EllipticTheta4

Notations

Traditional name

Jacobi theta function ϑ_4

Traditional notation

$\vartheta_4(z, q)$

Mathematica StandardForm notation

`EllipticTheta[4, z, q]`

Primary definition

09.04.02.0001.01

$$\vartheta_4(z, q) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2} \cos(2 k z) \;/;\; |q| < 1$$

Specific values

Specialized values

For fixed *z*

09.04.03.0001.01

$$\vartheta_4(z, 0) = 1$$

For fixed *q*

09.04.03.0006.01

$$\vartheta_4(0, q) = \frac{1}{\eta\left(-\frac{i \log(q)}{\pi}\right)} \eta\left(-\frac{i \log(q)}{2 \pi}\right)^2$$

09.04.03.0003.01

$$\vartheta_4\left(0, e^{\pi i \tau}\right) = \frac{1}{\eta(\tau)} \eta\left(\frac{\tau}{2}\right)^2 \;/;\; \operatorname{Im}(\tau) > 0$$

09.04.03.0007.01

$$\vartheta_4(0, q) = \sqrt{\frac{2}{\pi}} \sqrt{K\left(q^{-1}(-q)\right)}$$

09.04.03.0004.01

$$\vartheta_4\left(0, e^{-\frac{i\pi}{\tau}}\right) == \frac{\sqrt{\tau}}{\sqrt{i}}\,\vartheta_2\left(0, e^{i\pi\tau}\right)$$

09.04.03.0008.01

$$\vartheta_4(0, q) == \vartheta_3(0, -q)$$

09.04.03.0005.02

$$\vartheta_4(0, q) == \sqrt{\frac{2}{\pi}}\,\sqrt[4]{1 - q^{-1}(q)}\,\sqrt{K\left(q^{-1}(q)\right)}$$

09.04.03.0009.01

$$\vartheta_4\left(\frac{\pi}{2}, q\right) == \frac{1}{\eta\left(-\frac{2\,i\log(q)}{\pi}\right)^2 \eta\left(-\frac{i\log(q)}{2\pi}\right)^2}\,\eta\left(-\frac{i\log(q)}{\pi}\right)^5$$

09.04.03.0010.01

$$\vartheta_4(\pi\,m, q) == \frac{1}{\eta\left(-\frac{i\log(q)}{\pi}\right)}\,\eta\left(-\frac{i\log(q)}{2\pi}\right)^2 \;/;\; m \in \mathbb{Z}$$

09.04.03.0011.01

$$\vartheta_4\left(\pi\,m + \frac{\pi}{2}, q\right) == \frac{1}{\eta\left(-\frac{2\,i\log(q)}{\pi}\right)^2 \eta\left(-\frac{i\log(q)}{2\pi}\right)^2}\,\eta\left(-\frac{i\log(q)}{\pi}\right)^5 \;/;\; m \in \mathbb{Z}$$

09.04.03.0002.01

$$\vartheta_4\left(m\,\pi + (2\,n+1)\,\frac{\pi\,\tau}{2}, q\right) == 0 \;/;\; \{m, n\} \in \mathbb{Z} \bigwedge q == e^{i\pi\tau}$$

General characteristics

Domain and analyticity

$\vartheta_4(z, q)$ is an analytic function of z and q for $z, q \in \mathbb{C}$ and $|q| < 1$.

09.04.04.0001.01

$$(4 * z * q) \longrightarrow \vartheta_4(z, q) :: (\{4\} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

$\vartheta_4(z, q)$ is an even function with respect to z.

09.04.04.0002.01

$$\vartheta_4(-z, q) == \vartheta_4(z, q)$$

09.04.04.0003.02

$$\vartheta_4(z, -q) == \vartheta_3(z, q)$$

Mirror symmetry

09.04.04.0004.01

$\vartheta_4(\bar{z}, \bar{q}) = \overline{\vartheta_4(z, q)}$

Periodicity

The function $\vartheta_4(z, q)$ is a periodic function with respect to z with period π and a quasi-period $i \log(q)$.

