ExpIntegralE

Notations

Traditional name

Exponential integral E

Traditional notation

$E_\nu(z)$

Mathematica StandardForm notation

`ExpIntegralE[ν, z]`

Primary definition

06.34.02.0001.01

$$E_\nu(z) = \int_1^\infty \frac{e^{-z t}}{t^\nu} \, dt \; /; \; \operatorname{Re}(z) > 0$$

Specific values

Specialized values

For fixed ν

06.34.03.0001.01

$$E_\nu(0) = \tilde{\infty} \; /; \; \operatorname{Re}(\nu) < 1$$

06.34.03.0002.01

$$E_\nu(0) = \frac{1}{\nu - 1} \; /; \; \operatorname{Re}(\nu) > 1$$

06.34.03.0017.01

$$E_z(-1) = -e \, (-1)^z \operatorname{Subfactorial}(-z)$$

For fixed z

06.34.03.0003.01

$$E_0(z) = \frac{e^{-z}}{z}$$

06.34.03.0004.01

$$E_{\frac{1}{2}}(z) = \frac{\sqrt{\pi}}{\sqrt{z}} \left(1 - \operatorname{erf}\left(\sqrt{z}\right)\right)$$

06.34.03.0005.01

$$E_{\frac{1}{2}}(z) = \frac{\sqrt{\pi}}{\sqrt{z}} \operatorname{erfc}\left(\sqrt{z}\right)$$

06.34.03.0006.01

$$E_{n+\frac{1}{2}}(z) = \frac{(-1)^n \sqrt{\pi} \, z^{n-\frac{1}{2}}}{(1/2)_n} \operatorname{erfc}\left(\sqrt{z}\right) - e^{-z} \sum_{k=0}^{n-1} \frac{z^k}{(1/2-n)_{k+1}} \;/;\; n \in \mathbb{N}$$

06.34.03.0007.01

$$E_{-\frac{1}{2}}(z) = \frac{\sqrt{\pi}}{2\, z^{3/2}} \operatorname{erfc}\left(\sqrt{z}\right) + \frac{e^{-z}}{z}$$

06.34.03.0008.01

$$E_{\frac{1}{2}-n}(z) = \Gamma\left(n+\frac{1}{2}\right) z^{-n-\frac{1}{2}} \operatorname{erfc}\left(\sqrt{z}\right) - e^{-z} \sum_{k=0}^{n-1} \left(\frac{1}{2}-n\right)_{n-k-1} (-z)^{k-n} \;/;\; n \in \mathbb{N}$$

06.34.03.0009.01

$$E_1(z) = -\operatorname{Ei}(-z) + \frac{1}{2}\left(\log(-z) - \log\left(-\frac{1}{z}\right)\right) - \log(z)$$

06.34.03.0010.01

$$E_1(z) - E_1(-z) = 2\operatorname{Shi}(z) + \log(-z) - \log(z)$$

06.34.03.0011.01

$$E_n(z) = -\frac{(-z)^{n-1}}{(n-1)!}\left(\operatorname{Ei}(-z) - \frac{1}{2}\left(\log(-z) - \log\left(-\frac{1}{z}\right)\right) + \log(z)\right) - e^{-z} \sum_{k=1}^{n-1} \frac{z^{k-1}}{(1-n)_k} \;/;\; n \in \mathbb{N}^+$$

06.34.03.0018.01

$$E_n(z) = \frac{e^{-z}}{z} - n\, z^{n-1}\, \Gamma(-n, z) \;/;\; n \in \mathbb{N}^+$$

06.34.03.0012.01

$$E_{-1}(z) = \frac{e^{-z}(z+1)}{z^2}$$

06.34.03.0013.01

$$E_{-n}(z) = n!\, e^{-z} \sum_{k=0}^{n} \frac{z^{k-n-1}}{k!} \;/;\; n \in \mathbb{N}$$

06.34.03.0015.01

$$E_n(z) = z^{n-1}\left(\frac{(-1)^n}{(n-1)!}\left(\operatorname{Ei}(-z) + \frac{1}{2}\left(\log\left(-\frac{1}{z}\right) - \log(-z)\right) + \log(z)\right) + e^{-z} \sum_{k=0}^{-n} \frac{z^k}{(1-n)_{k+n}} - e^{-z} \sum_{k=1-n}^{-1} \frac{z^k}{(1-n)_{k+n}}\right) /;\; n \in \mathbb{Z}$$

06.34.03.0016.01

$$E_{n+\frac{1}{2}}(z) = z^{n-\frac{1}{2}}\left(\operatorname{erfc}\left(\sqrt{z}\right)\Gamma\left(\frac{1}{2}-n\right) + e^{-z} \sum_{k=0}^{-n-1} \frac{z^{k+\frac{1}{2}}}{\left(\frac{1}{2}-n\right)_{k+n+1}} - e^{-z} \sum_{k=-n}^{-1} \frac{z^{k+\frac{1}{2}}}{\left(\frac{1}{2}-n\right)_{k+n+1}}\right) /;\; n \in \mathbb{Z}$$

Values at infinities

06.34.03.0014.01

$$E_\nu(\infty) = 0$$

General characteristics

Domain and analyticity

$E_\nu(z)$ is an analytical function of ν and z which is defined in $\mathbb{C}^2$. For fixed z, it is an entire function of ν.

