EulerPhi

Notations

Traditional name

Euler totient function

Traditional notation

$\phi(n)$

Mathematica StandardForm notation

`EulerPhi[n]`

Primary definition

13.06.02.0001.01

$$\phi(n) == \sum_{k=1}^{n} \delta_{\gcd(n,k),1} \;/;\; n \in \mathbb{N}$$

For nonnegative integer n, the Euler totient function $\phi(n)$ is the number of positive integers less than n and relatively prime to n.

13.06.02.0002.01

$$\phi(-n) == \phi(n) \;/;\; n \in \mathbb{N}$$

Example: There are exist only 4 positive integers less than 10 and relatively prime to 10; they are 1, 3, 7, and 9 (because, for example, $\gcd(10, 9) == 1$ but $\gcd(10, 8) == 2 \neq 1$); so the Euler totient function $\phi(10) == 4$. By definition, $\phi(-10) == 4$.

Specific values

Specialized values

13.06.03.0001.01

$$\phi(p^n) == p^n - p^{n-1} \;/;\; p \in \mathbb{P} \bigwedge n \in \mathbb{N}^+$$

Values at fixed points

13.06.03.0002.01

$$\phi(0) == 0$$

13.06.03.0003.01

$\phi(1) == 1$

13.06.03.0004.01

$\phi(2) == 1$

13.06.03.0005.01

$\phi(3) == 2$

13.06.03.0006.01

$\phi(4) == 2$

13.06.03.0007.01

$\phi(5) == 4$

13.06.03.0008.01

$\phi(6) == 2$

13.06.03.0009.01

$\phi(7) == 6$

13.06.03.0010.01

$\phi(8) == 4$

13.06.03.0011.01

$\phi(9) == 6$

13.06.03.0012.01

$\phi(10) == 4$

13.06.03.0013.01

$\phi(11) == 10$

13.06.03.0014.01

$\phi(12) == 4$

13.06.03.0015.01

$\phi(13) == 12$

13.06.03.0016.01

$\phi(14) == 6$

13.06.03.0017.01

$\phi(15) == 8$

13.06.03.0018.01

$\phi(16) == 8$

13.06.03.0019.01

$\phi(17) == 16$

13.06.03.0020.01

$\phi(18) == 6$

13.06.03.0021.01

$\phi(19) == 18$

13.06.03.0022.01

$\phi(20) == 8$

13.06.03.0023.01

$\phi(21) == 12$

13.06.03.0024.01

$\phi(22) == 10$

13.06.03.0025.01

$\phi(23) == 22$

13.06.03.0026.01

$\phi(24) == 8$

13.06.03.0027.01

$\phi(25) == 20$

13.06.03.0028.01

$\phi(26) == 12$

13.06.03.0029.01

$\phi(27) == 18$

13.06.03.0030.01

$\phi(28) == 12$

13.06.03.0031.01

$\phi(29) == 28$

13.06.03.0032.01

$\phi(30) == 8$

13.06.03.0033.01

$\phi(31) == 30$

13.06.03.0034.01

$\phi(32) == 16$

13.06.03.0035.01

$\phi(33) == 20$

13.06.03.0036.01

$\phi(34) == 16$

13.06.03.0037.01

$\phi(35) == 24$

13.06.03.0038.01

$\phi(36) == 12$

13.06.03.0039.01

$\phi(37) == 36$

13.06.03.0040.01

$\phi(38) == 18$

13.06.03.0041.01

$\phi(39) == 24$

13.06.03.0042.01

$\phi(40) == 16$

13.06.03.0043.01

$\phi(41) == 40$

13.06.03.0044.01

$\phi(42) == 12$

13.06.03.0045.01

$\phi(43) == 42$

13.06.03.0046.01

$\phi(44) == 20$

13.06.03.0047.01

$\phi(45) == 24$

13.06.03.0048.01

$\phi(46) == 22$

13.06.03.0049.01

$\phi(47) == 46$

13.06.03.0050.01

$\phi(48) == 16$

13.06.03.0051.01

$\phi(49) == 42$

13.06.03.0052.01

$\phi(50) == 20$

13.06.03.0053.01

$\phi(100) == 40$

13.06.03.0054.01

$\phi(1000) == 400$

13.06.03.0055.01

$\phi(10\,000) == 4000$

13.06.03.0056.01

$\phi(-100) == 40$

General characteristics

Domain and analyticity

$\phi(n)$ is a nonanalytical function which is defined only for integer n.