09.04.04.0005.01

$\vartheta_4(z + \pi, q) = \vartheta_4(z, q)$

09.04.04.0017.01

$\vartheta_4(z + m\pi, q) = \vartheta_4(z, q)$

09.04.04.0006.01

$$\vartheta_4(z + \pi\tau, q) = -\frac{e^{-2iz}}{q} \vartheta_4(z, q) \,/;\, q = e^{i\pi\tau} \bigwedge \operatorname{Im}(\tau) > 0$$

09.04.04.0008.01

$$\vartheta_4(z + i\log(q), q) = -\frac{e^{2iz}}{q} \vartheta_4(z, q)$$

09.04.04.0018.01

$$\vartheta_4(z + m\pi\tau, q) = (-1)^m q^{-m} e^{-i(2mz+(m-1)m\pi\tau)} \vartheta_4(z, q) \,/;\, m \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$$

09.04.04.0009.01

$$\vartheta_4(z + i\, m \log(q), q) = (-1)^m q^{-m^2} e^{2miz} \vartheta_4(z, q) \,/;\, m \in \mathbb{Z}$$

09.04.04.0007.01

$$\vartheta_4(z + m\pi + n\pi\tau, q) = (-1)^n q^{-n^2} e^{-2nzi} \vartheta_4(z, q) \,/;\, \{m, n\} \in \mathbb{Z} \bigwedge q = e^{i\pi\tau}$$

Poles and essential singularities

With respect to *q*

The function $\vartheta_4(z, q)$ does not have poles and essential singularities inside of the unit circle $|q| < 1$

09.04.04.0010.01

$\mathcal{Sing}_q(\vartheta_4(z, q)) = \{\}$

With respect to *z*

09.04.04.0011.01

$\mathcal{Sing}_z(\vartheta_4(z, q)) = \{\}$

Branch points

With respect to *q*

For fixed z, the function $\vartheta_4(z, q)$ does not have branch points.

09.04.04.0012.01

$\mathcal{BP}_q(\vartheta_4(z, q)) = \{\}$

With respect to *z*

For fixed q, the function $\vartheta_4(z, q)$ does not have branch points.

09.04.04.0013.01

$$\mathcal{BP}_z(\vartheta_4(z, q)) == \{\}$$

Branch cuts

With respect to *q*

For fixed z, the function $\vartheta_4(z, q)$ does not have branch cuts.

09.04.04.0014.01

$$\mathcal{BC}_q(\vartheta_4(z, q)) == \{\}$$

With respect to *z*

For fixed q, the function $\vartheta_4(z, q)$ does not have branch cuts.

09.04.04.0015.01

$$\mathcal{BC}_z(\vartheta_4(z, q)) == \{\}$$

Natural boundary of analyticity

The unit circle $|q| == 1$ is the natural boundary of the region of analyticity.

09.04.04.0016.01

$$\mathcal{AB}_z(\vartheta_4(q, z)) == \left\{e^{i(-\pi,\pi)}\right\}$$

Series representations

q-series

Expansions at generic point $z == z_0$

09.04.06.0015.01

$$\vartheta_4(z, q) \propto \vartheta_4(z_0, q) + \vartheta_4^{(1,0)}(z_0, q)\,(z - z_0) + \frac{\vartheta_4^{(2,0)}(z_0, q)}{2}\,(z - z_0)^2 + \frac{\vartheta_4^{(3,0)}(z_0, q)}{6}\,(z - z_0)^3 + O\left((z - z_0)^4\right)$$

09.04.06.0016.01

$$\vartheta_4(z, q) \propto \vartheta_4(z_0, q) + \vartheta_4'(z_0, q)\,(z - z_0) + \frac{\vartheta_4^{(2,0)}(z_0, q)}{2}\,(z - z_0)^2 + \frac{\vartheta_4^{(3,0)}(z_0, q)}{6}\,(z - z_0)^3 + O\left((z - z_0)^4\right)$$

09.04.06.0017.01

$$\vartheta_4(z, q) = \sum_{k=0}^{\infty} \frac{\vartheta_4^{(k,0)}(z_0, q)}{k!}\,(z - z_0)^k$$