06.34.04.0001.01

$$(\nu * z) \longrightarrow E_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

06.34.04.0002.02

$$E_{\bar{\nu}}(\bar{z}) = \overline{E_\nu(z)} \;/;\; z \notin (-\infty, 0)$$

Periodicity

No periodicity

Poles and essential singularities

With respect to z

For fixed ν, the function $E_\nu(z)$ has an essential singularity at $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is a branch point for generic ν.

06.34.04.0003.01

$$\mathcal{S}ing_z(E_\nu(z)) = \{\{\tilde{\infty}.\infty\}\}$$

With respect to ν

For fixed z, the function $E_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point.

06.34.04.0004.01

$$\mathcal{S}ing_\nu(E_\nu(z)) = \{\{\tilde{\infty}, \infty\}\}$$

Branch points

With respect to z

For fixed ν, not being a nonpositive integer, the function $E_\nu(z)$ has two branch points: $z = 0$, $z = \tilde{\infty}$. At the same time, the point $z = \tilde{\infty}$ is an essential singularity.

06.34.04.0005.01

$$\mathcal{BP}_z(E_\nu(z)) = \{0, \tilde{\infty}\}$$

06.34.04.0006.01

$$\mathcal{R}_z(E_\nu(z), 0) = \log \;/;\; \nu \in \mathbb{Z} \bigvee \nu \notin \mathbb{Q}$$

06.34.04.0007.01

$$\mathcal{R}_z\left(E_{\frac{p}{q}}(z), 0\right) = q \;/;\; p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$$

06.34.04.0008.01

$$\mathcal{R}_z(E_\nu(z), \tilde{\infty}) = \log /; \nu \in \mathbb{Z} \bigvee \nu \notin \mathbb{Q}$$

06.34.04.0009.01

$$\mathcal{R}_z\left(E_{\frac{p}{q}}(z), \tilde{\infty}\right) = q /; p \in \mathbb{Z} \bigwedge q - 1 \in \mathbb{N}^+ \bigwedge \gcd(p, q) = 1$$

With respect to ν

For fixed z, the function $E_\nu(z)$ does not have branch points.

06.34.04.0010.01

$$\mathcal{BP}_\nu(E_\nu(z)) = \{\}$$

Branch cuts

With respect to z

For fixed ν, not being a nonpositive integer, the function $E_\nu(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where it is continuous from above.

06.34.04.0011.01

$$\mathcal{BC}_z(E_\nu(z)) = \{\{(-\infty, 0), -i\}\}$$

06.34.04.0012.01

$$\lim_{\epsilon \to +0} E_\nu(x + i\,\epsilon) = E_\nu(x) /; x < 0$$

06.34.04.0013.01

$$\lim_{\epsilon \to +0} E_\nu(x - i\,\epsilon) = E_\nu(x) - \frac{2\,\pi\,i\,e^{-\pi\,i\,\nu}\,x^{\nu-1}}{\Gamma(\nu)} /; x < 0$$

With respect to ν

For fixed z, the function $E_\nu(z)$ does not have branch cuts.

06.34.04.0014.01

$$\mathcal{BC}_\nu(E_\nu(z)) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $\nu = \nu_0$

For the function itself

06.34.06.0017.01

$$E_\nu(z) \propto E_{\nu_0}(z) - \frac{\Gamma(1-\nu_0)}{z}\left(z\,\Gamma(1-\nu_0)\,{}_2\tilde{F}_2(1-\nu_0, 1-\nu_0; 2-\nu_0, 2-\nu_0; -z) - z^{\nu_0}\left(-H_{-\nu_0} + \log(z) + \gamma\right)\right)(\nu - \nu_0) + \frac{\Gamma(1-\nu_0)}{2\,z}\left(z^{\nu_0}\left(\left(-H_{-\nu_0} + \log(z) + \gamma\right)^2 + \psi^{(1)}(1-\nu_0)\right) - 2\,z\,\Gamma(1-\nu_0)^2\,{}_3\tilde{F}_3(1-\nu_0, 1-\nu_0, 1-\nu_0; 2-\nu_0, 2-\nu_0, 2-\nu_0; -z)\right)(\nu-\nu_0)^2 + \ldots /; (\nu \to \nu_0)$$