13.06.04.0001.01

$n \longrightarrow \phi(n) :: \mathbb{Z} \longrightarrow \mathbb{Z}$

Symmetries and periodicities

Parity

$\phi(n)$ is an even function.

13.06.04.0002.01

$$\phi(-n) == \phi(n)$$

Symmetry

No symmetry

Periodicity

No periodicity

Series representations

Other series representations

13.06.06.0004.01

$$\phi(n) == \sum_{k=1}^{n} \left\lfloor \frac{1}{\gcd(k,\, n)} \right\rfloor \text{/; } n \in \mathbb{N}$$

13.06.06.0005.01

$$\phi(n) == \sum_{k=1}^{n} \delta_{\gcd(n,k),1} \text{/; } n \in \mathbb{N}$$

13.06.06.0006.01

$$\phi(n) == \sum_{j=1}^{p-1} \prod_{d_j \mid p} \left(1 - \frac{1}{d_j} \sum_{k=0}^{d_j-1} \exp\left(\frac{2\,\pi\, i\, j\, k}{d_j}\right)\right) \text{/; } p \in \mathbb{P} \bigwedge d_j \in \text{divisors}(n)$$

13.06.06.0007.01

$$\phi(n) == \sum_{d_j|n} d_j\, \mu\left(\frac{n}{d_j}\right) \text{/; } d_j \in \text{divisors}(n)$$

13.06.06.0008.01

$$\phi(n) == n \sum_{d_j|n} \frac{\mu(d_j)}{d_j} \text{/; } d_j \in \text{divisors}(n)$$

Product representations

13.06.08.0001.01

$$\phi(n) == n \prod_{k=1}^{m} \left(1 - \frac{1}{p_k}\right) \text{/; factors}(n) == \{\{p_1,\, n_1\},\, \ldots,\, \{p_m,\, n_m\}\} \bigwedge p_k \in \mathbb{P}$$

Identities

Functional identities

13.06.17.0001.01

$$\phi(n) == \frac{n}{c(n)}\, \phi(c(n)) \text{/; } c(n) == \sum_{d_j|n} \left|\mu(d_j)\right| \phi(d_j) == \prod_{k=1}^{r} p_k^{n_k} \bigwedge p_k \in \mathbb{P} \bigwedge n_k \in \mathbb{N}^+$$

13.06.17.0002.01

$$\phi(n^k) = \phi(n)\, n^{k-1} \,/;\, n \in \mathbb{N}^+ \bigwedge k \in \mathbb{N}^+$$

13.06.17.0003.01

$$\phi(n)\, \phi(m) = \frac{\phi(m\, n)\, \phi(\gcd(n, m))}{\gcd(n, m)} \,/;\, n \in \mathbb{N} \land m \in \mathbb{N}$$

13.06.17.0004.01

$$\phi(n) \sum_{d_j | n} \frac{\mu(d_j)^2}{\phi(d_j)} = n \,/;\, d_j \in \text{divisors}(n) \bigwedge n \in \mathbb{N}$$

Summation

Finite summation

13.06.23.0001.01

$$\sum_{l=1}^{n} \sum_{k=1}^{\left\lfloor \frac{n}{l} \right\rfloor} \phi(k) = \frac{n(n+1)}{2}$$

13.06.23.0002.01

$$\sum_{d_j | n} \phi(d_j) = n \,/;\, d_j \in \text{divisors}(n)$$

13.06.23.0003.01

$$\sum_{d_j | n} \phi(d_j)\, \sigma_0\!\left(\frac{n}{d_j}\right) = \sigma_1(n) \,/;\, d_j \in \text{divisors}(n)$$