09.04.06.0018.01

$$\vartheta_4(z, q) \propto \vartheta_4(z_0, q)\,(1 + O(z - z_0))$$

Expansions at generic point $q == q_0$

09.04.06.0019.01

$$\vartheta_4(z,q)\propto\vartheta_4(z,q_0)+\vartheta_4^{(0,1)}(z,q_0)\,(q-q_0)+\frac{\vartheta_4^{(0,2)}(z,q_0)}{2}\,(q-q_0)^2+\frac{\vartheta_4^{(0,3)}(z,q_0)}{6}\,(q-q_0)^3+O\left((q-q_0)^4\right)$$

09.04.06.0020.01

$$\vartheta_4(z,q)=\sum_{k=0}^{\infty}\frac{\vartheta_4^{(0,k)}(z,q_0)}{k!}\,(q-q_0)^k$$

09.04.06.0021.01

$$\vartheta_4(z,q)\propto\vartheta_4(z,q_0)\,(1+O(q-q_0))$$

Expansions at $q == 0$

09.04.06.0022.01

$$\vartheta_4(z,q)=1-2\cos(2z)\,q+2\cos(4z)\,q^4-2\cos(6z)\,q^9+2\cos(8z)\,q^{16}+\ldots\;/;\;(q\to 0)$$

09.04.06.0001.01

$$\vartheta_4(z,q)==1+2\sum_{k=1}^{\infty}(-1)^k\,q^{k^2}\cos(2kz)\;/;\;|q|<1$$

09.04.06.0002.01

$$\vartheta_4(z,q)==\sum_{k=-\infty}^{\infty}(-1)^k\,q^{k^2}\,e^{2kiz}$$

09.04.06.0003.01

$$\vartheta_4(0,q)==1+2\sum_{k=1}^{\infty}(-1)^k\,q^{k^2}$$

09.04.06.0023.01

$$\vartheta_4(z,q)\propto 1+O(q)\;/;\;q\to 0$$

Expansions at $q == 1$

09.04.06.0024.01

$$\vartheta_4(z,q)\propto-\frac{2i\sqrt{\pi}}{\sqrt{q-1}}\,e^{-i\pi\left\lfloor-\frac{\arg(q-1)}{2\pi}\right\rfloor}\left(1+\frac{q-1}{4}-\frac{7}{96}(q-1)^2+\ldots\right)e^{\frac{4z^2+\pi^2}{4\log(q)}}\left(\cosh\left(\frac{\pi z}{\log(q)}\right)+e^{\frac{2\pi^2}{\log(q)}}\cosh\left(\frac{3\pi z}{\log(q)}\right)+\ldots\right)/;$$
$$(q\to 1)\bigwedge|q|<1$$

09.04.06.0025.01

$$\vartheta_4(z,q)=\frac{-2i\sqrt{\pi}}{\sqrt{q-1}}\,e^{-i\pi\left\lfloor-\frac{\arg(q-1)}{2\pi}\right\rfloor}\,e^{\frac{4z^2+\pi^2}{4\log(q)}}\sum_{k=0}^{\infty}\binom{k+\frac{1}{2}}{k}\sum_{j=0}^{k}\frac{(-1)^j}{2j+1}\binom{k}{j}p_{j,k}\,(q-1)^k\sum_{m=0}^{\infty}e^{\frac{m(m+1)\pi^2}{\log(q)}}\cosh\left(\frac{(2m+1)\pi z}{\log(q)}\right)/;$$
$$(|q|<1\land|q-1|<1)\bigwedge c_k=\frac{(-1)^{k-1}}{k+1}\bigwedge p_{j,0}=1\bigwedge p_{j,k}=-\frac{1}{k}\sum_{m=1}^{k}(jm-k+m)\,c_m\,p_{j,k-m}\bigwedge k\in\mathbb{N}^+$$

09.04.06.0026.01

$$\vartheta_4(z,q)\propto-\frac{2i\sqrt{\pi}}{\sqrt{q-1}}\,e^{-i\pi\left\lfloor-\frac{\arg(q-1)}{2\pi}\right\rfloor}(1+O(q-1))\,e^{\frac{4z^2+\pi^2}{4\log(q)}}\left(\cosh\left(\frac{\pi z}{\log(q)}\right)+O\left(e^{\frac{2\pi^2}{\log(q)}}\cosh\left(\frac{3\pi z}{\log(q)}\right)\right)\right)/;\;(q\to 1)\bigwedge|q|<1$$

