EllipticK

Notations

**Traditional name**

Complete elliptic integral of the first kind

**Traditional notation**

\( K(z) \)

**Mathematica StandardForm notation**

\( \text{EllipticK}[z] \)

Primary definition

\[ K(z) = F\left( \frac{\pi}{2} \middle| z \right) \]

Specific values

**Values at fixed points**

\[ K(0) = \frac{\pi}{2} \]

\[ K\left( \frac{1}{2} \right) = \frac{8 \pi^{3/2}}{\Gamma\left( -\frac{1}{4} \right)^2} \]

\[ K\left( 17 - 12 \sqrt{2} \right) = \frac{2 \left( 2 + \sqrt{2} \right) \pi^{3/2}}{\Gamma\left( -\frac{1}{4} \right)^2} \]

\[ K(1) = \infty \]

\[ K(-1) = \frac{\Gamma\left( \frac{1}{2} \right)^2}{4 \sqrt{2 \pi}} \]
Values at infinities

08.02.03.0005.01
\( K(\infty) = 0 \)

08.02.03.0006.01
\( K(-\infty) = 0 \)

08.02.03.0007.01
\( K(i \infty) = 0 \)

08.02.03.0008.01
\( K(-i \infty) = 0 \)

08.02.03.0009.01
\( K(\bar{\infty}) = 0 \)

Singular values

08.02.03.0010.01
\[
\frac{K(1 - z^2)}{K(z)} = \sqrt{1} \;/\; z = \frac{1}{\sqrt{2}}
\]

08.02.03.0011.01
\[
\frac{K(1 - z^2)}{K(z)} = \sqrt{2} \;/\; z = \sqrt{2} - 1
\]

08.02.03.0012.01
\[
\frac{K(1 - z^2)}{K(z)} = \sqrt{3} \;/\; z = \frac{\sqrt{3} - 1}{2\sqrt{2}}
\]

08.02.03.0013.01
\[
\frac{K(1 - z^2)}{K(z)} = \sqrt{4} \;/\; z = 3 - 2\sqrt{2}
\]

08.02.03.0014.01
\[
\frac{K(1 - z^2)}{K(z)} = \sqrt{5} \;/\; z = \sqrt{\frac{1}{2} - \sqrt{-2 + \sqrt{5}}}
\]

08.02.03.0015.01
\[
\frac{K(1 - z^2)}{K(z)} = \sqrt{6} \;/\; z = -3 - 2\sqrt{2} + 2\sqrt{3} + \sqrt{6}
\]

08.02.03.0016.01
\[
\frac{K(1 - z^2)}{K(z)} = \sqrt{7} \;/\; z = \frac{1}{4}\sqrt{8 - 3\sqrt{7}}
\]

08.02.03.0017.01
\[
\frac{K(1 - z^2)}{K(z)} = \sqrt{8} \;/\; z = 5 + 4\sqrt{2} - 2\sqrt{2(7 + 5\sqrt{2})}
\]
\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{9} / z = \frac{1}{\sqrt{194 + 112 \sqrt{3} + 4 \sqrt{4680 + 2702 \sqrt{3}}}}
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{10} / z = -9 + 3 \sqrt{10} - 2 \sqrt{38 - 12 \sqrt{10}}
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{11} / z = \frac{1}{2} \left( \left( 32 \cdot 3^{2/3} + 3 \left( 9 + 7 \sqrt{33} \right)^{1/3} - 4 \cdot 3^{1/3} \left( 9 + 7 \sqrt{33} \right)^{2/3} \right) \right)
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{12} / z = \sqrt{833 - 340 \sqrt{6} - 12 \sqrt{9602 - 3920 \sqrt{6}}}
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{13} / z = \frac{1}{\sqrt{2 \left( 649 + 180 \sqrt{13} + 6 \sqrt{23382 + 6485 \sqrt{13}} \right)}}
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{14} / z = \sqrt{\left( z; 1 - 7960 \cdot z - 3364 \cdot z^2 - 42152 \cdot z^3 + 107206 \cdot z^4 - 42152 \cdot z^5 - 3364 \cdot z^6 - 7960 \cdot z^7 + z^8 \right)^{1/2}}
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{15} / z = \frac{1}{\sqrt{2 \left( 376 + 168 \sqrt{5} + 6 \sqrt{47067 + 21049 \sqrt{5}} \right)}}
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{16} / z = \sqrt{4481 + 3168 \sqrt{2} - 24 \sqrt{69708 + 49291 \sqrt{2}}}
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{17} / z = \frac{1}{\sqrt{2 \left( -412 - 100 \sqrt{17} + 5 \sqrt{13598 + 3298 \sqrt{17}} \right)}}
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{18} / z = \frac{1}{\sqrt{1649 + 400 \sqrt{17} - 20 \sqrt{13598 + 3298 \sqrt{17}}}}
\]
\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{18} / z = \sqrt{9603 - 6790 \sqrt{2} - 56 \sqrt{58803 - 41580 \sqrt{2}}}
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{19} / z = \frac{1}{2} \left[ \frac{96 + (1 + 3 \sqrt{57})^{1/3} - 12 (1 + 3 \sqrt{57})^{2/3}}{2 (1 + 3 \sqrt{57})^{1/3} + \sqrt{3 (4 + 12 \sqrt{57} - 32 (1 + 3 \sqrt{57})^{1/3} + (1 + 3 \sqrt{57})^{2/3})}} \right]
\]

\[
\frac{K(1 - z^2)}{K(z^2)} = \sqrt{20} / z;
\]

\[z = \left( z \in \mathbb{C} \right)^{-1/4}
\]

\[
\sqrt{1 - 78984 z - 290020 z^2 - 2454456 z^3 + 6695494 z^4 - 2454456 z^5 - 290020 z^6 - 78984 z^7 + z^8}
\]

**General characteristics**

**Domain and analyticity**

\(K(z)\) is an analytical function of \(z\) which is defined over the whole complex \(z\)-plane.

\(z \mapsto K(z) : \mathbb{C} \rightarrow \mathbb{C}\)

**Symmetries and periodicities**

**Mirror symmetry**

\(K(z) = K(\overline{z}) / z \notin (1, \infty)\)

**Periodicity**

The function \(K(z)\) is not periodic.

**Poles and essential singularities**

The function \(K(z)\) does not have poles and essential singularities.

\(\text{Sing}_z(K(z)) = \{\}\)

**Branch points**

The function \(K(z)\) has two branch points: \(z = 1, \ z = \infty\).

\(\text{BP}_z(K(z)) = \{1, \ \infty\}\)

\(\mathcal{R}_z(K(z), 1) = \log\)
\( R_\varepsilon(K(z), \infty) = \log \)

**Branch cuts**

The function \( K(z) \) is a single-valued function on the \( z \)-plane cut along the interval \((1, \infty)\). The function \( K(z) \) is continuous from below on the interval \((1, \infty)\).

\[ \mathcal{BC}_\varepsilon(K(z)) = \{(1, \infty), i\} \]

\[
\lim_{\varepsilon \to 0^+} K(x - i \varepsilon) = K(x); \quad x > 1
\]

\[
\lim_{\varepsilon \to 0^+} K(x + i \varepsilon) = 2i K(1 - x) + K(x); \quad x > 1
\]

**Series representations**

**Generalized power series**

Expansions at generic point \( z = z_0 \)

**For the function itself**

\[
K(z) \propto K(z_0) - 2i K(1 - z_0) \left[ \frac{\arg(1 - z_0) + \pi}{2\pi} + \frac{\arg(z_0 - z)}{2\pi} \right] +
\]

\[
\frac{1}{2\pi} \left[ G^{\pm \nu}_{2,2} \left( 1 - z_0 \left| \begin{array}{c} -\frac{1}{2}, -\frac{1}{2} \\ 0, -1 \end{array} \right. \right) + 2i \pi \arg(1 - z_0) + \pi \right] \left( K(1 - z_0) z_0 - E(1 - z_0) \right) (z - z_0) +
\]

\[
\frac{1}{4\pi} \left[ G^{\pm \nu}_{2,2} \left( 1 - z_0 \left| \begin{array}{c} -\frac{3}{2}, -\frac{3}{2} \\ 0, -2 \end{array} \right. \right) + \frac{i}{2\pi} \arg(1 - z_0) + \pi \right] \left( K(1 - z_0) z_0 - E(1 - z_0) \right) (z - z_0) +
\]

\[
( -3K(1 - z_0) z_0^2 + 4E(1 - z_0) + K(1 - z_0)) z_0 - 2E(1 - z_0) \right) (z - z_0)^2 + \ldots; (z \to z_0)
\]

\[
K(z) \propto K(z_0) - 2i K(1 - z_0) \left[ \frac{\arg(1 - z_0) + \pi}{2\pi} + \frac{\arg(z_0 - z)}{2\pi} \right] +
\]

\[
\frac{1}{2\pi} \left[ G^{\pm \nu}_{2,2} \left( 1 - z_0 \left| \begin{array}{c} -\frac{1}{2}, -\frac{1}{2} \\ 0, -1 \end{array} \right. \right) + 2i \pi \arg(1 - z_0) + \pi \right] \left( K(1 - z_0) z_0 - E(1 - z_0) \right) (z - z_0) +
\]