13.06.23.0004.01

$$\sum_{d_j | n} \phi\!\left(\frac{n}{d_j}\right) \sigma_k(d_j) = n\, \sigma_{k-1}(n) \,/;\, d_j \in \text{divisors}(n) \bigwedge k \in \mathbb{Z}$$

Infinite summation

13.06.23.0005.01

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} \,/;\, \text{Re}(s) > 1$$

13.06.23.0006.01

$$n^2 \sum_{k=1}^{n} \phi(k)\, f\!\left(\frac{k}{n}\right) = \int_0^1 f(x)\, dx \,/;\, (n \to \infty)$$

in case $x\, f(x) \in C[0, 1]$.

Asymptotic finite summation

13.06.23.0007.01

$$\sum_{k=1}^{n} \phi(k) \propto \frac{3\, n^2}{\pi^2} + O\!\left(n \log^{\frac{2}{3}}(n) \log^{\frac{4}{3}}(\log(n))\right) /;\, (n \to \infty)$$

13.06.23.0008.01

$$\sum_{k=1}^{n}\frac{\phi(k)}{k}\propto\frac{6}{\pi^2}\,x+O\left(\log^{\frac{2}{3}}(n)\log^{\frac{4}{3}}(\log(n))\right)/;\,(n\to\infty)$$

13.06.23.0009.01

$$\sum_{k=1}^{n}\frac{1}{\phi(k)}\propto\frac{315\,\zeta(3)}{2\,\pi^4}\log(n)+\frac{315\,\gamma\,\zeta(3)}{2\,\pi^4}-\sum_{k=1}^{\infty}\frac{|\mu(k)|^2\log(k)}{k\,\phi(k)}+O\left(\frac{\log(n)}{n}\right)/;\,(n\to\infty)$$

13.06.23.0010.01

$$\sum_{k=3}^{n}\frac{1}{\log(\phi(k))}\propto\frac{n}{\log(n)}+a_2\,\frac{n}{\log(n)}+O\left(\frac{n}{\log^2(n)}\right)/;\,\left((n\to\infty)\bigwedge a_2==1-\sum_{k=1}^{\infty}\frac{1}{p_k}\log\left(1-\frac{1}{p_k}\right)\bigwedge p_k\in\mathbb{P}\right)$$

Asymptotic infinite summation

13.06.23.0011.01

$$\sum_{k=1}^{\infty}\theta(x-\phi(k))\propto\frac{\zeta(2)\,\zeta(3)}{\zeta(6)}\,x\,+R(x)\,/;\,(x\to\infty)\bigwedge R(x)\ll x\,e^{-(1-\varepsilon)\sqrt{\frac{1}{2}\log(x)\log(\log(x))}}\bigwedge\varepsilon>0$$

Operations

Limit operation

13.06.25.0001.01

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\phi(k)==\frac{3}{\pi^2}$$

13.06.25.0002.01

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\phi(k)==\frac{6}{\pi^2}\int_0^1 t\,f(t)\,dt$$

13.06.25.0003.01

$$\lim_{n\to\infty}\min\left(\frac{\phi(2)}{\phi(1)},\,\frac{\phi(3)}{\phi(2)},\,\ldots,\,\frac{\phi(n+1)}{\phi(n)}\right)==\infty$$

13.06.25.0004.01

$$\lim_{n\to\infty}\max\left(\frac{\phi(2)}{\phi(1)},\,\frac{\phi(3)}{\phi(2)},\,\ldots,\,\frac{\phi(n+1)}{\phi(n)}\right)==\infty$$

13.06.25.0005.01

$$\lim_{n\to\infty}\max\left(\frac{1}{2}\,\phi(\phi(2)),\,\frac{1}{3}\,\phi(\phi(3)),\,\ldots,\,\frac{1}{n}\,\phi(\phi(n))\right)==\frac{1}{2}$$