Other q-series representations

09.04.06.0004.01

$$\frac{\vartheta_4'(z, q)}{\vartheta_4(z, q)} = 4 \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \sin(2 n z)$$

09.04.06.0005.01

$$\log\left(\frac{\vartheta_4(a + b, q)}{\vartheta_4(a - b, q)}\right) = 4 \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 - q^{2n}} \sin(2 n a) \sin(2 n b)$$

09.04.06.0006.01

$$\log(\vartheta_4(z, q)) = \log(\vartheta_4(0, q)) + 4 \sum_{r=1}^{\infty} \frac{q^r}{r \left(1 - q^{2r}\right)} \sin^2(r z)$$

09.04.06.0007.01

$$\frac{\vartheta_1'(0, q) \vartheta_4(z, q)}{4 \vartheta_2(0, q) \vartheta_3(z, q)} = \frac{1}{4} + \sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1 + q^{2n}} \cos(2 n z) /; |\mathrm{Im}(z)| < \frac{1}{2} \mathrm{Im}(\tau) \bigwedge q = e^{i \pi \tau}$$

09.04.06.0008.01

$$\frac{\vartheta_1'(0, q) \vartheta_4(z, q)}{4 \vartheta_2(0, q) \vartheta_3(z, q)} = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n \left(q^{4n-2} + \cos(2 z) q^{2n-1}\right)}{q^{4n-2} + 2 \cos(2 z) q^{2n-1} + 1} /; |\mathrm{Im}(z)| < \frac{1}{2} \mathrm{Im}(\tau) \bigwedge q = e^{i \pi \tau}$$

09.04.06.0009.01

$$\frac{\vartheta_1'(0, q) \vartheta_4(z, q)}{4 \vartheta_4(0, q) \vartheta_1(z, q)} = \frac{1}{4} \csc(z) + \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1 - q^{2n-1}} \sin((2 n - 1) z) /; |\mathrm{Im}(z)| < \mathrm{Im}(\tau) \bigwedge q = e^{i \pi \tau}$$

09.04.06.0010.01

$$\frac{\vartheta_1'(0, q) \vartheta_4(z, q)}{4 \vartheta_4(0, q) \vartheta_1(z, q)} = \frac{1}{4} \csc(z) + \sum_{n=1}^{\infty} \frac{\left(1 + q^{2n}\right) q^n \sin(z)}{1 - 2 \cos(2 z) q^{2n} + q^{4n}} /; |\mathrm{Im}(z)| < \mathrm{Im}(\tau) \bigwedge q = e^{i \pi \tau}$$

09.04.06.0011.01

$$\frac{\vartheta_1'(0, q) \vartheta_1(z, q)}{4 \vartheta_4(0, q) \vartheta_4(z, q)} = \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1 - q^{2n-1}} \sin((2 n - 1) z) /; |\mathrm{Im}(z)| < \frac{\mathrm{Im}(\tau)}{2} \bigwedge q = e^{i \pi \tau}$$

09.04.06.0012.01

$$\frac{\vartheta_1'(0, q) \vartheta_1(z, q)}{4 \vartheta_4(0, q) \vartheta_4(z, q)} = \sum_{n=1}^{\infty} \frac{\left(1 + q^{2n-1}\right) q^{n-\frac{1}{2}} \sin(z)}{1 - 2 \cos(2 z) q^{2n-1} + q^{4n-2}} /; |\mathrm{Im}(z)| < \frac{\mathrm{Im}(\tau)}{2} \bigwedge q = e^{i \pi \tau}$$

Other series representations

09.04.06.0027.01

$$\vartheta_4(z, q) = \frac{2 \sqrt{\pi}}{\sqrt{-\log(q)}} e^{\frac{4 z^2 + \pi^2}{4 \log(q)}} \sum_{k=0}^{\infty} e^{\frac{k (k+1) \pi^2}{\log(q)}} \cosh\left(\frac{(2 k + 1) (\pi z)}{\log(q)}\right)$$

09.04.06.0013.01

$$\vartheta_4(z, q) = \exp\left(-\frac{i z^2}{\pi \tau}\right) \sum_{n=-\infty}^{\infty} (-1)^n \exp\left(i \pi \tau \left(n + \frac{z}{\pi \tau}\right)^2\right) /; q = e^{i \pi \tau}$$