\[
\frac{1}{4\pi} \left[ G^{\pm \nu}_{2,2} \left( 1 - z_0 \left| \begin{array}{c} -\frac{3}{2}, -\frac{3}{2} \\ 0, -2 \end{array} \right. \right) + \frac{i}{2\pi} \arg(1 - z_0) + \pi \right] \left( K(1 - z_0) z_0 - E(1 - z_0) \right) (z - z_0) +
\]

\[
( -3K(1 - z_0) z_0^2 + 4E(1 - z_0) + K(1 - z_0)) z_0 - 2E(1 - z_0) \right) (z - z_0)^2 + O((z - z_0)^3)
\]
\[
K(z) = \frac{1}{2\pi} \sum_{k=0}^{\infty} \frac{1}{k!} \left( G_{2,2}^{2,2} \left( 1 - z \left| \begin{array}{c} \frac{1}{2} - k, \frac{1}{2} - k \\ 0, -k \end{array} \right. \right) - 2\pi i (-1)^k \left[ \frac{\arg(1 - z_0) + \pi}{2\pi} \right] \left[ \frac{\arg(z_0 - z)}{2\pi} \right] \Gamma \left( k + \frac{1}{2} \right) \right) 2F_1 \left( k + \frac{1}{2}, k + \frac{1}{2}; 1; 1 - z_0 \right) \right) \]
\]

\[
K(z) = \pi \sum_{k=0}^{\infty} \left( \frac{1}{k!} \right)^2 \left( 2F_1 \left( k + \frac{1}{2}, k + \frac{1}{2}; k + 1; z_0 \right) \right) (z - z_0)^k
\]

\[
K(z) \propto K(z_0) - 2i K(1 - z_0) \left( \frac{\arg(1 - z_0) + \pi}{2\pi} \right) \left( \frac{\arg(z_0 - z)}{2\pi} \right) + O(z - z_0)
\]

**Expansions on branch cuts**

**For the function itself**

\[
K(z) \propto K(x) - 2i K(1 - x) \left( \frac{\arg(x - z) + \pi}{2\pi} \right) + \frac{1}{2\pi} \left( \frac{2i}{(x - 1) x} \left( (K(1 - x) x - E(1 - x)) \left( \frac{\arg(x - z)}{2\pi} \right) + G_{2,2}^{2,2} \left( 1 - x \left| \begin{array}{c} \frac{1}{2}, -\frac{1}{2} \\ 0, -1 \end{array} \right. \right) \right) \right) \]
\]

\[
\frac{1}{4\pi} \left( \frac{i}{(x - 1)^2 x^2} \left( \frac{\arg(x - z)}{2\pi} \right) \right) (-3K(1 - x)x^2 + (4E(1 - x) + K(1 - x)x - 2E(1 - x)) + G_{2,2}^{2,2} \left( 1 - x \left| \begin{array}{c} \frac{3}{2}, -\frac{3}{2} \\ 0, -2 \end{array} \right. \right)) (z - x)^2 + O((z - x)^3); \ x \in \mathbb{R} \land x > 1
\]

**Expansions at \( z = 0 \)**

**For the function itself**
\[ K(z) \propto \frac{\pi}{2} \left( 1 + \frac{z}{4} + \frac{9 z^2}{64} + \ldots \right); (z \to 0) \]

\[ K(z) \propto \frac{\pi}{2} \left( 1 + \frac{z}{4} + \frac{9 z^2}{64} + O(z^3) \right) \]

\[ K(z) = \frac{\pi}{2} \sum_{k=0}^{n} \left( \frac{1}{2} \right)^k \frac{z^k}{k!^2} / |z| < 1 \]

\[ K(z) = \frac{\pi}{2} F\left( \frac{1}{2}, 1; -1; z \right); |z| < 1 \]

\[ K(z) = \frac{\pi}{2} + O(z) \]

\[ K(z) = F_\omega(z) / \left( F_\omega(z) = \frac{\pi}{2} \sum_{k=0}^{n} \left( \frac{1}{2} \right)^k \frac{z^k}{k!^2} = K(z) - \frac{z^{n+1} \Gamma\left(n + \frac{3}{2}\right)^2}{2 (n+1)!^2} 3F_2\left(1, n + \frac{3}{2}, n + \frac{3}{2}; n + 2, n + 2; z \right) \right) \bigwedge n \in \mathbb{N} \]

Summed form of the truncated series expansion.

**Expansions at \( z = 1 \)**

**For the function itself**

\[ K(z) \propto -\frac{1}{2} \log(1 - z) \left( 1 - \frac{z-1}{4} + \frac{9}{64} (z-1)^2 + \ldots \right) + \log(4) + \frac{1}{4} (1 - \log(4))(z-1) + \frac{3}{128} (6 \log(4) - 7) (z-1)^2 + \ldots ; (z \to 1) \]

\[ K(z) \propto -\frac{1}{2} \log(1 - z) \left( 1 - \frac{z-1}{4} + \frac{9}{64} (z-1)^2 + O((z-1)^3) \right) + \]

\[ \log(4) + \frac{1}{4} (1 - \log(4))(z-1) + \frac{3}{128} (6 \log(4) - 7) (z-1)^2 + O((z-1)^3) \]

\[ K(z) = -\frac{1}{2} \log(1 - z) \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{5}{2} \right)^2}{(k!)^2} (z-1)^k - \frac{z-1}{4} + 2 \log(2) \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} \right)^2}{(k!)^2} (z-1)^k + \]

\[ \frac{9}{16} \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{5}{2} \right)^2}{((k+2)!)^2} \left( \frac{1}{k+1} + \frac{1}{k+2} - \frac{2}{2k+3} - \sum_{i=k+1}^{\infty} \frac{2}{i} \right) (z-1)^{k+2} / |z-1| < 1 \]
\[ K(z) = \frac{1}{2} \log(1 - z) - 2 \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} \right)_k^2}{k!^2} (z - 1)^k + \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} \right)_k^2}{k!^2} \left( \psi(k + 1) - \psi \left( k + \frac{1}{2} \right) \right) (z - 1)^k / |z - 1| < 1 \]

\[ K(z) = -\frac{1}{\pi} \log(1 - z) - \sum_{k=0}^{\infty} \frac{\left( \frac{1}{2} \right)_k^2}{k!^2} \left( \psi(k + 1) - \psi \left( k + \frac{1}{2} \right) \right) (1 - z)^k / |z - 1| < 1 \]

\[ K(z) \propto -\frac{1}{2} \log(1 - z) \left( 1 + O(z - 1) \right) + \log(4) \left( 1 + O(z - 1) \right) \]

\[ K(z) = \frac{1}{2} \log(z) / (-\log(1 - z) - 2 \psi(k + 1) - 2 \psi \left( k + \frac{1}{2} \right)) (1 - z)^k = K(z) - \frac{1}{2} \log(4) + \frac{3 (6 \log(4) - 7)}{128 z^2} + \ldots | n \in \mathbb{N} \]

Summed form of the truncated series expansion.

**Expansions at** \( z = \infty \)

**For the function itself**

\[ K(z) \propto \frac{\log( - z )}{2 \sqrt{- z}} \left( 1 + \frac{1}{4 z} + \frac{9}{64 z^2} + \ldots \right) + \frac{1}{\sqrt{- z}} \left( \log(4) + \frac{\log(4) - 1}{4 z} + \frac{3 (6 \log(4) - 7)}{128 z^2} + \ldots / |z| \to \infty \right) \]

\[ K(z) \propto \frac{\log( - z )}{2 \sqrt{- z}} \left( 1 + \frac{1}{4 z} + \frac{9}{64 z^2} + \frac{1}{z^3} \left( \log(4) + \frac{\log(4) - 1}{4 z} + \frac{3 (6 \log(4) - 7)}{128 z^2} + \frac{1}{z^3} \right) \right) \]

\[ K(z) = \frac{\log( - z )}{2 \sqrt{- z}} \sum_{k=0}^{\infty} \frac{\left( \frac{1}{2} \right)_k^2 z^{-k}}{\left( k \right)_2^2} + \frac{1}{\sqrt{- z}} \left( \log(4) + \frac{1}{64 z^2} + \frac{9}{16 z^3} \right) / |z| > 1 \]

\[ K(z) = \frac{\log( - z )}{2 \sqrt{- z}} \sum_{k=0}^{\infty} \frac{\left( \frac{1}{2} \right)_k^2 z^{-k}}{k \left( k + 1 \right)_2^2} + \frac{1}{\sqrt{- z}} \sum_{k=0}^{\infty} \frac{\left( \frac{1}{2} \right)_k^2}{k \left( k + 1 \right)_2^2} + \psi(k + 1) - \psi \left( k + \frac{1}{2} \right) z^{-k} / |z| > 1 \]
\[ K(z) = \frac{\log(-z)}{\pi \sqrt{-z}} K\left( \frac{1}{z} \right) + \frac{1}{\sqrt{-z}} \sum_{k=0}^{\infty} \frac{\left( \frac{1}{2} \right)_k^2}{k!^2} \left( \psi(k+1) - \psi\left( \frac{1}{2} + k \right) \right) z^{-k}; \ |z| > 1 \]

\[ K(z) \propto \frac{\log(4)}{\sqrt{-z}} \left( 1 + O\left( \frac{1}{z} \right) \right) + \frac{\log(-z)}{2 \sqrt{-z}} \left( 1 + O\left( \frac{1}{z} \right) \right) \]

\[ K(z) \propto \left\{ \begin{array}{ll} \frac{-i \log(z)}{2 \sqrt{z}} & \text{arg}(z) \leq 0 \\ \frac{i \log(z)}{2 \sqrt{z}} & \text{True; } (|z| \to \infty) \end{array} \right. \]

\[ K(z) = F_{\infty}(z) / \left\{ F_n(z) = \frac{1}{2 \sqrt{-z}} \sum_{k=0}^{m} \frac{\left( \frac{1}{2} \right)_k^2}{k!^2} \left( \log(-z) + 2 \psi(k+1) - \psi\left( \frac{1}{2} + k \right) - \psi\left( \frac{1}{2} + k \right) \right) z^{-k} \right\} \]

\[ K(z) = \frac{1}{2} G_{4,4} \left( \begin{array}{c} \begin{array}{c} -m - \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{2}, -m - \frac{1}{2} \\ 0, -m - \frac{1}{2}, -m - \frac{1}{2}, 0 \end{array} \\ \begin{array}{c} -z \\ \in \mathbb{N} \end{array} \end{array} \right) \]

Summed form of the truncated series expansion.

