EulerGamma

Notations

Traditional name

Euler–Mascheroni constant

Traditional notation

γ

Mathematica StandardForm notation

`EulerGamma`

Primary definition

02.06.02.0001.01

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log(n) \right)$$

Euler formula

Specific values

02.06.03.0001.01

$$\gamma = 0.577215664901532860606512090082402431042159335939923598805767234884867726777664670936947063 \ldots$$

Above approximate numerical value of γ shows 90 decimal digits.

General characteristics

The Euler-Mascheroni number γ is a constant. It is a positive real number. Whether γ is irrational or transcendental over $\mathbb{Q}$ are not known.

Series representations

Generalized power series

02.06.06.0019.01

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \log\left(\frac{k+1}{k} \right) \right)$$

02.06.06.0001.01

$$\gamma = \sum_{k=2}^{\infty}\left(\log\left(1-\frac{1}{k}\right)+\frac{1}{k}\right)+1$$

02.06.06.0002.01

$$\gamma = \frac{\log(2)}{2}+\frac{1}{\log(2)}\sum_{k=2}^{\infty}\frac{(-1)^k\log(k)}{k}$$

02.06.06.0003.01

$$\gamma = \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left\lfloor\log_2(k)\right\rfloor$$

02.06.06.0004.01

$$\gamma = \log(2)-\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{4^k(2k+1)}$$

02.06.06.0005.01

$$\gamma = \sum_{k=2}^{\infty}\frac{(-1)^k}{k}\zeta(k)$$

02.06.06.0006.01

$$\gamma = 1-\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k}$$

02.06.06.0007.01

$$\gamma = \sum_{k=2}^{\infty}\frac{(k-1)(\zeta(k)-1)}{k}$$

02.06.06.0008.01

$$\gamma = 1-\frac{\log(2)}{2}-\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{2k+1}$$

02.06.06.0010.01

$$\gamma = 1-\log\left(\frac{3}{2}\right)-\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{4^k(2k+1)}$$

02.06.06.0011.01

$$\gamma = 1-\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(k+1)(2k+1)}$$

02.06.06.0012.01

$$\gamma = 1-\log(2)+\sum_{k=2}^{\infty}\frac{(-1)^k(\zeta(k)-1)}{k}$$

02.06.06.0013.02

$$\gamma = \frac{5}{4}-\log(2)-\frac{1}{2}\sum_{k=3}^{\infty}\frac{(-1)^k(k-2)(\zeta(k)-1)}{k}$$

02.06.06.0014.01

$$\gamma == \frac{3}{2} - \log(2) - \sum_{k=2}^{\infty} \frac{(-1)^k (k-1)}{k} (\zeta(k) - 1)$$

02.06.06.0015.01

$$\gamma == \log\left(\frac{4}{\pi}\right) - \sum_{k=1}^{\infty} \frac{(-1)^k \zeta(k+1)}{2^k (k+1)}$$

02.06.06.0016.01

$$\gamma == 1 + \log\left(\frac{16}{9\pi}\right) + 2 \sum_{k=2}^{\infty} \frac{(-1)^k (\zeta(k) - 1)}{2^k k}$$

Other series representations

02.06.06.0017.01

$$\gamma == \sum_{k=1}^{\infty} k \sum_{j=2^k}^{2^{k+1}-1} \frac{(-1)^j}{j}$$

02.06.06.0018.01

$$\gamma == \frac{1}{2} + \frac{1}{2} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\delta_{1,\gcd(l,n)}}{(k\, l\, m\, n\, (k+m)\, (l+n))^2}$$

Product representations

02.06.08.0001.01

$$\gamma == \log\left(\prod_{n=0}^{\infty} \left(\prod_{k=0}^{n} (k+1)^{(-1)^{k+1} \binom{n}{k}}\right)^{\frac{1}{n+1}}\right)$$