06.34.06.0018.01

$$E_\nu(z) \propto E_{\nu_0}(z) - \frac{\Gamma(1-\nu_0)}{z}\left(z\,\Gamma(1-\nu_0)\,{}_2\tilde{F}_2(1-\nu_0, 1-\nu_0; 2-\nu_0, 2-\nu_0; -z) - z^{\nu_0}\left(-H_{-\nu_0} + \log(z) + \gamma\right)\right)(\nu-\nu_0) + \frac{\Gamma(1-\nu_0)}{2\,z}\left(z^{\nu_0}\left(\left(-H_{-\nu_0} + \log(z) + \gamma\right)^2 + \psi^{(1)}(1-\nu_0)\right) - 2\,z\,\Gamma(1-\nu_0)^2\,{}_3\tilde{F}_3(1-\nu_0, 1-\nu_0, 1-\nu_0; 2-\nu_0, 2-\nu_0, 2-\nu_0; -z)\right)(\nu-\nu_0)^2 + O\left((\nu-\nu_0)^3\right)$$

06.34.06.0019.01

$$E_\nu(z) = z^{\nu_0-1}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{s=0}^{k}(-1)^s\binom{k}{s}\log^{k-s}(z)\left(\Gamma^{(s)}(1-\nu_0) - z^{1-\nu_0}\sum_{j=0}^{s}(-1)^{s-j}\binom{s}{j}(s-j)!\,\Gamma(1-\nu_0)^{-j+s+1}\log^j(z)\,{}_{s-j+1}\tilde{F}_{s-j+1}\left(a_1, a_2, \ldots, a_{s-j+1}; a_1+1, a_2+1, \ldots, a_{s-j+1}+1; -z\right)\right)(\nu-\nu_0)^k \,/;\, a_1 = a_2 = \ldots = a_{k+1} = 1-\nu_0 \bigwedge k \in \mathbb{N}$$

06.34.06.0020.01

$$E_\nu(z) \propto E_{\nu_0}(z)\,(1 + O(\nu-\nu_0))$$

Expansions at generic point $z = z_0$

For the function itself

06.34.06.0021.01

$$E_\nu(z) \propto E_\nu(z_0) + \Gamma(1-\nu)\,z_0^{\nu-1}\left(\left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 1\right) - z_0^{\nu-2}\left(\Gamma(2-\nu, z_0) + \Gamma(2-\nu)\left(\left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 1\right)\right)(z-z_0) + \frac{1}{2}\,z_0^{\nu-3}\left(\Gamma(3-\nu, z_0) + \Gamma(3-\nu)\left(\left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 1\right)\right)(z-z_0)^2 + \ldots \,/;\, (z \to z_0)$$

06.34.06.0022.01

$$E_\nu(z) \propto E_\nu(z_0) + \Gamma(1-\nu)\,z_0^{\nu-1}\left(\left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 1\right) - z_0^{\nu-2}\left(\Gamma(2-\nu, z_0) + \Gamma(2-\nu)\left(\left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 1\right)\right)(z-z_0) + \frac{1}{2}\,z_0^{\nu-3}\left(\Gamma(3-\nu, z_0) + \Gamma(3-\nu)\left(\left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} - 1\right)\right)(z-z_0)^2 + O\left((z-z_0)^3\right)$$

06.34.06.0023.01

$$E_\nu(z) = \Gamma(1-\nu)\sum_{k=0}^{\infty}\frac{z_0^{-k}}{k!}\left(\left(\frac{1}{z_0}\right)^{\nu\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor} z_0^{\left\lfloor\frac{\arg(z-z_0)}{2\pi}\right\rfloor\nu+\nu-1}(-1)^k\,(1-\nu)_k - {}_2\tilde{F}_2(1, 1-\nu; 1-k, 2-\nu; -z_0)\right)(z-z_0)^k$$

06.34.06.0024.01

$$E_\nu(z) \propto E_\nu(z_0) + \Gamma(1-\nu)\, z_0^{\nu-1} \left(\left(\frac{1}{z_0}\right)^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} z_0^{\nu \left\lfloor \frac{\arg(z-z_0)}{2\pi} \right\rfloor} - 1 \right) + O(z-z_0)$$

Expansions on branch cuts

For the function itself

06.34.06.0025.01

$$E_\nu(z) \propto E_\nu(x) + x^{\nu-1}\left(e^{2\pi i \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} - 1\right)\Gamma(1-\nu) - x^{\nu-2}\left(\left(e^{2\pi i \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} - 1\right)\Gamma(2-\nu) + \Gamma(2-\nu, x)\right)(z-x) + \frac{1}{2} x^{\nu-3}\left(\left(e^{2\pi i \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} - 1\right)\Gamma(3-\nu) + \Gamma(3-\nu, x)\right)(z-x)^2 + \ldots \;/;\; (z \to x) \land x \in \mathbb{R} \land x < 0$$