09.04.06.0014.01

$$\vartheta_4(z, q) = \frac{\sqrt{i}}{\sqrt{\tau}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{\pi i}{\tau} \left(\frac{z}{\pi} + n - \frac{1}{2}\right)^2\right) /; q = e^{i \pi \tau}$$

Product representations

09.04.08.0001.01

$$\vartheta_4(0, q) == \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \left(1 - q^{2n-1}\right)^2$$

09.04.08.0002.01

$$\vartheta_4(z, q) == \prod_{k=1}^{\infty} \left(1 - q^{2j}\right) \left(1 - 2\, q^{2k-1} \cos(2\, z) + q^{4k-2}\right)$$

Differential equations

Ordinary nonlinear differential equations

09.04.13.0001.01

$$w'(z)^2 == \left(\vartheta_2(0, q)^2 - w(z)^2\, \vartheta_3(0, q)^2\right) \left(\vartheta_3(0, q)^2 - w(z)^2\, \vartheta_2(0, q)^2\right) /;\ w(z) == \frac{\vartheta_1(z, q)}{\vartheta_4(z, q)}$$

09.04.13.0002.01

$$\pi^2 \left(w(\tau)\, w''(\tau) - 3\, w'(\tau)^2\right)^2 w(\tau)^{10} - 32 \left(3\, w'(\tau)^2 - w(\tau)\, w''(\tau)\right)^3 + \left(30\, w'(\tau)^3 - 15\, w(\tau)\, w''(\tau)\, w'(\tau) + w(\tau)^2\, w^{(3)}(\tau)\right)^2 == 0 /;\ w(\tau) == \vartheta_4\left(0, e^{i \pi \tau}\right)$$

Partial differential equations

The elliptic theta functions satisfy the (one-dimensional) heat equation:

09.04.13.0003.01

$$\frac{\partial \vartheta_4(z, q)}{\partial \tau} == -\frac{\pi\, i}{4} \frac{\partial^2 \vartheta_4(z, q)}{\partial z^2} /;\ q == e^{i \pi \tau}$$

09.04.13.0004.01

$$4\, q \frac{\partial \vartheta_4(z, q)}{\partial q} + \frac{\partial^2 \vartheta_4(z, q)}{\partial z^2} == 0$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.04.16.0005.01

$$\vartheta_4(z, q) = \frac{\sqrt{\pi}\, e^{\frac{4 z^2 + \pi^2}{4 \log(q)}}}{\sqrt[4]{e^{\frac{\pi^2}{\log(q)}}}\, \sqrt{-\log(q)}}\, \vartheta_2\left(\frac{i\, \pi\, z}{\log(q)}, e^{\frac{\pi^2}{\log(q)}}\right)$$

09.04.16.0001.01

$$\vartheta_4\left(\frac{z}{\tau}, e^{-\frac{i \pi}{\tau}}\right) == \frac{\sqrt{\tau}}{\sqrt{i}} \exp\left(\frac{i\, z^2}{\pi\, \tau}\right) \vartheta_2(z, q) /;\ q == e^{i \pi \tau}$$

n th root of *q*

09.04.16.0002.01

$$\vartheta_4\left(z, q^{1/n}\right) = \left(\prod_{r=1}^{\infty} \frac{1-q^{\frac{2r}{n}}}{\left(1-q^{2r}\right)^n}\right) \prod_{r=-\frac{n-1}{2}}^{\frac{n-1}{2}} \vartheta_4\left(z + \frac{i\, r \log(q)}{n}, q\right) /; \frac{n+1}{2} \in \mathbb{Z}^+$$

Multiple angle formulas

09.04.16.0003.01

$$\vartheta_4(n\, z, q^n) = \frac{\prod_{s=1}^{\infty}\left(1-q^{2ns}\right)}{\prod_{s=1}^{\infty}\left(1-q^{2s}\right)^n} \prod_{r=0}^{n-1} \vartheta_4\left(z + \frac{\pi\, r}{n}, q\right) /; n \in \mathbb{Z}^+$$