Residue representations

\[ K(z) = \frac{1}{2} \sum_{j=0}^{\infty} \text{res}_{s=j} \left( \frac{\Gamma\left( \frac{1}{2} - s \right)^2 (-z)^{-s}}{\Gamma(1-s)} \Gamma(s) \right) (-j); \ |z| < 1 \]

\[ K(z) = -\frac{1}{2} \sum_{j=0}^{\infty} \text{res}_{s=j} \left( \frac{\Gamma(s) (-z)^{-s}}{\Gamma(1-s)} \Gamma\left( \frac{1}{2} - s \right)^2 \right) (j + \frac{1}{2}); \ |z| > 1 \]

Other series representations

\[ K(z) = \pi \left( \frac{1}{2} \sum_{k=1}^{\infty} q(z)^k + 1 \right)^2 \]

\[ K(z) = \pi \left( \frac{1}{2} \sum_{k=1}^{\infty} \frac{q(z)^k}{q(z)^{2k} + 1} \right) \]

Integral representations

On the real axis
Of the direct function

\[
K(z) = \int_0^2 \frac{1}{\sqrt{1 - z \sin^2(t)}} \; dt \; / \; |\arg(1 - z)| < \pi
\]

\[
K(z) = \int_0^1 \frac{1}{\sqrt{1 - t^2} \sqrt{1 - z t^2}} \; dt \; / \; |\arg(1 - z)| < \pi
\]

\[
K(z) = \int_1^\infty \frac{1}{\sqrt{t^2 - 1} \sqrt{t^2 - z}} \; dt \; / \; |\arg(1 - z)| < \pi
\]

Contour integral representations

\[
K(z) = \frac{1}{4 \pi i} \int_{\gamma'} \Gamma(s) \frac{(1/s - 1) \Gamma(s)}{\Gamma(1 - s)} (-z)^s \; ds
\]

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

\[
(1 - z) w''(z) + (1 - 2 z) w'(z) - \frac{1}{4} w(z) = 0 \; / \; w(z) = c_1 K(z) + c_2 K(1 - z)
\]

\[
W_d(K(z), K(1 - z)) = \frac{\pi}{4(z - 1) z}
\]

\[
w''(z) + \frac{(2 g(z) - 1) g'(z) - g''(z)}{(g(z) - 1) g(z)} w'(z) + \frac{g'(z)^2}{4(g(z) - 1) g(z)} w(z) = 0 \; / \; w(z) = c_1 K(g(z)) + c_2 K(1 - g(z))
\]

\[
W_d(K(g(z)), (1 - g(z))) = \frac{\pi g'(z)}{4(g(z) - 1) g(z)}
\]

\[
w''(z) + \frac{(2 g(z) - 1) g'(z)}{(g(z) - 1) g(z)} w'(z) + \left[ \frac{g'(z)^2}{4(g(z) - 1) g(z)} + \frac{1 - 2 g(z) h'(z) g'(z)}{(g(z) - 1) g(z) h(z)} + \frac{2 h'(z)^2}{h(z)^2} + \frac{h'(z) g''(z)}{h(z) g'(z)} - \frac{h''(z)}{h(z)} \right] w(z) = 0 \; / \; w(z) = c_1 h(z) K(g(z)) + c_2 h(z) K(1 - g(z))
\]
\begin{align*}
W_{\eta}(h(z), K(g(z)), h(z) K(1 - g(z))) &= \frac{\pi h(z)^2 g'(z)}{4 (g(z) - 1) g(z)} \\
W_{\eta}(z' K(a z'), z' K(1 - a z')) &= \frac{\pi r z'^{r-1}}{4 (a z' - 1)} \\
w'(z) + \left( 1 + \frac{1}{a r^2 - 1} \right) \log(r) + 2 \log(s) \right) w'(z) + \frac{a r^2 (\log(r) - 2 \log(s))^2 - 4 \log^2(s)}{4 (a r^2 - 1)} w(z) = 0; \\
w(z) = c_1 s' K(a r') + c_2 s' K(1 - a r') \\
W_{\eta}(s' K(a r'), s' K(1 - a r')) &= \frac{\pi s'^{r} \log(r)}{4 (a r^2 - 1)}
\end{align*}

**Identities**

**Functional identities**

\begin{align*}
K(z) &= \frac{1}{\sqrt{1 - z}} K\left( \frac{z}{z - 1} \right); |\arg(1 - z)| < \pi \\
K(z) &= \frac{1}{\sqrt{1 - z}} K\left( \frac{z}{z - 1} \right); z \in \mathbb{R} \\
K(z) &= \frac{1}{\sqrt{1 - z}} K\left( \frac{z}{z - 1} \right) + i \left( \frac{1}{\sqrt{1 - z}} - \frac{1}{\sqrt{1 - z}} \right) K\left( \frac{1}{z - 1} \right) \\
\text{KK}(z) &= \sqrt{\frac{1}{z}} K\left( \frac{1}{z} \right) + \sqrt{\frac{1}{1 - z}} \sqrt{1 - z} \sqrt{- \frac{1}{z}} K\left( \frac{1 - 1}{z} \right) \\
\text{KK}(z) &= \frac{1}{\sqrt{z}} K\left( \frac{1}{z} \right) + \sqrt{\frac{1}{1 - z}} \sqrt{z(1 - z)} \sqrt{- \frac{1}{z}} K(1 - z) \\
K\left( \frac{1}{z} \right) &= \sqrt{z} \left( K(z) - \sqrt{- \frac{1}{z}} \sqrt{1 - z} \sqrt{\frac{1}{z}} \sqrt{z(1 - z)} K(1 - z) \right)
\end{align*}
08.02.17.0003.01

\[ K(z) = \frac{2}{1 + \sqrt{1 - z}} K\left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right) \]

Complex characteristics

Real part

08.02.19.0001.01

\[ \text{Re}(K(x + iy)) = \frac{\pi}{2} F_{2 \times 4}^{1 \times 0}(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}; \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}; -y^2, x^2) + \frac{\pi x}{8} F_{2 \times 1}^{1 \times 1}(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; -y^2, x^2); x \in \mathbb{R} \land y \in \mathbb{R} \]

08.02.19.0003.01

\[ \text{Re}(K(e^{iz})) = \frac{1}{2} \cos\left(\frac{x}{4}\right) K\left(1 - \sin^2\left(\frac{x}{4}\right) + \frac{1}{2} K\left(\sin^2\left(\frac{x}{4}\right)\right) \sin\left(\frac{x}{4}\right); x \in \mathbb{R} \land 0 < x < \pi \]

Imaginary part

08.02.19.0002.01

\[ \text{Im}(K(x + iy)) = \frac{\pi y}{8} F_{2 \times 4}^{1 \times 0}(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}; \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}; -y^2, x^2) + \frac{9 \pi x y}{64} F_{2 \times 1}^{1 \times 1}(\frac{5}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; 2, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}; -y^2, x^2); x \in \mathbb{R} \land y \in \mathbb{R} \]

08.02.19.0004.01

\[ \text{Im}(K(e^{iz})) = \frac{1}{2} \cos\left(\frac{x}{4}\right) K\left(\sin^2\left(\frac{x}{4}\right) - \frac{1}{2} K\left(1 - \sin^2\left(\frac{x}{4}\right)\right) \sin\left(\frac{x}{4}\right); x \in \mathbb{R} \land 0 < x < \pi \]

Differentiation

Low-order differentiation

08.02.20.0001.01

\[ \frac{\partial K(z)}{\partial z} = \frac{E(z) - (1 - z) K(z)}{2(1 - z) z} \]

08.02.20.0002.01

\[ \frac{\partial^2 K(z)}{\partial z^2} = \frac{2(2z - 1) E(z) + (3z^2 - 5z + 2) K(z)}{4(z - 1)^2 z^2} \]

Symbolic differentiation

08.02.20.0003.02

\[ \frac{\partial^n K(z)}{\partial z^n} = \frac{\pi \left(\frac{1}{2}\right)_n^2}{2n!} F_{1 \times 2}^{2 \times 1}(\frac{n}{2}, \frac{n + 1}{2}; n + 1; z); n \in \mathbb{N} \]