13.06.25.0006.01

$$\lim_{n\to\infty}\log\left(\max\left(\frac{\phi(3)}{\frac{3}{\log(\log(3))}},\,\frac{\phi(4)}{\frac{4}{\log(\log(4))}},\,\ldots,\,\frac{\phi(n)}{\frac{n}{\log(\log(n))}}\right)\right)==\gamma$$

13.06.25.0007.01

$$\lim_{n\to\infty}\min\left(\left\{\frac{\phi(n)}{\frac{n}{\log(\log(n))}}\right\}_{k,1,n}\right)==e^{-\gamma}$$

Representations through equivalent functions

With related functions

13.06.27.0001.01

$$\phi(n) == \sum_{k=1}^{n}\left(1-\operatorname{sgn}\left(\sum_{i=1}^{m}\sum_{j=1}^{s}\delta_{p_i,q_j}\right)\right) /;\ \text{factors}(n) == \{\{p_1, n_1\}, \ldots, \{p_m, n_m\}\} \land \text{factors}(k) == \{\{q_1, n_1\}, \ldots, \{q_s, n_s\}\} \land n \in \mathbb{N}^+$$

13.06.27.0002.01

$$\phi(p^n) == \lambda(p^n) /;\ p \in \mathbb{P} \land p > 2 \land n \in \mathbb{N}^+$$

13.06.27.0003.01

$$\pi(x) == \sum_{k=2}^{\lfloor x \rfloor}\left\lfloor \frac{\phi(k)}{k-1} \right\rfloor /;\ x \geq 0$$

13.06.27.0004.01

$$\sigma_0(n) == \frac{1}{n}\sum_{d_j|n}\sigma_1\left(\frac{n}{d_j}\right)\phi(d_j) /;\ d_j \in \text{divisors}(n)$$

13.06.27.0005.01

$$\sigma_1(n) == \sum_{d_j|n}\phi(d_j)\,\sigma_0\left(\frac{n}{d_j}\right) /;\ d_j \in \text{divisors}(n)$$

13.06.27.0006.01

$$\phi(n)\sum_{d_j|n}\frac{d_j}{\phi(d_j)}\,\delta_{\gcd(m,d_j),1}\,\mu\left(\frac{n}{d_j}\right) == \mu(n)\sum_{d_j|\gcd(m,n)} d_j\,\mu\left(\frac{n}{d_j}\right) /;\ d_j \in \text{divisors}(n) \land n \in \mathbb{N}^+ \land m \in \mathbb{N}^+$$

Inequalities

13.06.29.0001.01

$$\phi(n) \geq \sqrt{n} /;\ n \neq 2 \land n \neq 6$$

13.06.29.0002.01

$$\phi(n) \geq n^{2/3} /;\ n > 42$$

13.06.29.0003.01

$$\phi(n) > \frac{n}{e^{\gamma}\log(\log(n)) + \frac{2.50637}{\log(\log(n))}} /;\ n \geq 2$$

13.06.29.0004.01

$$\phi(n) \geq \frac{\log(x\,n)}{\log(x)} /;\ n > 2 \land x \in \mathbb{R} \land x > 6$$

13.06.29.0005.01

$$\phi(n) \leq n - \sqrt{n} /;\ n == \prod_{k=1}^{m} p_k^{n_k} \bigwedge p_k \in \mathbb{P} \bigwedge n_k \in \mathbb{N}^+ \bigwedge p_k < p_{k+1} \bigwedge 1 \leq k \leq m-1 \bigwedge m \geq 2$$