J. Sondow

Integral representations

On the real axis

Of the direct function

02.06.07.0001.01

$$\gamma == -\int_0^{\infty} e^{-t} \log(t)\, dt$$

02.06.07.0002.01

$$\gamma == -\int_0^1 \log(-\log(t))\, dt$$

02.06.07.0003.01

$$\gamma == -\int_0^1 \log\left(\log\left(\frac{1}{t}\right)\right) dt$$

02.06.07.0004.01

$$\gamma = -\frac{4}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \log(t)\, dt - \log(4)$$

02.06.07.0005.01

$$\gamma = -\int_0^1 \frac{e^{1-\frac{1}{t}} - t}{t\,(1-t)}\, dt$$

02.06.07.0006.01

$$\gamma = \int_0^1 \frac{1 - e^{-t} - e^{-1/t}}{t}\, dt$$

Barnes formula

02.06.07.0007.01

$$\gamma = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t\, e^t} \right) dt$$

02.06.07.0008.01

$$\gamma = 2 \int_0^\infty \frac{e^{-t^2} - e^{-t}}{t}\, dt$$

02.06.07.0009.01

$$\gamma = \frac{\alpha\, \beta}{\alpha - \beta} \int_0^\infty \frac{e^{-t^\alpha} - e^{-t^\beta}}{t}\, dt \;/;\; \alpha > 0 \bigwedge \beta > 0$$

02.06.07.0010.01

$$\gamma = -\int_x^\infty \frac{e^{-t}}{t}\, dt + \int_0^x \frac{1 - e^{-t}}{t}\, dt - \log(x) \;/;\; x > 0$$

02.06.07.0011.01

$$\gamma = -\int_{-\infty}^\infty t\, e^{t - e^t}\, dt$$

02.06.07.0012.01

$$\gamma = \int_0^\infty \frac{1}{t} \left(\frac{1}{t+1} - e^{-t} \right) dt$$

02.06.07.0013.01

$$\gamma = \frac{1}{2} + 2 \int_0^\infty \frac{t}{(t^2 + 1)\left(e^{2\pi t} - 1\right)}\, dt$$

Hermite's formula

02.06.07.0014.01

$$\gamma = 2 \int_0^\infty \frac{t}{(n^2 + t^2)\left(e^{2\pi t} - 1\right)}\, dt - \log(n) + \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{2\,n}$$

02.06.07.0015.01

$$\gamma = 1 - \int_0^1 \frac{1}{t+1} \sum_{k=1}^\infty t^{2^k}\, dt$$

Catalan's formula

02.06.07.0016.01

$$\gamma == -\int_0^\infty \frac{1}{t}\left(\cos(t) - \frac{1}{t^2+1}\right) dt$$

02.06.07.0017.01

$$\gamma == \int_0^x \frac{1-\cos(t)}{t}\, dt - \int_x^\infty \frac{\cos(t)}{t}\, dt - \log(x) \text{ /; } x > 0$$

02.06.07.0018.01

$$\gamma == \int_0^1 \left(\frac{1}{\log(t)} + \frac{1}{1-t}\right) dt$$

02.06.07.0019.01

$$\gamma == \int_0^1 \left(\frac{1}{\log(1-t)} + \frac{1}{t}\right) dt$$

02.06.07.0020.01

$$\gamma == \int_0^\infty \frac{1}{t}\left(\frac{1}{t^2+1} - J_0(2\,t)\right) dt$$

02.06.07.0021.01

$$\gamma == \frac{1}{2} - \int_1^\infty t^{-n-1}\, B_n(t - \lfloor t \rfloor)\, dt + \sum_{k=2}^n \frac{B_k}{k} \text{ /; } n \in \mathbb{N}^+$$

Multiple integral representations

02.06.07.0023.01

$$\gamma == \int_0^1 \int_0^1 \frac{x-1}{(1-x\,y)\log(x\,y)}\, dy\, dx$$

02.06.07.0022.01

$$\gamma == \log(2) - \pi \int_0^{\frac{1}{2}} \int_0^1 \tan\left(\frac{\pi\,t}{2}\right)\left(\frac{\sin(\pi\,t\,u)}{\sin(\pi\,u)} - t\right) dt\, du$$