09.04.16.0004.01

$$\vartheta_4(n\, z, q^n) = \frac{\prod_{s=1}^{\infty}\left(1-q^{2ns}\right)}{\prod_{s=1}^{\infty}\left(1-q^{2s}\right)^n} \prod_{r=\left\lfloor -\frac{n-1}{2}\right\rfloor}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \vartheta_4\left(z + \frac{\pi\, r}{n}, q\right) /; n \in \mathbb{Z}^+$$

Identities

Functional identities

09.04.17.0001.01

$$\left(\frac{3\,\vartheta_4\left(0, q^9\right)}{\vartheta_4(0, q)} - 1\right)^3 = \frac{9\,\vartheta_4\left(0, q^3\right)^4}{\vartheta_4(0, q)^4} - 1$$

Differentiation

Low-order differentiation

With respect to *z*

09.04.20.0001.01

$$\frac{\partial \vartheta_4(z, q)}{\partial z} = \vartheta_4'(z, q)$$

09.04.20.0002.01

$$\frac{\partial^2 \vartheta_4(z, q)}{\partial z^2} = 8 \sum_{k=1}^{\infty} (-1)^{k-1} q^{k^2} k^2 \cos(2\, k\, z) /; |q| < 1$$

With respect to *q*

09.04.20.0009.01

$$\frac{\partial \vartheta_4(z, q)}{\partial q} = -\frac{1}{4\, q} \vartheta_2(0, q)^2 \vartheta_4(0, q)^2 \frac{\vartheta_3(z, q)^2 \vartheta_4(z, q)}{\vartheta_1(z, q)^2} - \frac{\vartheta_1'(z, q)^2 \vartheta_4(z, q)}{4\, q\, \vartheta_1(z, q)^2} + \frac{1}{2\, q} \vartheta_4(0, q)^2 \frac{\vartheta_1'(z, q)}{\vartheta_1(z, q)^2} \vartheta_2(z, q)\, \vartheta_3(z, q) + \frac{1}{q\, \pi^2} \vartheta_4(z, q) \left(\frac{\pi^2}{12}\left(\vartheta_3(0, q)^4 + \vartheta_4(0, q)^4\right) + \zeta\left(1; g_2\left(1, \frac{\log(q)}{\pi\, i}\right), g_3\left(1, \frac{\log(q)}{\pi\, i}\right)\right)\right)$$

09.04.20.0003.01

$$\frac{\partial \vartheta_4(z, q)}{\partial q} == 2 \sum_{k=1}^{\infty} (-1)^k k^2 q^{k^2-1} \cos(2 k z) /; |q| < 1$$

09.04.20.0004.01

$$\frac{\partial^2 \vartheta_4(z, q)}{\partial q^2} == \frac{2}{q^2} \sum_{k=2}^{\infty} (-1)^k q^{k^2} k^2 \left(k^2 - 1\right) \cos(2 k z) /; |q| < 1$$

Symbolic differentiation

With respect to *z*

09.04.20.0005.01

$$\frac{\partial^n \vartheta_4(z, q)}{\partial z^n} == 2^{n+1} \sum_{k=1}^{\infty} (-1)^k q^{k^2} k^n \cos\left(\frac{\pi n}{2} + 2 k z\right) /; |q| < 1 \bigwedge n \in \mathbb{N}^+$$

With respect to *q*

09.04.20.0006.01

$$\frac{\partial^n \vartheta_4(z, q)}{\partial q^n} == 2 \sum_{k=1}^{\infty} (-1)^k q^{k^2-n} \left(k^2 - n + 1\right)_n \cos(2 k z) /; |q| < 1 \bigwedge n \in \mathbb{N}^+$$

Fractional integro-differentiation

With respect to *z*

09.04.20.0007.01

$$\frac{\partial^\alpha \vartheta_4(z, q)}{\partial z^\alpha} == 2^{\alpha+1} \sqrt{\pi}\, z^{-\alpha} \sum_{k=1}^{\infty} (-1)^k q^{k^2} {}_1\tilde{F}_2\left(1; \frac{1-\alpha}{2}, 1 - \frac{\alpha}{2}; -k^2 z^2\right) /; |q| < 1$$

With respect to *q*

09.04.20.0008.01

$$\frac{\partial^\alpha \vartheta_4(z, q)}{\partial q^\alpha} == 2 q^{-\alpha} \sum_{k=1}^{\infty} \frac{(-1)^k q^{k^2} \Gamma\left(k^2 + 1\right) \cos(2 k z)}{\Gamma\left(k^2 - \alpha + 1\right)} + \frac{q^{-\alpha}}{\Gamma(1-\alpha)} /; |q| < 1$$