08.02.20.0004.02

\[ \frac{\partial^n K(z)}{\partial z^n} = \frac{\pi z^n}{2} F_{1 \times 2}^{2 \times 1}(\frac{1}{2}, \frac{1}{2}; 1 - n; z); n \in \mathbb{N} \]
Fractional integro-differentiation

\[ \frac{\partial^n K(z)}{\partial z^n} = \frac{\pi z^{-n}}{2} \, _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1 - \alpha; z \right) \]

Integration

Indefinite integration

Involving only one direct function

\[ \int K(a z) \, dz = \frac{2(E(a z) + (a z - 1) K(a z))}{a} \]

Involving one direct function and elementary functions

Involving power function

Involving power

Linear argument

\[ \int \frac{\partial^n K(a z)}{\partial z^n} \, dz = \frac{\pi z^{-n}}{2} \, _3F_2 \left( \frac{1}{2}, \frac{1}{2}, \alpha; 1, \alpha + 1; a z \right) \]

\[ \int \frac{\partial^n K(z)}{\partial z^n} \, dz = \frac{2(E(z) + (z - 1) K(z))}{a} \]

\[ \int \frac{K(a z)}{\sqrt{z}} \, dz = \pi \sqrt{z} \, _3F_2 \left( \frac{1}{2}, \frac{1}{2}, \alpha; 1, \alpha + 1; a z \right) \]

\[ \int \frac{K(a z)}{z} \, dz = \frac{1}{8} \pi \left( a z \, _3F_2 \left( 1, 1, \frac{3}{2}; 2, 2, 2; a z \right) - 4 \gamma^2 - 8 \log(4) + 4 \log(-a z) + 4 \right) \]

\[ \int \frac{K(a z)}{z^2} \, dz = \frac{1}{128 z^2} \left( \pi \left( 9 \, a^2 \, _3F_2 \left( 1, 1, \frac{5}{2}; 2, 2, 2; a z \right) \, z^2 + 16 (-2 a z \log(4) - 1) + a z \log(-a z) - 4 \right) \right) \]
Power arguments

\[ \int z^{a-1} K(a z^2) \, dz = \frac{\pi}{2} z \; \Phi\left( \frac{1}{2}, -\frac{1}{2}; 1, -\frac{1}{2}; a z^2 \right) \]

\[ \int z K(a z^2) \, dz = \frac{E(a z^2) + (a z^2 - 1) K(a z^2)}{a} \]

\[ \int z^2 K(a z^2) \, dz = \frac{(a z^2 + 4) E(a z^2) + (3 a^2 z^4 + a z^2 - 4) K(a z^2)}{9 a^2} \]

\[ \int z^5 K(a z^2) \, dz = \frac{1}{225 a^3} \left( (9 a^2 z^6 + 16 a z^2 + 45 a^4 z^6 + 3 a^2 z^4 + 64 a^3 z^2 - 64) K(a z^2) \right) \]

\[ \int K(a z^2) \, dz = -\pi z \; \Phi\left( \frac{1}{2}, 1, 1; 1, 3; a z^2 \right) \]

\[ \int \frac{K(z^2)}{z} \, dz = \frac{E(z^2)}{z} \]

\[ \int \frac{K(z^2)}{z^3} \, dz = \frac{1}{256 z^3} \left( \pi \left( 9 a^2 \Phi\left( 1, 1, \frac{5}{2}; 2, 3, a z^2 \right) z^4 + 16 (\log(4) - 1) z^2 + a \log(-a z^2) z^2 - 4 \right) \right) \]

Involving algebraic functions

\[ \int \frac{z E(z^2)}{(1 - z^2)^{3/2}} \, dz = \frac{E(z^2) + (z^2 - 1) K(z^2)}{\sqrt{1 - z^2}} \]

Definite integration

For the direct function itself

\[ \int_0^1 t^{-1} K(t) \, dt = \frac{\pi}{2} \; \Phi\left( \frac{1}{2}, 1; 1, -\frac{1}{2}; a + 1; 1 \right); \text{Re}(a) > 0 \]

\[ \int_0^\infty t^{-1} K(-t) \, dt = \frac{\Gamma\left( \frac{1}{2} - a \right)^2 \Gamma(a)}{2 \; \Gamma(1 - a)}; \text{Re}(a) < \frac{1}{2} \]
\[
\int_0^\infty \frac{\sqrt{a^2 + x^2} - a}{a^2 + x^2} \frac{1}{b + \sqrt{b^2 + x^2}} K \left( \frac{\sqrt{b^2 + x^2} - b}{b + \sqrt{b^2 + x^2}} \right)^2 \, dx = \frac{\sqrt{a - \sqrt{a^2 - b^2}}}{b} \text{sech}^2(\alpha) K(\text{sech}^2(\alpha)) K(\tanh^2(\alpha)) /; \\
\text{Re}(\alpha) \geq \text{Re}(\beta) > 0 \sqrt[3]{\cosh^{-1} \left( \frac{b + \sqrt{2 a^2 - 2 a \sqrt{a^2 - b^2}}}{\sqrt{2 b}} \right)}
\]

Representations through more general functions

Through hypergeometric functions

Involving \( \text{}_2F_1 \)

Through Meijer \( G \)

Classical cases for the direct function itself

Classical cases involving algebraic functions

\[
\frac{1}{\sqrt{z + 1}} K \left( \frac{1}{z + 1} \right) = \frac{1}{2} G_{2,2}^{1,2} \left( \begin{array}{c} 1, 1 \\ 0, 0 \end{array} \right) /; z \neq (-\infty, -1)
\]

\[
\frac{1}{\sqrt{z + 1}} K \left( \frac{1}{z + 1} \right) = \frac{1}{2} G_{2,2}^{2,1} \left( \begin{array}{c} 1, 1 \\ 0, 0 \end{array} \right) /; z \neq (-1, 0)
\]

\[
\frac{1}{\sqrt{z + 1}} K \left( \frac{1}{z + 1} \right) = \frac{1}{2} G_{2,2}^{1,2} \left( \begin{array}{c} 1, 1 \\ 0, 0 \end{array} \right) /; z \neq (-\infty, -1)
\]

\[
K \left( -\frac{\sqrt{z - 1}^2}{4 \sqrt{z}} \right) = \frac{1}{2 \pi} G_{2,2}^{2,2} \left( \begin{array}{c} 3, 3 \\ 1, 1 \end{array} \right) /; z \neq (-\infty, 0)
\]
\begin{align*}
\frac{1}{\sqrt{z}+1} K \left( \frac{4\sqrt{z}}{(\sqrt{z}+1)^2} \right) & = \frac{\pi}{2} G^{1,1}_{2,2} \left( z \left| \frac{1}{2}, \frac{1}{2} \right| 0, 0 \right) \\
\frac{1}{\sqrt{z}+1} K \left( \frac{1-\sqrt{z}}{(1+\sqrt{z})^2} \right) & = \frac{1}{4\pi} G^{1,1}_{2,2} \left( z \left| \frac{1}{2}, \frac{1}{2} \right| 0, 0 \right) \\
K \left( \frac{1}{2} (1-\sqrt{z}+1) \right) & = \frac{\pi}{2 \Gamma \left( \frac{1}{2} \right)^2} G^{1,1}_{2,2} \left( z \left| \frac{3}{4}, \frac{3}{4} \right| 0, 0 \right) \\
\frac{1}{\sqrt{z}+1} K \left( \frac{\sqrt{z}+1 - 1}{z} \right) & = \frac{\pi}{2 \Gamma \left( \frac{1}{2} \right)^2} G^{1,1}_{2,2} \left( z \left| \frac{1}{2}, \frac{1}{2} \right| 0, 0 \right) ; z \notin (-1, 0) \\
\frac{\sqrt{z}+1 - 1}{\sqrt{z}+1} K \left( \frac{2(\sqrt{z}+1 - 1)}{z} \right) & = \frac{\pi}{2 \Gamma \left( \frac{3}{2} \right)^2} G^{1,1}_{2,2} \left( z \left| \frac{3}{4}, \frac{3}{4} \right| \frac{1}{2}, \frac{1}{2} \right) ; z \notin (-1, 0) \\
\frac{1}{\sqrt{z}+1} K \left( \frac{2(\sqrt{z}+1 + 1)}{z} \right) & = \frac{\pi}{2 \Gamma \left( \frac{3}{2} \right)^2} G^{1,1}_{2,2} \left( z \left| \frac{1}{2}, \frac{1}{2} \right| 0, 0 \right) ; z \notin (-1, 0) \\
\frac{1}{\sqrt{z}+1 \sqrt{\sqrt{z}+1 - 1}} K \left( \frac{-2(\sqrt{z}+1 + 1)}{z} \right) & = \frac{\pi}{2 \Gamma \left( \frac{3}{2} \right)^2} G^{1,1}_{2,2} \left( z \left| \frac{1}{2}, \frac{1}{2} \right| 0, 0 \right) ; z \notin (-1, 0) \\
\frac{1}{\sqrt{z}+1} K \left( \frac{2}{1-\sqrt{z}+1} \right) & = \frac{\pi}{2 \Gamma \left( \frac{1}{2} \right)^2} G^{1,1}_{2,2} \left( z \left| \frac{1}{2}, \frac{1}{2} \right| 0, 0 \right) ; z \notin (-1, 0) \\
\frac{1}{\sqrt{z}+1} K \left( \frac{2}{1-\sqrt{z}+1} \right) & = \frac{\pi}{2 \Gamma \left( \frac{1}{2} \right)^2} G^{1,1}_{2,2} \left( z \left| \frac{1}{2}, \frac{1}{2} \right| 0, 0 \right) ; z \notin (-1, 0) \\
K \left( \frac{\sqrt{z} - \sqrt{z}+1}{2 \sqrt{z}} \right) & = \frac{\pi}{2 \Gamma \left( \frac{1}{2} \right)^2} G^{1,1}_{2,2} \left( z \left| \frac{1}{2}, \frac{1}{2} \right| 0, 0 \right) ; z \notin (-1, 0) 
\end{align*}
\[
\frac{1}{\sqrt{z+1}} \left( \frac{\sqrt{z} - \sqrt{z+1}}{2\sqrt{z}} \right) = \frac{\pi}{2\Gamma\left(\frac{3}{4}\right)^2} G_{2,2}^{2,1} \left( z \left| \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \end{array} \right. \right) / z \notin (-1, 0)
\]