Limit representations

02.06.09.0001.01

$$\gamma == \lim_{s \to 1} \left(\zeta(s) - \frac{1}{s-1}\right)$$

02.06.09.0002.01

$$\gamma == \lim_{s \to \infty} \left(s - \Gamma\left(\frac{1}{s}\right)\right)$$

02.06.09.0003.01

$$\gamma == \lim_{x \to 1^+} \left(\sum_{k=1}^\infty \left(k^{-x} - x^{-k}\right)\right)$$

02.06.09.0004.01

$$\gamma == \lim_{n \to \infty} (H_{n-1} - \log(n))$$

02.06.09.0011.01

$$\gamma == \lim_{n\to\infty} \frac{A_n - L_n}{\binom{2n}{n}} /; n \in \mathbb{N}^+ \bigwedge A_n == \sum_{i=0}^{n} \binom{n}{i}^2 H_{i+n} \bigwedge L_n == \frac{\log(S_n)}{d(2n)} \bigwedge S_n == \prod_{k=1}^{n} \prod_{i=0}^{\mathrm{Min}(k-1,n-k)} \prod_{j=i+1}^{n-i} (k+n)^{2 d_{2n} \binom{n}{i}^2 / j} \bigwedge d_n == \mathrm{lcm}(1, 2, \ldots, n)$$

02.06.09.0005.01

$$\gamma == \lim_{n\to\infty} \frac{1}{2\log(n)} \left(\sum_{k=1}^{n} \frac{\sigma_0(k)}{k} - \frac{\log^2(n)}{2} \right)$$

02.06.09.0006.01

$$\gamma == \lim_{\alpha\to 0} (\mathrm{li}(e^{\alpha x}) - \log(\alpha)) - \log(x) /; x > 0$$

02.06.09.0007.01

$$\gamma == \lim_{n\to\infty} \left(\log(p_n) - \sum_{k=1}^{n} \frac{\log(p_k)}{p_k - 1} \right) /; p_k \in \mathbb{P}$$

02.06.09.0012.01

$$\gamma == \lim_{n\to\infty} -\log\left(\log(p_n) \prod_{k=1}^{n} \left(1 - \frac{1}{p_k} \right) \right) /; p_k \in \mathbb{P}$$

Mertens theorem

02.06.09.0008.01

$$\gamma == \lim_{n\to\infty} \left(-\log(\log(n)) - \sum_{k=1}^{\infty} \theta(n - p_k) \log\left(1 - \frac{1}{p_k} \right) \right) /; p_k \in \mathbb{P}$$

02.06.09.0009.01

$$\gamma == \log\left(\frac{1}{6} \pi^2 \left(\lim_{n\to\infty} \frac{1}{\log(n)} \prod_{k=1}^{\infty} \theta(n - p_k) \left(1 + \frac{1}{p_k} \right) \right) \right) /; p_k \in \mathbb{P}$$

02.06.09.0010.01

$$\gamma == \lim_{x\to 0} \left(\mathrm{Ei}(\log(x)) - \mathrm{Ei}(\log(x+1)) - \log\left(1 - \frac{1}{x} \right) + \pi i \right)$$

A. Radovi█

02.06.09.0013.01

$$\gamma = \lim_{n\to\infty} \left(\frac{2n - 1}{2n} - \log(n) + \sum_{k=2}^{n} \left(\frac{1}{k} - \frac{\zeta(1-k)}{n^k} \right) \right)$$

02.06.09.0014.01

$$\gamma = \lim_{n\to\infty} \left(\frac{n^2}{n+1} - \frac{\Gamma\left(\frac{1}{n}\right) \Gamma(n+1)\, n^{1+\frac{1}{n}}}{\Gamma\left(n + 2 + \frac{1}{n}\right)} \right)$$

02.06.09.0015.01

$$\gamma = \lim_{n\to\infty} \frac{\sum_{k=1}^{n}\left(\left\lceil \frac{n}{k}\right\rceil - \frac{n}{k}\right)}{n}$$

02.06.09.0016.01

$$\gamma = \lim_{n\to\infty}\left(-\log(n) + \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=2}^{\infty}\frac{\zeta(k, n+1)}{k}\right)$$

02.06.09.0017.01

$$\gamma == \lim_{x\to\infty}\left(\frac{1}{\sum_{k=0}^{\infty}\frac{x^k}{(k!)^2}}\sum_{k=0}^{\infty}\sum_{i=1}^{k}\frac{x^k}{i\,(k!)^2} - \frac{\log(x)}{2}\right)$$

The above formula is used for the numerical computation of Euler-Mascheroni constant in *Mathematica*. The algorithm which is based on this formula is the fastest known algorithm for computing this constant.