13.06.29.0006.01

$$\phi(n) > \frac{\log(2)\,n}{2\log(n)} /;\ n \geq 3$$

13.06.29.0007.01

$$\phi(m)\,\phi(n) \leq \phi(m\,n) \leq n\,\phi(m)$$

13.06.29.0008.01

$$\phi(m)\,\phi(n) \leq \phi(m^2)\,\phi(n^2)$$

13.06.29.0009.01

$$\phi(n) > \frac{n}{\sigma_0(n)} \;/;\; n \geq 3$$

13.06.29.0010.01

$$\phi(n) \leq \frac{n \log(n)}{\log(\sigma_0(n))} \;/;\; n \geq 2$$

13.06.29.0011.01

$$\phi(n) > \frac{n}{\sigma_1(n)} \;/;\; n \geq 2$$

13.06.29.0012.01

$$\phi(n) > \pi(n) \;/;\; n \geq 91$$

13.06.29.0013.01

$$\phi\left(n \left\lfloor \frac{\sigma_1(n)}{n} \right\rfloor\right) < n \;/;\; n \geq 2$$

13.06.29.0014.01

$$\phi\left(n \left\lfloor \frac{n}{\sigma_0(n)} \right\rfloor\right) \leq \phi(n)^2 \;/;\; n \geq 2$$

13.06.29.0015.01

$$\min\left(\left\{ \frac{\sigma_1(\phi(2))}{2}, \frac{\sigma_1(\phi(3))}{3}, \ldots \right\}\right) = \frac{1}{2}$$

13.06.29.0016.01

$$\max\left(\left\{ \frac{\phi(\phi(2))}{2}, \frac{\phi(\phi(3))}{3}, \ldots \right\}\right) = \frac{1}{2}$$

Zeros

13.06.30.0001.01

$$\phi(0) = 0$$

Other identities

Congruence properties

13.06.32.0001.01

$$\phi(n) \bmod 2 = 0 \;/;\; n > 2$$

Theorems

Fermat-Euler theorem

$$a^{\phi(m)} \equiv 1 \pmod{m} \;/;\; \gcd(a, m) = 1$$

Generalized Fermat-Euler theorem

$a^{k+\phi(b)} \bmod b = a^k \bmod b$ /; $\gcd(a^k, b) = 1 \land \gcd(a^{k+1}, b) = 1 \land k \in \mathbb{N}^+$.

The solution of a linear congruence

The solution of $(a\,x - b) \bmod m = 0$ is given by $x = \frac{b}{d} \left(\frac{a}{d}\right)^{\phi\left(\frac{m}{d}\right)-1} \bmod \frac{m}{d}$ /; $d = \gcd(a, m) \bigwedge \frac{b}{d} \in \mathbb{N}$.

Number of different values of $\phi(k)$ for $k \leq n$

The number $V(x)$ of different values occurring in the list $\{\phi(1), \phi(2), \ldots, \phi(x)\}$ behaves asymptotically as

where

Farey fractions

The number of irreducible fractions between 0 and 1 with denominator n is $\phi(n)$.

Generators of cyclic groups

$\phi(n)$ is the number of generators of a cyclic group of order n.

Carmichael conjecture

For every n it is possible to find an m ($n \neq m$), such that $\phi(n) = \phi(m)$.

Number of necklaces

The number n of unique fixed necklaces of length l which can be made out of b different beads is $n = \frac{1}{l} \sum_{d|l} \phi(d)\, b^{\frac{l}{d}}$.

Probability that two randomly choosen positive integers are relatively prime

The probability p_2 that two randomly choosen positive integers $n_1$and n_2 (n_1, $n_2 < x$) are relatively prime ($\gcd(n_1, n_2) = 1$) is given by

$$p_2 = \frac{1}{x^2} \left(-1 + 2 \sum_{k=1}^{\lfloor x \rfloor} \phi(k)\right) \underset{x \to \infty}{\approx} \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Prime roots

All integers n of the form 1, 2, 4, p^k, $2\,p^k$ where $2\,p + 1 \in \mathbb{P}$, $k \in \mathbb{N}$, have $\phi(\phi(n))$ different primitive roots.

History

– L. Euler (1760, 1763)
– C. F. Gauss (1801) introduced the symbol ϕ
– J. J. Sylvester (1879) introduced the name "totient function"
– E. Cesaro (1888) evaluated asymptotics for cumulative sums
– E. Landau (1900)

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