\[
\frac{1}{\sqrt{z+1}} K \left( \frac{\sqrt{z+1} - 1}{2\sqrt{z+1}} \right) = \frac{1}{2\sqrt{2}} G_{2,2}^{1,2} \left( z \left| \begin{array}{c} \frac{3}{4}, \frac{3}{4} \\ 0, 0 \end{array} \right. \right) / z \notin (-1, 0)
\]

\[
\frac{1}{\sqrt{z+1}} K \left( \frac{\sqrt{z} - \sqrt{z+1}}{2\sqrt{z}} \right) = \frac{1}{2\sqrt{2}} G_{2,2}^{2,1} \left( z \left| \begin{array}{c} \frac{3}{4}, \frac{3}{4} \\ 0, 0 \end{array} \right. \right) / z \notin (-1, 0)
\]

\[
\frac{1}{\sqrt{z+1}} K \left( \frac{1}{2(\sqrt{z} + \sqrt{z+1})} \right) = \frac{1}{2\sqrt{2}} G_{2,2}^{2,1} \left( z \left| \begin{array}{c} \frac{3}{4}, \frac{3}{4} \\ 0, 0 \end{array} \right. \right) / z \notin (-1, 0)
\]

\[
\frac{1}{\sqrt{z+1}} K \left( \frac{\sqrt{z} + 1}{2(\sqrt{z+1} + 1)} \right) = \frac{1}{2\sqrt{2}} G_{2,2}^{1,2} \left( z \left| \begin{array}{c} \frac{3}{4}, \frac{3}{4} \\ 0, 0 \end{array} \right. \right)
\]

\[
\frac{1}{\sqrt{z+1}} K \left( \frac{\sqrt{z+1} - 1}{\sqrt{z+1} + 1} \right) = \frac{\pi}{2\sqrt{2} \Gamma\left(\frac{1}{4}\right)^2} G_{2,2}^{1,2} \left( z \left| \begin{array}{c} \frac{1}{4}, \frac{1}{4} \\ 0, 0 \end{array} \right. \right)
\]

\[
\frac{1}{\sqrt{z+1}} K \left( \frac{1 - \sqrt{z+1}}{1 + \sqrt{z+1}} \right) = \frac{1}{4} G_{2,2}^{1,2} \left( z \left| \begin{array}{c} \frac{1}{4}, \frac{1}{4} \\ 0, 0 \end{array} \right. \right)
\]

\[
\frac{1}{\sqrt{z+1}} K \left( \frac{\sqrt{z+1} - \sqrt{z}}{\sqrt{z+1} + \sqrt{z}} \right) = \frac{\pi}{2\sqrt{2} \Gamma\left(\frac{1}{4}\right)^2} G_{2,2}^{2,1} \left( z \left| \begin{array}{c} \frac{1}{4}, \frac{1}{4} \\ 0, 0 \end{array} \right. \right) / z \notin (-1, 0)
\]

\[
\frac{1}{\sqrt{z+1}} K \left( \frac{\sqrt{z+1} - \sqrt{z}}{\sqrt{z+1} + \sqrt{z} + 1} \right) = \frac{\pi}{2\sqrt{2} \Gamma\left(\frac{3}{4}\right)^2} G_{2,2}^{2,1} \left( z \left| \begin{array}{c} \frac{1}{4}, \frac{1}{4} \\ 0, 0 \end{array} \right. \right) / z \notin (-1, 0)
\]

\[
\frac{1}{\sqrt{z+1}} K \left( \frac{\sqrt{z} - \sqrt{z+1}}{\sqrt{z} + \sqrt{z+1}} \right) = \frac{1}{4} G_{2,2}^{1,2} \left( z \left| \begin{array}{c} \frac{3}{4}, \frac{3}{4} \\ 0, 0 \end{array} \right. \right) / z \notin (-1, 0)
\]
\[
\sqrt{z+1} - \sqrt{z} \quad K\left(2\left(\sqrt{z^2 + z} - z\right)\right) = \frac{\pi}{2 \Gamma\left(\frac{3}{4}\right)} G^{1,2}_{2,2}\left(z \left| \begin{array}{c}
\frac{3}{4}, \frac{3}{4} \\
0, 0
\end{array} \right. \right); \quad \text{Re}(z) \geq 0
\]

\[
\frac{\sqrt{z+1} - \sqrt{z}}{\sqrt{z+1}} \quad K\left(\frac{2\left(\sqrt{z^2 + z} - z\right)}{\sqrt{z+1}}\right) = \frac{\pi}{2 \Gamma\left(\frac{3}{4}\right)} G^{1,2}_{2,2}\left(z \left| \begin{array}{c}
\frac{1}{4}, \frac{1}{4} \\
0, 0
\end{array} \right. \right); \quad \text{Re}(z) \geq 0
\]

\[
\frac{1}{\sqrt{z+1} - \sqrt{z}} \quad K\left(\frac{-2\left(z + \sqrt{z^2 + z}\right)}{\sqrt{z+1} - \sqrt{z}}\right) = \frac{\pi}{2 \Gamma\left(\frac{3}{4}\right)} G^{1,2}_{2,2}\left(z \left| \begin{array}{c}
\frac{1}{4}, \frac{1}{4} \\
0, 0
\end{array} \right. \right); \quad \text{Re}(z) \geq 0
\]

\[
(\sqrt{z+1} - 1) K\left(\frac{4 \sqrt{z+1}}{(\sqrt{z+1} + 1)^2}\right) = \frac{1}{2} G^{2,1}_{2,2}\left(z \left| \begin{array}{c}
\frac{3}{2}, \frac{3}{2} \\
1, 1
\end{array} \right. \right); \quad \text{Re}(z) \geq 0
\]

\[
(\sqrt{z+1} + 1) K\left(\frac{-4 \sqrt{z+1}}{(1 - \sqrt{z+1})^2}\right) = \frac{1}{2} G^{2,1}_{2,2}\left(z \left| \begin{array}{c}
\frac{3}{2}, \frac{3}{2} \\
1, 1
\end{array} \right. \right); \quad \text{Re}(z) \geq 0
\]

\[
(\sqrt{z+1} - \sqrt{z}) K\left(\frac{4 \sqrt{z^2 + z}}{(\sqrt{z} + \sqrt{z+1})^2}\right) = \frac{1}{2} G^{2,1}_{22}\left(z \left| \begin{array}{c}
\frac{1}{2}, \frac{1}{2} \\
0, 0
\end{array} \right. \right); \quad \text{Re}(z) \geq 0
\]

\[
(\sqrt{z} + \sqrt{z+1}) K\left(\frac{-4 \sqrt{z^2 + z}}{(\sqrt{z} - \sqrt{z+1})^2}\right) = \frac{1}{2} G^{2,1}_{22}\left(z \left| \begin{array}{c}
\frac{1}{2}, \frac{1}{2} \\
0, 0
\end{array} \right. \right); \quad \text{Re}(z) \geq 0
\]

\[
\frac{1}{\sqrt{z+1}} \quad K\left(\frac{\left(\sqrt{z+1} - 1\right)^2}{4 \sqrt{z+1}}\right) = \frac{1}{2} G^{1,2}_{2,2}\left(z \left| \begin{array}{c}
\frac{1}{2}, \frac{1}{2} \\
0, 0
\end{array} \right. \right)
\]

\[
\frac{1}{\sqrt{z+1}} \quad K\left(\frac{\left(\sqrt{z+1} - \sqrt{z}\right)^2}{4 \sqrt{z} \sqrt{z+1}}\right) = \frac{1}{2} G^{1,2}_{2,2}\left(z \left| \begin{array}{c}
\frac{3}{4}, \frac{3}{4} \\
\frac{1}{2}, \frac{1}{2}
\end{array} \right. \right); \quad z \notin (-1, 0)
\]
\[(\sqrt{z+1} - \sqrt{z}) K \left( \frac{1}{(\sqrt{z+1} + \sqrt{z})^4} \right) = \frac{1}{4} G^{2,1}_{2,2} \left( \frac{1}{z^2} \bigg| \frac{3}{2}, \frac{3}{2}, 1, 1 \right) ; z \notin (-1, 0)\]

\[\frac{1}{\sqrt{z+1} + \sqrt{z+1}} K \left( \frac{1}{(\sqrt{z} + \sqrt{z+1})^4} \right) = \frac{1}{4} G^{2,1}_{2,2} \left( \frac{1}{z^2} \bigg| \frac{3}{2}, \frac{3}{2}, 0, 0 \right) ; z \notin (-1, 0)\]