Complex characteristics

Real part

02.06.19.0001.01

$$\mathrm{Re}(\gamma) == \gamma$$

Imaginary part

02.06.19.0002.01

$$\mathrm{Im}(\gamma) == 0$$

Absolute value

02.06.19.0003.01

$$|\gamma| == \gamma$$

Argument

02.06.19.0004.01

$$\arg(\gamma) == 0$$

Conjugate value

02.06.19.0005.01

$$\overline{\gamma} == \gamma$$

Signum value

02.06.19.0006.01

$$\mathrm{sgn}(\gamma) == 1$$

Differentiation

Low-order differentiation

02.06.20.0001.01

$$\frac{\partial \gamma}{\partial z} == 0$$

Fractional integro-differentiation

02.06.20.0002.01

$$\frac{\partial^{\alpha} \gamma}{\partial z^{\alpha}} == \frac{z^{-\alpha} \gamma}{\Gamma(1-\alpha)}$$

Integration

Indefinite integration

02.06.21.0001.01

$$\int \gamma \, dz == \gamma z$$

02.06.21.0002.01

$$\int z^{\alpha-1} \gamma \, dz == \frac{z^{\alpha} \gamma}{\alpha}$$

Integral transforms

Fourier exp transforms

02.06.22.0001.01

$$\mathcal{F}_t[\gamma](z) == \sqrt{2\pi} \, \gamma \, \delta(z)$$

Inverse Fourier exp transforms

02.06.22.0002.01

$$\mathcal{F}_t^{-1}[\gamma](z) == \sqrt{2\pi} \, \gamma \, \delta(z)$$

Fourier cos transforms

02.06.22.0003.01

$$\mathcal{F}c_t[\gamma](z) == \sqrt{\frac{\pi}{2}} \, \gamma \, \delta(z)$$

Fourier sin transforms

02.06.22.0004.01

$$\mathcal{F}s_t[\gamma](z) == \sqrt{\frac{2}{\pi}} \, \frac{\gamma}{z}$$

Laplace transforms

02.06.22.0005.01

$$\mathcal{L}_t[\gamma](z) == \frac{\gamma}{z}$$

Inverse Laplace transforms

02.06.22.0006.01

$$\mathcal{L}_t^{-1}[\gamma](z) == \gamma\,\delta(z)$$

Representations through more general functions

Through Meijer *G*

02.06.26.0003.01

$$\gamma = \gamma\, G_{0,1}^{1,0}(z \mid 0) + \gamma\, G_{1,2}^{1,1}\left(z \,\middle|\, \begin{matrix} 1 \\ 1, 0 \end{matrix}\right)$$

Through other functions

02.06.26.0001.01

$$\gamma == -\psi(1)$$

02.06.26.0002.01

$$\gamma == \gamma_0$$

Inequalities

02.06.29.0001.01

$$\frac{1}{2} < \gamma < \frac{3}{5}$$

Theorems

The value of the sum of all integers whose squares divide an integer

The expected value of the sum of all integers whose squares divide an integer n is given asymptotically by $\frac{1}{2}\log(n) + \frac{3}{2}\gamma$.

History

– L. Euler (1735, 1740) introduced this constant, denoted it through symbol C, and initially calculated its value to 6 decimal places;
– L. Euler (1781) calculated it to 16 digits;
– Lorenzo Mascheroni (1790) first used symbol γ for this constant and calculated it to 19 correct digits;
– Soldner (1809) calculated γ to 40 correct digits;
– Gauss and Nicolai (1812) verified calculation γ to 40 correct digits;

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