \rceil) == \tan^{-1}(\lceil x \rceil, \lceil y \rceil)$

04.02.19.0009.01

$$\arg(\lceil z \rceil) = \tan^{-1}(\lceil \operatorname{Re}(z) \rceil, \lceil \operatorname{Im}(z) \rceil)$$

Conjugate value

04.02.19.0005.01

$$\overline{\lceil x + i\, y \rceil} = \lceil x \rceil - i \lceil y \rceil$$

04.02.19.0010.01

$$\overline{\lceil z \rceil} = \lceil \operatorname{Re}(z) \rceil + i \lceil \operatorname{Im}(z) \rceil$$

Signum value

04.02.19.0011.01

$$\operatorname{sgn}(\lceil x + i\, y \rceil) = \frac{\lceil x + i\, y \rceil}{|\lceil x + i\, y \rceil|}$$

04.02.19.0012.01

$$\operatorname{sgn}(\lceil z \rceil) = \frac{\lceil z \rceil}{|\lceil z \rceil|}$$

Differentiation

Low-order differentiation

04.02.20.0001.01

$$\frac{\partial \lceil z \rceil}{\partial z} = 0$$

04.02.20.0002.01

$$\frac{\partial \lceil x \rceil}{\partial x} = \sum_{k=-\infty}^{\infty} \delta(x - k)$$

In a distributional sense, for $x \in \mathbb{R}$.

Fractional integro-differentiation

04.02.20.0003.01

$$\frac{\partial^{\alpha} \lceil z \rceil}{\partial z^{\alpha}} = \frac{\lceil z \rceil\, z^{-\alpha}}{\Gamma(1 - \alpha)}$$

Integration

Indefinite integration

Involving only one direct function

04.02.21.0001.01

$$\int \lceil z \rceil\, dz = z \lceil z \rceil$$

Involving one direct function and elementary functions

Involving power function

04.02.21.0002.01

$$\int z^{\alpha-1} \lceil z \rceil \, dz = \frac{z^{\alpha} \lceil z \rceil}{\alpha}$$

04.02.21.0003.01

$$\int \frac{\lceil z \rceil}{z} \, dz = \log(z) \lceil z \rceil$$

Definite integration

For the direct function itself

In the following formulas $a \in \mathbb{R}$.

04.02.21.0004.01

$$\int_0^n \lceil t \rceil \, dt = \frac{n\,(n+1)}{2} \;/;\; n \in \mathbb{N}$$

04.02.21.0005.01

$$\int_0^a \lceil t \rceil \, dt = \frac{1}{2} (2\,a - \lceil a \rceil + 1) \lceil a \rceil$$

04.02.21.0006.01

$$\int_0^a t^{\alpha-1} \lceil t \rceil \, dt = \frac{\lceil a \rceil\, a^{\alpha} - \zeta(-\alpha) + \zeta(-\alpha, \lceil a \rceil)}{\alpha} \;/;\; \mathrm{Re}(\alpha) > 0$$

04.02.21.0007.01

$$\int_a^\infty t^{\alpha-1} \lceil t \rceil \, dt = -\frac{1}{\alpha} (\lceil a \rceil\, a^{\alpha} + \zeta(-\alpha, \lceil a \rceil)) \;/;\; \mathrm{Re}(\alpha) < -1$$

04.02.21.0008.01

$$\int_1^\infty t^{\alpha-1} \lceil t \rceil \, dt = -\frac{\zeta(-\alpha)+1}{\alpha} \;/;\; \mathrm{Re}(\alpha) < -1$$

04.02.21.0009.01

$$\int_{-a}^a \lceil t \rceil \, dt = a$$

Integral transforms

Fourier exp transforms

04.02.22.0001.01

$$\mathcal{F}_t[\lceil t \rceil](z) = \sqrt{\frac{\pi}{2}}\, \delta(z) + \frac{i}{\sqrt{2\,\pi}} \sum_{k=1}^{\infty} \frac{\delta(2\,k\,\pi - z) - \delta(2\,\pi\,k + z)}{k} - i\,\sqrt{2\,\pi}\; \delta'(z)$$