Classical cases involving unit step \( \theta \)

\[\theta(1 - |z|) K (1 - z) = \frac{\pi}{2} G^{2,0}_{2,2} \left( \frac{1}{z^2} \bigg| \frac{1}{2}, \frac{1}{2}, 0, 0 \right) ; z \notin (-1, 0)\]
\[ \theta(z - 1) K \left( \frac{1}{z} \right) = \frac{\pi}{2} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} 1, 1 \\ \frac{1}{2}, \frac{1}{2} \end{array} \right. \right) / z \notin (-\infty, -1) \]

\[ \theta(1 - |z|) K \left( \frac{1 - \sqrt{z}}{2} \right) = \frac{\pi}{2} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} 3, 3 \\ \frac{1}{4}, \frac{1}{4} \end{array} \right. \right) / z \notin (-1, 0) \]

\[ \theta(z - 1) K \left( \frac{1 - \sqrt{z}}{2} \right) = \frac{\pi}{2} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 0, \frac{1}{2} \end{array} \right. \right) \]

\[ \theta(1 - |z|) K \left( \frac{\sqrt{z} - 1}{2 \sqrt{z}} \right) = \frac{\pi}{2} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} 1, 1 \\ \frac{1}{4}, \frac{1}{4} \end{array} \right. \right) / z \notin (-1, 0) \]

\[ \theta(z - 1) K \left( \frac{\sqrt{z} - 1}{2 \sqrt{z}} \right) = \frac{\pi}{2} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 0, \frac{1}{2} \end{array} \right. \right) \]

\[ \frac{\theta(1 - |z|)}{\sqrt{z} + 1} K \left( \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right) = \frac{\pi}{2 \sqrt{2}} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} 1, 3 \\ \frac{1}{4}, \frac{1}{4} \end{array} \right. \right) / z \notin (-1, 0) \]

\[ \frac{\theta(z - 1)}{\sqrt{z} + 1} K \left( \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right) = \frac{\pi}{2 \sqrt{2}} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} 1, 3 \\ 0, \frac{1}{2} \end{array} \right. \right) \]

\[ \frac{\theta(1 - |z|)}{\sqrt{z} + 1} K \left( \frac{\sqrt{z} - 1}{\sqrt{z} + 1} \right) = \frac{\pi}{2 \sqrt{2}} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} 3, 3 \\ \frac{1}{4}, \frac{1}{4} \end{array} \right. \right) / z \notin (-1, 0) \]

\[ \frac{\theta(z - 1)}{\sqrt{z} + 1} K \left( \frac{\sqrt{z} - 1}{\sqrt{z} + 1} \right) = \frac{\pi}{2 \sqrt{2}} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} 3, 3 \\ 0, \frac{1}{2} \end{array} \right. \right) \]

\[ \frac{\theta(1 - |z|)}{4 \sqrt{z}} K \left( \frac{(\sqrt{z} - 1)^2}{4 \sqrt{z}} \right) = \frac{\pi}{2} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} 3, 3 \\ \frac{1}{4}, \frac{1}{4} \end{array} \right. \right) / z \notin (-1, 0) \]

\[ \frac{\theta(z - 1)}{4 \sqrt{z}} K \left( \frac{(\sqrt{z} - 1)^2}{4 \sqrt{z}} \right) = \frac{\pi}{2} G_{2,2}^{0,2} \left( z \left| \begin{array}{c} 3, 3 \\ 0, \frac{1}{2} \end{array} \right. \right) \]
\[
\frac{\theta(1-|z|)}{\sqrt{1-|z|}} K \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right)^2 = \frac{\pi}{4} \frac{G_{1,2}^2 \left( \frac{1}{z - 2\zeta} \right)}{G_{1,2}^2 \left( \frac{1}{z + 2\zeta} \right)}; \ z \notin (-1, 0)
\]

\[
\frac{\theta(|z|-1)}{\sqrt{1-|z|}} K \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right)^2 = \frac{\pi}{4} \frac{G_{1,2}^2 \left( \frac{1}{z - 2\zeta} \right)}{G_{1,2}^2 \left( \frac{1}{z + 2\zeta} \right)}; \ z \notin (-1, 0)
\]

\[
\frac{\theta(1-|z|)}{\sqrt{1-|z|}} K \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right)^2 = \frac{\pi}{2} \frac{G_{1,2}^2 \left( \frac{1}{z - 2\zeta} \right)}{G_{1,2}^2 \left( \frac{1}{z + 2\zeta} \right)}; \ z \notin (-1, 0)
\]

\[
\frac{\theta(1-|z|)}{\sqrt{1-|z|}} K \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right)^2 = \frac{\pi}{2} \frac{G_{1,2}^2 \left( \frac{1}{z - 2\zeta} \right)}{G_{1,2}^2 \left( \frac{1}{z + 2\zeta} \right)}; \ z \notin (-1, 0)
\]

\[
\frac{\theta(|z|-1)}{\sqrt{1-|z|}} K \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right)^2 = \frac{\pi}{2} \frac{G_{1,2}^2 \left( \frac{1}{z - 2\zeta} \right)}{G_{1,2}^2 \left( \frac{1}{z + 2\zeta} \right)}; \ z \notin (-1, 0)
\]

\[
\frac{\theta(1-|z|)}{\sqrt{1-|z|}} K \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right)^2 = \frac{\pi}{2} \frac{G_{1,2}^2 \left( \frac{1}{z - 2\zeta} \right)}{G_{1,2}^2 \left( \frac{1}{z + 2\zeta} \right)}; \ z \notin (-1, 0)
\]

\[
\frac{\theta(1-|z|)}{\sqrt{1-|z|}} K \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right)^2 = \frac{\pi}{2} \frac{G_{1,2}^2 \left( \frac{1}{z - 2\zeta} \right)}{G_{1,2}^2 \left( \frac{1}{z + 2\zeta} \right)}; \ z \notin (-1, 0)
\]

\[
\frac{\theta(1-|z|)}{\sqrt{1-|z|}} K \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right)^2 = \frac{\pi}{2} \frac{G_{1,2}^2 \left( \frac{1}{z - 2\zeta} \right)}{G_{1,2}^2 \left( \frac{1}{z + 2\zeta} \right)}; \ z \notin (-1, 0)
\]
08.02.26.0071.01
\begin{equation}
\theta(z) \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right); \quad z \notin (-\infty, -1)
\end{equation}

08.02.26.0072.01
\begin{equation}
\theta(z) \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right); \quad z \notin (-\infty, -1)
\end{equation}

08.02.26.0073.01
\begin{equation}
\theta(z) \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right); \quad z \notin (-\infty, 0)
\end{equation}

08.02.26.0074.01
\begin{equation}
\theta(z) \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right); \quad z \notin (-\infty, -1)
\end{equation}

08.02.26.0075.01
\begin{equation}
\theta(z) \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right); \quad z \notin (-1, 0)
\end{equation}

08.02.26.0076.01
\begin{equation}
\theta(z) \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right); \quad z \notin (-1, 0)
\end{equation}

08.02.26.0077.01
\begin{equation}
\theta(z) \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right)
\end{equation}

08.02.26.0078.01
\begin{equation}
\theta(z) \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right)
\end{equation}

08.02.26.0079.01
\begin{equation}
\theta(z) \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right); \quad \text{Re}(z) > 0
\end{equation}

08.02.26.0080.01
\begin{equation}
\theta(z) \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right); \quad \text{Re}(z) > 0
\end{equation}

Classical cases involving \text{sgn}

08.02.26.0081.01
\begin{equation}
\text{sgn} \frac{1}{z} \sqrt{z - \sqrt{z - 1}} \left( \frac{2}{z - \sqrt{z - 1}} \right)^{1/2} = \frac{\pi}{2} G_{2,2}^{2,0} \left( z \left| \begin{array}{c}
\frac{3}{4}, \frac{1}{4} \\
0, \frac{1}{2}
\end{array} \right. \right)
\end{equation}

Classical cases involving powers of complete elliptic integral \text{K}
\[ K \left( \frac{1 - \sqrt{z + 1}}{2} \right)^2 = \frac{\sqrt{\pi}}{4} G^{1, 3}_{3, 3} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0 \right) \]

\[ K \left( \frac{\sqrt{z} - \sqrt{z + 1}}{2 \sqrt{z}} \right)^2 = \frac{\sqrt{\pi}}{4} G^{1, 3}_{3, 3} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0 \right); \quad z \notin (-1, 0) \]

\[ \frac{1}{\sqrt{z} + \sqrt{z + 1}} \left( \sqrt{z + 1} - \sqrt{z} \right)^2 = \frac{\sqrt{\pi}}{8} G^{1, 3}_{3, 3} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0 \right); \quad z \notin (-1, 0) \]

\[ \frac{1}{\sqrt{z} + 1 + \frac{1}{z}} \left( \sqrt{z + 1} - 1 \right)^2 = \frac{\sqrt{\pi}}{8} G^{1, 3}_{3, 3} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0 \right); \quad z \notin (-1, 0) \]