Integration

Indefinite integration

Involving only one direct function

09.04.21.0001.01

$$\int \vartheta_4(z, q)\, dz == z + \sum_{k=1}^{\infty} \frac{(-1)^k q^{k^2} \sin(2 k z)}{k} /; |q| < 1$$

Involving only one direct function with respect to *q*

09.04.21.0002.01

$$\int \vartheta_4(z, q)\, dq = q + 2 \sum_{k=1}^{\infty} \frac{(-1)^k q^{k^2+1} \cos(2 k z)}{k^2 + 1} \;/;\; |q| < 1$$

Representations through equivalent functions

With related functions

Involving theta functions

Involving $\vartheta_1(z, q)$

09.04.27.0003.02

$$\vartheta_4(z, q) = i\, e^{i z} \sqrt[4]{q}\, \vartheta_1\left(z - \frac{\pi \tau}{2}, q\right) /;\; q = e^{-i \pi \tau}$$

09.04.27.0004.02

$$\vartheta_4(z, q) = i^{2 m+1} e^{i (2 m+1) z} q^{\left(m+\frac{1}{2}\right)^2} \vartheta_1\left(z - \frac{\pi \tau}{2} (2 m + 1), q\right) /;\; m \in \mathbb{Z} \bigwedge q = e^{-i \pi \tau}$$

09.04.27.0005.02

$$\vartheta_4(z, q) = i\, e^{-i z} \sqrt[4]{q}\, \vartheta_1\left(z + \frac{1}{2} i \log(q), q\right)$$

09.04.27.0006.02

$$\vartheta_4(z, q) = i^{2 m+1} e^{-i (2 m+1) z} q^{\frac{1}{4} (2 m+1)^2} \vartheta_1\left(z + \frac{1}{2} (i \log(q)) (2 m + 1), q\right) /;\; m \in \mathbb{Z}$$

Involving $\vartheta_2(z, q)$

09.04.27.0007.02

$$\vartheta_4(z, q) = -i \sqrt[4]{q}\, e^{-i z} \vartheta_2\left(z - \frac{1}{2} \pi (\tau + 1), q\right) /;\; q = e^{i \pi \tau}$$

09.04.27.0014.01

$$\vartheta_4(z, q) = i\, q^{m^2+m+\frac{1}{4}} e^{i (2 m+1) z} \vartheta_2\left(z - \frac{\pi}{2} (2 m + 1) (1 - \tau), q\right) /;\; m \in \mathbb{Z} \bigwedge q = e^{i \pi \tau}$$

09.04.27.0015.01

$$\vartheta_4(z, q) = i \sqrt[4]{q}\, e^{-i z} \vartheta_2\left(z - \frac{1}{2} (\pi - i \log(q)), q\right)$$

09.04.27.0016.01

$$\vartheta_4(z, q) = -i\, q^{m^2+m+\frac{1}{4}} e^{i (2 m+1) z} \vartheta_2\left(z - \frac{1}{2} (2 m + 1) (i \log(q) + \pi), q\right) /;\; m \in \mathbb{Z}$$

Involving $\vartheta_3(z, q)$

09.04.27.0017.01

$$\vartheta_4(z, q) = \vartheta_3(z, -q)$$

09.04.27.0001.02

$$\vartheta_4(z, q) = \vartheta_3\left(z - \frac{\pi}{2}, q\right)$$

09.04.27.0018.01

$$\vartheta_4(z, q) = \vartheta_3\left(z + \frac{\pi}{2}, q\right)$$

09.04.27.0002.02

$$\vartheta_4(z, q) = \vartheta_3\left(\frac{\pi}{2}(2m+1) + z, q\right) /; m \in \mathbb{Z}$$

Involving Jacobi functions

09.04.27.0008.02

$$\frac{\vartheta_4(z, q(m))}{\vartheta_1(z, q(m))} = \frac{1}{\sqrt[4]{m}} \operatorname{ns}\left(\frac{2K(m)z}{\pi} \middle| m\right)$$