Fourier cos transforms

04.02.22.0002.01

$$\mathcal{F}c_t[\lceil t \rceil](z) = \sqrt{\frac{\pi}{2}}\,\delta(z) - \frac{1}{\sqrt{2\pi}\,z}\cot\left(\frac{z}{2}\right)$$

Fourier sin transforms

04.02.22.0003.01

$$\mathcal{F}s_t[\lceil t \rceil](z) = \frac{1}{\sqrt{2\pi}\,z} - \sqrt{2\pi}\,\delta'(z) + \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\frac{\delta(2k\pi - z) - \delta(2\pi k + z)}{k}$$

Laplace transforms

04.02.22.0004.01

$$\mathcal{L}_t[\lceil t \rceil](z) = \frac{e^z}{(e^z - 1)z} \;/;\; \mathrm{Re}(z) > 0$$

Summation

Finite summation

04.02.23.0001.01

$$\sum_{k=0}^{y}\left\lceil x - \frac{k}{y}\right\rceil = \lceil x\,y - \lceil x-1 \rceil (y + \lceil -y \rceil)\rceil \;/;\; x \in \mathbb{R} \land y \in \mathbb{R} \land 0 < x < 1 \land 0 < y < 1$$

04.02.23.0002.01

$$\sum_{k=0}^{n-1}\left\lceil \frac{1}{p}\left\lfloor \frac{k}{m}\right\rfloor\right\rceil = \left(n - m - \frac{p\,m}{2}\left\lfloor \frac{n-m}{p\,m}\right\rfloor\right)\left\lceil \frac{1}{p}\left\lfloor \frac{n}{m}\right\rfloor\right\rceil \;/;\; n \in \mathbb{N}^+ \land m \in \mathbb{N}^+ \land p \in \mathbb{N}^+$$

Representations through equivalent functions

With related functions

With Floor

For real arguments

04.02.27.0009.01

$$\lceil x \rceil = \lfloor x \rfloor + 1 \;/;\; x \in \mathbb{R} \land x \notin \mathbb{Z}$$

04.02.27.0010.01

$$\lceil x \rceil = \lfloor x \rfloor \;/;\; x \in \mathbb{Z}$$

04.02.27.0011.01

$$\lceil x \rceil = \lfloor x \rfloor - \theta(\chi_{\mathbb{Z}}(x) - 1) + 1 \;/;\; x \in \mathbb{R}$$

For complex arguments

04.02.27.0012.01

$$\lceil z \rceil = \lfloor z \rfloor \;/;\; \mathrm{Re}(z) \in \mathbb{Z} \land \mathrm{Im}(z) \in \mathbb{Z}$$

04.02.27.0013.01

$$\lceil z \rceil = \lfloor z \rfloor + 1 \;/;\; \mathrm{Re}(z) \notin \mathbb{Z} \bigwedge \mathrm{Im}(z) \in \mathbb{Z}$$

04.02.27.0014.01

$$\lceil z \rceil = \lfloor z \rfloor + i \;/;\; \mathrm{Re}(z) \in \mathbb{Z} \bigwedge \mathrm{Im}(z) \notin \mathbb{Z}$$

04.02.27.0015.01

$$\lceil z \rceil = \lfloor z \rfloor + 1 + i \;/;\; \mathrm{Re}(z) \notin \mathbb{Z} \bigwedge \mathrm{Im}(z) \notin \mathbb{Z}$$

04.02.27.0002.01

$$\lceil z \rceil = \lfloor z \rfloor - \theta(\chi_{\mathbb{Z}}(\mathrm{Re}(z)) - 1) + i\,\theta(-\chi_{\mathbb{Z}}(\mathrm{Im}(z))) + 1$$

04.02.27.0016.01

$$\lceil z \rceil = -\lfloor -z \rfloor$$

With Round

For real arguments

04.02.27.0017.01

$$\lceil x \rceil = \left\lfloor x + \frac{1}{2} \right\rceil \;/;\; x \in \mathbb{R} \bigwedge \frac{x+1}{2} \notin \mathbb{Z}$$