**Generalized cases involving algebraic functions**

\[ K \left( \frac{(z - 1)^2}{4z} \right) = \frac{1}{2 \pi} G^{1, 3}_{3, 3} \left( \frac{1}{2}, 0, \frac{3}{2}, \frac{1}{2} \right); \quad z \notin (-\infty, 0) \]

\[ \frac{1}{z + 1} K \left( \frac{4z}{(z + 1)^2} \right) = \frac{\pi}{2} G^{1, 3}_{3, 3} \left( \frac{1}{2}, 0, \frac{1}{2} \right) \]

\[ \frac{1}{z + 1} K \left( \frac{(1 - z)^2}{(1 + z)^2} \right) = \frac{1}{4 \pi} G^{2, 3}_{3, 3} \left( \frac{1}{2}, 0, \frac{1}{2} \right) \]

\[ K \left( \frac{z - \sqrt{z^2 + 1}}{2z} \right) = \frac{\pi}{2 \Gamma \left( \frac{3}{2} \right)^2} G^{2, 1}_{2, 2} \left( \frac{1}{2}, 0, \frac{1}{2} \right); \quad \text{Re}(z) > 0 \]

\[ \frac{1}{\sqrt{z^2 + 1}} K \left( \frac{z - \sqrt{z^2 + 1}}{2z} \right) = \frac{\pi}{2 \Gamma \left( \frac{3}{2} \right)^2} G^{2, 1}_{2, 2} \left( \frac{1}{2}, 0, \frac{1}{2} \right); \quad \text{Re}(z) > 0 \]

\[ \frac{1}{\sqrt{z^2 + 1}} K \left( \frac{\sqrt{z^2 + 1} - z}{2 \sqrt{z^2 + 1}} \right) = \frac{1}{2 \sqrt{2}} G^{2, 1}_{2, 2} \left( \frac{3}{2}, 0, \frac{1}{2} \right); \quad \text{Re}(z) > 0 \]

\[ \frac{1}{\sqrt{z^2 + 1}} \left( \frac{1}{z + \sqrt{z^2 + 1}} \right) = \frac{1}{2 \sqrt{2}} G^{2, 1}_{2, 2} \left( \frac{3}{2}, 0, \frac{1}{2} \right); \quad \text{Re}(z) > 0 \]
\[
\frac{1}{\sqrt{z + \sqrt{z^2 + 1}}} K \left( \frac{\sqrt{z^2 + 1} - z}{\sqrt{z^2 + 1} + z} \right) = \frac{\pi}{2 \sqrt{\Gamma(\frac{1}{4})^2}} G_{2,2}^{1,1} \left( z, \frac{1}{2} \left| \frac{1}{2}, \frac{1}{2}, 0, 0 \right. \right) ; \quad \text{Re}(z) > 0
\]

\[
\frac{1}{\sqrt{z^2 + 1} \sqrt{z + \sqrt{z^2 + 1}}} K \left( \frac{\sqrt{z^2 + 1} - z}{\sqrt{z^2 + 1} + z} \right) = \frac{\pi}{2 \sqrt{\Gamma(\frac{1}{4})^2}} G_{2,2}^{1,1} \left( z, \frac{1}{2} \left| \frac{1}{2}, \frac{1}{2}, 0, 0 \right. \right) ; \quad \text{Re}(z) > 0
\]

\[
\sqrt{z^2 + 1} - z K \left( 2z \left( \sqrt{z^2 + 1} - z \right) \right) = \frac{\pi}{2 \Gamma\left( \frac{1}{2} \right)} G_{2,2}^{1,1} \left( z, \frac{1}{2} \left| \frac{1}{2}, \frac{1}{2}, 0, 0 \right. \right)
\]

\[
\sqrt{z^2 + 1} - z K \left( 2z \left( \sqrt{z^2 + 1} - z \right) \right) = \frac{\pi}{2 \Gamma\left( \frac{3}{4} \right)} G_{2,2}^{1,1} \left( z, \frac{1}{2} \left| \frac{1}{2}, \frac{1}{2}, 0, 0 \right. \right)
\]

\[
\frac{1}{\sqrt{z^2 + 1} - z} K \left( -2z \left( z + \sqrt{z^2 + 1} \right) \right) = \frac{\pi}{2 \Gamma\left( \frac{1}{4} \right)} G_{2,2}^{1,1} \left( z, \frac{1}{2} \left| \frac{1}{2}, \frac{1}{2}, 0, 0 \right. \right)
\]

\[
\frac{1}{\sqrt{z^2 + 1} - z} K \left( -2z \left( z + \sqrt{z^2 + 1} \right) \right) = \frac{\pi}{2 \Gamma\left( \frac{1}{4} \right)} G_{2,2}^{1,1} \left( z, \frac{1}{2} \left| \frac{1}{2}, \frac{1}{2}, 0, 0 \right. \right)
\]

\[
\left( \sqrt{z^2 + 1} - z \right) K \left( \frac{4z \sqrt{z^2 + 1}}{\left( z + \sqrt{z^2 + 1} \right)^2} \right) = \frac{1}{2} G_{2,2}^{1,1} \left( z, \frac{1}{2} \left| \frac{1}{2}, \frac{1}{2}, 0, 0 \right. \right) ; \quad i z \notin (-\infty, -1) \land i z \notin (1, \infty)
\]

\[
\left( z + \sqrt{z^2 + 1} \right) K \left( \frac{4z \sqrt{z^2 + 1}}{\left( z - \sqrt{z^2 + 1} \right)^2} \right) = \frac{1}{2} G_{2,2}^{1,1} \left( z, \frac{1}{2} \left| \frac{1}{2}, \frac{1}{2}, 0, 0 \right. \right) ; \quad i z \notin (-\infty, -1) \land i z \notin (1, \infty)
\]
Generalized cases involving unit step $\theta$

\[ \theta(1 - |z|) K \left( \frac{1 - z}{2} \right) = \frac{\pi}{2} G_{2,2}^{0,0} \left( z, \frac{1}{2} \left| \frac{3}{4}, \frac{3}{4} \right| 0, \frac{1}{2} \right) \]

\[ \theta(1 - |z|) K \left( \frac{z - 1}{2} \right) = \frac{\pi}{2} G_{2,2}^{0,0} \left( z, \frac{1}{2} \left| \frac{1}{2}, \frac{1}{4} \right| \right) \]

\[ \theta(|z| - 1) K \left( \frac{1 - z}{2} \right) = \frac{\pi}{2} G_{2,2}^{0,0} \left( z, \frac{1}{2} \left| \frac{3}{4}, \frac{3}{4} \right| 0, \frac{1}{2} \right) \]

\[ \theta(|z| - 1) K \left( \frac{z - 1}{2} \right) = \frac{\pi}{2} G_{2,2}^{0,0} \left( z, \frac{1}{2} \left| \frac{1}{2}, \frac{1}{4} \right| \right) \]
\[\frac{\theta(1-|z|)}{\sqrt{1+|z|}} \frac{1}{1+|z|} = \frac{\pi}{2\sqrt{2}} G^{1,0}_{2,2} \left( \frac{1}{2}, \frac{1}{2} \right) \left( \frac{1}{2}, 0, 0 \right) / \text{if } z \notin (-1, 0)\]

\[\frac{\theta(|z|-1)}{\sqrt{1+|z|}} \frac{1}{1+|z|} = \frac{\pi}{2\sqrt{2}} G^{1,0}_{2,2} \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2} \right) \left( \frac{1}{2}, 0, \frac{1}{2} \right) / \text{if } z \notin (-\infty, -1)\]

\[\frac{\theta(|z|-1)}{\sqrt{1+|z|}} \frac{1}{1+|z|} = \frac{\pi}{2\sqrt{2}} G^{1,0}_{2,2} \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2} \right) \left( \frac{1}{2}, 0, 0 \right) \]

\[\frac{\theta(1-|z|)}{\sqrt{1+|z|}} \frac{1}{1+|z|} = \frac{\pi}{2\sqrt{2}} G^{1,0}_{2,2} \left( \frac{1}{2}, \frac{1}{2} \right) \left( \frac{1}{2}, 0, 0 \right) / \text{if } z \notin (-1, 0)\]

\[\frac{\theta(|z|-1)}{\sqrt{1+|z|}} \frac{1}{1+|z|} = \frac{\pi}{2\sqrt{2}} G^{1,0}_{2,2} \left( \frac{1}{2}, \frac{1}{2} \right) \left( \frac{1}{2}, 0, 0 \right) / \text{if } z \notin (-\infty, -1)\]

\[\theta(|z|-1) \sqrt{z - \sqrt{z^2 - 1}} K \left( 2 \left( z - \sqrt{z^2 - 1} \right) \sqrt{z^2 - 1} \right) = \frac{\pi}{2} G^{1,0}_{2,2} \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2} \right) / \text{Re}(z) > 0\]

\[\frac{\theta(|z|-1)}{\sqrt{z - \sqrt{z^2 - 1}}} K \left( 2 \sqrt{z^2 - 1} \frac{z + \sqrt{z^2 - 1}}{z + \sqrt{z^2 - 1}} \right) = \frac{\pi}{2} G^{1,0}_{2,2} \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2} \right) / \text{Re}(z) > 0\]