09.04.27.0009.02

$$\frac{\vartheta_4(z, q(m))}{\vartheta_2(z, q(m))} = \frac{\sqrt[4]{1-m}}{\sqrt[4]{m}} \operatorname{nc}\left(\frac{2K(m)z}{\pi} \middle| m\right)$$

09.04.27.0010.02

$$\frac{\vartheta_4(z, q(m))}{\vartheta_3(z, q(m))} = \sqrt[4]{1-m} \operatorname{nd}\left(\frac{2K(m)z}{\pi} \middle| m\right)$$

Involving Weierstrass functions

09.04.27.0011.01

$$\vartheta_4(z, q) = \left(\prod_{n=1}^{\infty}\left(1 - q^{2n}\right)\right)\left(\prod_{n=1}^{\infty}\left(1 - q^{2n-1}\right)\right)^2 \exp\left(-\frac{2\eta_1\omega_1 z^2}{\pi^2}\right)\sigma_3\left(\frac{2\omega_1 z}{\pi}; g_2, g_3\right) /;$$

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \eta_1 = \zeta(\omega_1; g_2, g_3) \bigwedge q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

09.04.27.0012.01

$$\frac{\vartheta_4(z, q)}{\vartheta_4(0, q)} = \exp\left(-\frac{2\eta_1\omega_1 z^2}{\pi^2}\right)\sigma_3\left(\frac{2\omega_1 z}{\pi}; g_2, g_3\right) /;$$

$$\{\omega_1, \omega_3\} = \{\omega_1(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge \eta_1 = \zeta(\omega_1; g_2, g_3) \bigwedge q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right)$$

09.04.27.0013.01

$$\frac{\vartheta_4'(z, q)}{\vartheta_4(z, q)} = \frac{2\omega_1}{\pi}\zeta\left(\frac{2\omega_1}{\pi}\left(z + \frac{\pi\tau}{2}\right); g_2, g_3\right) - \frac{2\eta_3\omega_1}{\pi} - \frac{4\eta_1 z\omega_1}{\pi^2} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge q = \exp\left(\frac{\pi i \omega_3}{\omega_1}\right) \bigwedge \eta_n = \zeta(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

Zeros

09.04.30.0002.01

$$\vartheta_4\left(\frac{\pi\tau}{2}, q\right) = 0 /; q = e^{i\pi\tau}$$

09.04.30.0001.01

$$\vartheta_4\left(m\pi+(2n+1)\frac{\pi\tau}{2},q\right)==0\ /;\ \{m,n\}\in\mathbb{Z}\bigwedge q==e^{i\pi\tau}$$

Theorems

Mapping of the interior of the ellipse into the unit disk

The interior of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}==1$ is mapped into the unit disk by

$$w(x+i\,y)==w(z)\ ==\frac{\vartheta_1\left(\sin^{-1}\left(\frac{z}{e}\right),q\right)}{\vartheta_4\left(\sin^{-1}\left(\frac{z}{e}\right),q\right)}\ /;\ q==\left(\frac{a-b}{a+b}\right)^2\bigwedge e=\sqrt{a^2-b^2}\ .$$

Solution set of the Halphen equations

The functions $w_1(z)==2\,\frac{\partial\log\left(\vartheta_4\left(0,e^{i\pi\tau}\right)\right)}{\partial\tau}\bigwedge w_2(z)==2\,\frac{\partial\log\left(\vartheta_2\left(0,e^{i\pi\tau}\right)\right)}{\partial\tau}\bigwedge w_3(z)==2\,\frac{\partial\log\left(\vartheta_3\left(0,e^{i\pi\tau}\right)\right)}{\partial\tau}$ are a solution set of the Halphen equations

$$w_1'(z)==w_1(z)\,(w_2(z)+w_3(z))-w_2(z)\,w_3(z)\bigwedge w_2'(z)==w_2(z)\,(w_1(z)+w_3(z))-w_1(z)\,w_3(z)\bigwedge w_3'(z)==w_3(z)\,(w_1(z)+w_2(z))-w_1(z)\,w_2(z).$$

History

– J. Bernoulli (1713); L. Euler; J. Fourier; C. G. J. Jacobi (1827); C. W. Borchardt (1838); K. Weierstrass (1862–1863)

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