EllipticLog

Notations

Traditional name

Elliptic logarithm

Traditional notation

$\mathrm{elog}(z_1, z_2; a, b)$

Mathematica StandardForm notation

`EllipticLog[{`z_1`, `z_2`}, {`a`, `b`}]`

Primary definition

09.57.02.0001.01

$$\mathrm{elog}(z_1, z_2; a, b) = \frac{\sqrt{z_2^2}}{2 z_2} \int_{\infty}^{z_1} \frac{1}{\sqrt{t^3 + a t^2 + b t}} \, dt \,/;\, z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0 \bigwedge a \in \mathbb{R} \bigwedge b \in \mathbb{R}$$

General characteristics

Domain and analyticity

$\mathrm{elog}(z_1, z_2; a, b)$ is an analytical function of z_1, z_2, a, b, which is defined in $\mathbb{C}^4$.

09.57.04.0001.01

$$(\{z_1 * z_2\} * \{a * b\}) \longrightarrow \mathrm{elog}(z_1, z_2; a, b) :: (\{\mathbb{C} \otimes \mathbb{C}\} \otimes \{\mathbb{C} \otimes \mathbb{C}\}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.57.04.0002.01

$$\mathrm{elog}\left(\overline{z_1}, \overline{z_2}; \overline{a}, \overline{b}\right) = \overline{\mathrm{elog}(z_1, z_2; a, b)}$$

Periodicity

No periodicity

Poles and essential singularities

The function elog$(z_1, z_2; a, b)$ does not have poles and essential singularities.

09.57.04.0003.01

$$\mathcal{Sing}_{z_1}(\text{elog}(z_1, z_2; a, b)) == \{\}$$

09.57.04.0004.01

$$\mathcal{Sing}_{a}(\text{elog}(z_1, z_2; a, b)) == \{\}$$

09.57.04.0005.01

$$\mathcal{Sing}_{b}(\text{elog}(z_1, z_2; a, b)) == \{\}$$

Branch points

Branch points locations: complicated

Branch cuts

Branch cut locations: complicated

Integral representations

On the real axis

Of the direct function

09.57.07.0001.01

$$\text{elog}(z_1, z_2; a, b) == \frac{\sqrt{z_2^2}}{2 z_2} \int_{\infty}^{z_1} \frac{1}{\sqrt{t^3 + a t^2 + b t}} \, dt \,/;\, z_1^3 + a z_1^2 + b z_1 - z_2^2 == 0 \bigwedge a \in \mathbb{R} \bigwedge b \in \mathbb{R}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

09.57.13.0001.01

$$2 z (b + z (a + z)) w''(z) + (b + z (2 a + 3 z)) w'(z) == 1 \,/;\, w(z) == \text{elog}(z, z_2; a, b) \bigwedge z^3 + a z^2 + b z - z_2^2 == 0$$

Ordinary nonlinear differential equations

09.57.13.0002.01

$$4 \left(z^3 + a z^2 + b z\right) w'(z)^2 == 1 \,/;\, w(z) == \text{elog}(z, z_2; a, b)$$

Differentiation

Low-order differentiation

With respect to z_1

09.57.20.0001.01

$$\frac{\partial \operatorname{elog}(z_1, z_2; a, b)}{\partial z_1} = \frac{1}{2 z_2}$$

Representations through more general functions

Through other functions

Involving some hypergeometric-type functions

09.57.26.0001.01

$$\operatorname{elog}(z_1, z_2; a, b) = -\frac{\sqrt{z_2^2}}{\sqrt{z_1}\, z_2} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{-a - \sqrt{a^2 - 4b}}{2 z_1}, \frac{\sqrt{a^2 - 4b} - a}{2 z_1}\right) /; z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0$$

Representations through equivalent functions

With inverse function

09.57.27.0001.01

$$\operatorname{eexp}(\operatorname{elog}(z_1, z_2; a, b); a, b) = \{z_1, z_2\} /; z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0$$

09.57.27.0002.01

$$\xi = 2 z_2 /; \{\xi, \eta\} = \operatorname{eexp}_z'(\operatorname{elog}(z_1, z_2; a, b); a, b) \bigwedge z_1^3 + a z_1^2 + b z_1 - z_2^2 = 0$$

With related functions

Involving Weierstrass functions

09.57.27.0003.01

$$\operatorname{elog}\left(z_1, \sqrt{z_1^3 + a z_1^2 + b z_1}; a, b\right) = \frac{1}{\sqrt[3]{2}} \wp^{-1}\left(\frac{1}{6} \sqrt[3]{2}\,(a + 3 z_1); \sqrt[3]{4}\left(\frac{a^2}{3} - b\right), \frac{a b}{3} - \frac{2 a^3}{27}\right)$$

Involving incomplete elliptic integrals

09.57.27.0004.01

$$\operatorname{elog}\left(z_1, \sqrt{z_1^3 + a z_1^2 + b z_1}; a, b\right) = -\frac{1}{2} \sqrt{\frac{2\left(a - \sqrt{a^2 - 4b}\right)}{b}}\, F\left(\cot^{-1}\left(\sqrt{\frac{2 z_1}{a + \sqrt{a^2 - 4b}}}\right) \middle| \frac{2\sqrt{a^2 - 4b}}{a + \sqrt{a^2 - 4b}}\right)$$

History

– D. Masser (1975)

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