\[\frac{\theta(1-|z|)}{\sqrt{z + \sqrt{z^2 - 1}}} K \left( 2 \frac{z + \sqrt{z^2 - 1}}{z + \sqrt{z^2 - 1}} \right) = \frac{\pi}{2} G^{1,0}_{2,2} \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2} \right) / \text{Re}(z) > 0\]

\[\frac{\theta(|z|-1)}{\sqrt{z + \sqrt{z^2 - 1}}} K \left( 2 \sqrt{z^2 - 1} \frac{z + \sqrt{z^2 - 1}}{z + \sqrt{z^2 - 1}} \right) = \frac{\pi}{2} G^{1,0}_{2,2} \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2} \right) / \text{Re}(z) > 0\]
\[
\theta(1 - |z|) \sqrt{z + \sqrt{z^2 - 1}} K \left( \frac{2 \sqrt{z^2 - 1}}{\sqrt{z^2 - 1} - z} \right) = \frac{\pi}{2} G^{2,0}_{2,2} \left( \begin{array}{c|c}
\frac{3}{4} & \frac{3}{4} \\
\frac{1}{2} & \frac{1}{2}
\end{array} \right) ; \quad \text{Re}(z) > 0
\]

\[
\frac{\theta(1 - |z|)}{\sqrt{z - \sqrt{z^2 - 1}}} K \left( \frac{2 \sqrt{z^2 - 1}}{\sqrt{z^2 - 1} - z} \right) = \frac{\pi}{2} G^{2,0}_{2,2} \left( \begin{array}{c|c}
\frac{3}{4} & \frac{3}{4} \\
\frac{1}{2} & \frac{1}{2}
\end{array} \right) ; \quad \text{Re}(z) > 0
\]

\[
\frac{\theta(|z| - 1)}{\sqrt{z - \sqrt{z^2 - 1}}} K \left( \frac{2 \sqrt{z^2 - 1}}{\sqrt{z^2 - 1} - z} \right) = \frac{\pi}{2} G^{0,2}_{2,2} \left( \begin{array}{c|c}
\frac{3}{4} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{2}
\end{array} \right) ; \quad \text{Re}(z) > 0
\]

Generalized cases involving \( \text{sgn} \)

\[
\text{sgn} \left( 1 - |z| \right) \frac{4 z}{1 - z} K \left( - \frac{4 z}{(z - 1)^2} \right) = \frac{\pi}{2} G^{1,1}_{2,2} \left( \begin{array}{c|c}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{array} \right)
\]

Generalized cases involving powers of complete elliptic integral \( K \)

\[
K \left( \frac{z - \sqrt{z^2 + 1}}{2 z} \right)^2 = \frac{\sqrt{\pi}}{4} G^{3,1}_{3,3} \left( \begin{array}{c|c}
\frac{1}{2} & 1, 1, 1 \\
\frac{1}{2} & 1, 1, 1
\end{array} \right) ; \quad \text{Re}(z) > 0
\]

\[
K \left( \frac{\sqrt{z^2 + 1} - z}{z + \sqrt{z^2 + 1}} \right)^2 = \frac{\sqrt{\pi}}{8} G^{3,1}_{3,3} \left( \begin{array}{c|c}
\frac{1}{2} & 1, 1, 1 \\
\frac{1}{2} & 1, 1, 1
\end{array} \right) ; \quad \text{Re}(z) > 0
\]

Through other functions

Involving incomplete elliptic integrals

\[
K(z) = \Pi \left( 0 ; \frac{\pi}{2} \bigg| \frac{1}{2} z \right)
\]

\[
K(z) = \Pi(0 \bigg| \frac{1}{2} z)
\]

\[
K(z) = F \left( \frac{\pi}{2} \bigg| \frac{1}{2} z \right)
\]

\[
K(z) = \frac{1}{\sqrt{z}} F \left( \sin^{-1}(\sqrt{z}) \bigg| \frac{1}{2} \right)
\]
Involving elliptic theta functions

\[ K(m) = \frac{\pi}{2} \Theta(0, q(m))^2 \]

Involving inverse Jacobi functions

\[ K(z) = \text{sn}^{-1}(1 | z) \]
\[ K(z) = \text{dn}^{-1}\left(\sqrt{1 - z} | z\right) \]
\[ K(z) = \text{cn}^{-1}(0 | z) /; z \in \mathbb{R} \land z < 1 \]

Involving some elliptic-type functions

\[ K(z) = \frac{\pi}{2 \text{am}(1, \sqrt{1 - z})} \]
\[ K\left(q^{-1}\left(\exp\left(\frac{i \pi \omega_2}{\omega_1}\right)\right)\right) = \sqrt{e_1 - e_3} /; \]
\[ \{e_1, e_2, e_3\} = \{\text{sn}(\omega_1; g_2, g_3), \text{sn}(\omega_1 + \omega_2; g_2, g_3), \text{sn}(\omega_2; g_2, g_3)\} \land \{g_2, g_3\} = \{g_2(\omega_1, \omega_2), g_3(\omega_1, \omega_2)\} \]
\[ \frac{K(1 - z)}{K(z)} = -i \frac{\omega_2}{\omega_1} /; z = q^{-1}\left(\exp\left(\frac{i \pi \omega_2}{\omega_1}\right)\right) \land \{\omega_1, \omega_2\} = \{\omega_1(\omega_2, g_2), \omega_2(\omega_2, g_2)\} \]

Involving Legendre functions

\[ K(z) = \frac{\pi}{2 \text{P}^{-1}_1(1 - 2z)} \]
\[ K(z) = Q^{-1}_2(2z - 1) \]

Involving some hypergeometric-type functions

\[ K(z) = F_1\left( \begin{array}{c} 1/2, 1/2, 1/2; 3/2 \end{array}; 1, z \right) \]

Representations through equivalent functions

With inverse function

\[ \text{am}(K(m) | m) = -\frac{\pi}{2} \]
With related functions

\[ E(z) K(1 - z) - K(z) K(1 - z) + E(1 - z) K(z) = \frac{\pi}{2} \]

\[ K(2) = \frac{\sqrt{2} \, \pi^{3/2} \Gamma\left(\frac{3}{2}\right)^2 - \pi^2 E(2)}{2 \Gamma\left(\frac{3}{2}\right)^4} \]

Theorems

The period \( T \) of a mathematical pendulum in a gravitational field

The period \( T \) of a mathematical pendulum of length \( l \) in a gravitational field with acceleration \( g \) and maximal angle of excursion \( \alpha \) is given by \( T = 4 \sqrt{\frac{l}{g}} K(\sin^2(\frac{\alpha}{2})) \).

The partition function for a one-dimensional monatomic ideal classical gas

The partition function \( Z \) for a one-dimensional monatomic ideal classical gas of \( n \) atoms in a box of length \( l \) at temperature \( T \) is given by \( Z = \frac{1}{2^n n!} \left( \frac{2}{\pi} K\left( q^{-1} \left( \exp\left( -\frac{3\lambda T^3}{8\pi F} \right) \right) \right) - 1 \right)^n \), where \( \lambda(T) \) is the thermal de Broglie wavelength.

The magnetic induction of an infinitely long solenoid

The magnetic induction \( B \) of an infinitely long solenoid formed by a wire (parametrized by \( \phi \)) \( \{R \cos(\phi), R \sin(\phi), R \phi \tan(\alpha)\} \) carrying the current \( i_0 \) is at the center line is given by

\[ B \propto \{0, -i_0(\cot(\alpha) K_0(\cot(\alpha)) + K_1(\cot(\alpha))), i_0 \frac{\cot(\alpha)}{2\pi R} \}. \]

The lattice Green function for the body-centered cubic lattice

The lattice Green function \( G(\varepsilon) = \frac{1}{\pi^2} \int_0^{\beta} \int_0^{\beta} \int_0^{\beta} (1 - \frac{\varepsilon}{5} (\cos(x) \cos(y) \cos(z)))^{-1} \, dx \, dy \, dz \) for the simple cubic lattice can be expressed as

\[ \frac{4}{\pi^2} \sqrt{1 - \frac{3}{4} \alpha^2} \frac{1}{1 - \alpha} K\left( \frac{1}{2} + \frac{\beta}{4 \sqrt{4 - \beta}} - \frac{(2 - \beta)}{4} \sqrt{1 - \beta} \right) K\left( \frac{1}{2} - \frac{\beta}{4 \sqrt{4 - \beta}} - \frac{(2 - \beta)}{4} \sqrt{1 - \beta} \right) /; \]

\[ \alpha = \frac{1}{2} + \frac{\varepsilon^2}{6} - \frac{1}{2} \sqrt{1 - \varepsilon^2} \sqrt{1 - \frac{\varepsilon^2}{9}} \quad \text{and} \quad \beta = \frac{\alpha}{\alpha - 1}. \]
The probability that a random walk in three dimensions will return to its origin

The probability $p_0$ that a random walk in three dimensions will return to its point of origin is given by

$$p_0 = 1 - \frac{\pi^2}{72} \left(6 + 2\sqrt{3} + \sqrt{6}\right) K\left(35 + 24\sqrt{2} - 20\sqrt{3} - 14\sqrt{6}\right)^{-2} \approx 0.34053732955099914283 \ldots$$

History

– A. M. Legendre (1811, 1825)
– C. G. J. Jacobi (1829)
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