04.02.27.0018.01

$$\lceil x \rceil = \left\lfloor x + \frac{1}{2} \right\rceil - 1 \;/;\; \frac{x+1}{2} \in \mathbb{Z}$$

04.02.27.0019.01

$$\lceil x \rceil = \left\lfloor x + \frac{1}{2} \right\rceil - \chi_{\mathbb{Z}}\left(\frac{x+1}{2}\right) \;/;\; x \in \mathbb{R}$$

For complex arguments

04.02.27.0003.01

$$\lceil z \rceil = \left\lfloor \frac{1+i}{2} + z \right\rceil - \chi_{\mathbb{Z}}\left(\frac{\mathrm{Re}(z)+1}{2}\right) - i\,\chi_{\mathbb{Z}}\left(\frac{\mathrm{Im}(z)+1}{2}\right)$$

With IntegerPart

For real arguments

04.02.27.0020.01

$$\lceil x \rceil = \mathrm{int}(x) + 1 \;/;\; x \in \mathbb{R} \bigwedge x > 0 \bigwedge x \notin \mathbb{Z}$$

04.02.27.0021.01

$$\lceil x \rceil = \mathrm{int}(x) \;/;\; x \in \mathbb{R} \land x \leq 0 \lor x \in \mathbb{Z}$$

04.02.27.0022.01

$$\lceil x \rceil = \mathrm{int}(x) - \mathrm{sgn}(\chi_{\mathbb{Z}}(-x) + \theta(-x)) + 1 \;/;\; x \in \mathbb{R}$$

For complex arguments

04.02.27.0004.01

$$\lceil z \rceil = \mathrm{int}(z) + 1 + i - \mathrm{sgn}(\chi_{\mathbb{Z}}(-\mathrm{Re}(z)) + \theta(-\mathrm{Re}(z))) - i\,\mathrm{sgn}(\chi_{\mathbb{Z}}(-\mathrm{Im}(z)) + \theta(-\mathrm{Im}(z)))$$

With FractionalPart

For real arguments

04.02.27.0023.01

$\lceil x \rceil = x - \operatorname{frac}(x) + 1 /; x \in \mathbb{R} \bigwedge x > 0 \bigwedge x \notin \mathbb{Z}$

04.02.27.0024.01

$\lceil x \rceil = x - \operatorname{frac}(x) /; x \in \mathbb{R} \land x \leq 0 \lor x \in \mathbb{Z}$

04.02.27.0025.01

$\lceil x \rceil = x - \operatorname{frac}(x) - \operatorname{sgn}(\chi_{\mathbb{Z}}(-x) + \theta(-x)) + 1 /; x \in \mathbb{R}$

For complex arguments

04.02.27.0005.01

$\lceil z \rceil = z - \operatorname{frac}(z) + 1 + i - \operatorname{sgn}(\chi_{\mathbb{Z}}(-\operatorname{Re}(z)) + \theta(-\operatorname{Re}(z))) - i \operatorname{sgn}(\chi_{\mathbb{Z}}(-\operatorname{Im}(z)) + \theta(-\operatorname{Im}(z)))$

With Mod

04.02.27.0006.01

$\lceil z \rceil = z + -z \bmod 1$

With Quotient

04.02.27.0007.01

$\lceil z \rceil = -\operatorname{quotient}(-z, 1)$

With elementary functions

04.02.27.0008.01

$$\lceil z \rceil = z + \frac{\tan^{-1}(\cot(\pi z))}{\pi} + \frac{1}{2} /; z \in \mathbb{R} \bigwedge z \notin \mathbb{Z}$$

Zeros

04.02.30.0001.01

$\lceil z \rceil = 0 /; -1 < \operatorname{Re}(z) \leq 0 \land -1 < \operatorname{Im}(z) \leq 0$

History

– C. F. Gauss (1808)

– K. E. Iverson (1962) suggested the notation $\lceil z \rceil$

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