06.34.06.0026.01

$$E_\nu(z) \propto E_\nu(x) + x^{\nu-1}\left(e^{2\pi i \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} - 1\right)\Gamma(1-\nu) - x^{\nu-2}\left(\left(e^{2\pi i \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} - 1\right)\Gamma(2-\nu) + \Gamma(2-\nu, x)\right)(z-x) + \frac{1}{2} x^{\nu-3}\left(\left(e^{2\pi i \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} - 1\right)\Gamma(3-\nu) + \Gamma(3-\nu, x)\right)(z-x)^2 + O\left((z-x)^3\right) \;/;\; x \in \mathbb{R} \land x < 0$$

06.34.06.0027.01

$$E_\nu(z) = \Gamma(1-\nu) \sum_{k=0}^{\infty} \frac{x^{-k}}{k!} \left(e^{2\pi i \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} (-1)^k x^{\nu-1} (1-\nu)_k - {}_2\tilde{F}_2(1, 1-\nu; 1-k, 2-\nu; -x)\right)(z-x)^k \;/;\; x \in \mathbb{R} \land x < 0$$

06.34.06.0028.01

$$E_\nu(z) \propto E_\nu(x) + x^{\nu-1}\left(e^{2\pi i \nu \left\lfloor \frac{\arg(z-x)}{2\pi} \right\rfloor} - 1\right)\Gamma(1-\nu) + O(z-x) \;/;\; x \in \mathbb{R} \land x < 0$$

Expansions at $z = 0$

For the function itself

General case

06.34.06.0001.02

$$E_\nu(z) \propto \Gamma(1-\nu)\, z^{\nu-1} - \frac{1}{1-\nu} + \frac{z}{2-\nu} - \frac{z^2}{2(3-\nu)} + \ldots \;/;\; (z \to 0)$$

06.34.06.0029.01

$$E_\nu(z) \propto \Gamma(1-\nu)\, z^{\nu-1} - \frac{1}{1-\nu} + \frac{z}{2-\nu} - \frac{z^2}{2(3-\nu)} + O\left(z^3\right)$$

06.34.06.0002.01

$$E_\nu(z) = \Gamma(1-\nu)\, z^{\nu-1} - \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{(k-\nu+1)\, k!}$$

06.34.06.0003.01

$$E_\nu(z) = \Gamma(1-\nu)\, z^{\nu-1} - \frac{1}{1-\nu}\, {}_1F_1(1-\nu; 2-\nu; -z)$$

06.34.06.0004.02

$$E_\nu(z) \propto \Gamma(1-\nu)\, z^{\nu-1} - \frac{1}{1-\nu}\,(1+O(z))$$

06.34.06.0030.01

$$E_\nu(z) = F_\infty(z,\nu) \;/;\; \left(\left(F_n(z,\nu) = \Gamma(1-\nu)\, z^{\nu-1} - \sum_{k=0}^{n} \frac{(-1)^k z^k}{(k-\nu+1)\, k!} = \Gamma(1-\nu, z)\, z^{\nu-1} + \frac{(-1)^n z^{n+1}}{(-n+\nu-2)\,(n+1)!}\, {}_2F_2(1, n-\nu+2; n+2, n-\nu+3; -z)\right) \bigwedge n \in \mathbb{N}\right)$$

Summed form of the truncated series expansion.

Special cases

06.34.06.0031.01

$$E_1(z) \propto -\log(z) - \gamma + z + O\left(z^2\right)$$

06.34.06.0032.01

$$E_2(z) \propto 1 - \frac{z^2}{2} + (\log(z) + \gamma - 1)\, z + O\left(z^3\right)$$

06.34.06.0033.01

$$E_n(z) \propto \frac{(-z)^{n-1}}{(n-1)!}\,(\psi(n) - \log(z)) - \sum_{k=0}^{n-2} \frac{(-1)^k z^k}{(k-n+1)\, k!} - \frac{(-1)^n z^n}{n!} + O\left(z^{n+1}\right) \;/;\; n \in \mathbb{Z} \land n > 2$$

06.34.06.0034.01

$$E_1(z) = -\log(z) - \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k\, k!} - \gamma$$

06.34.06.0005.01

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}\,(\psi(n) - \log(z)) - \sum_{\substack{k=0 \\ k \neq n-1}}^{\infty} \frac{(-1)^k z^k}{(k-n+1)\, k!} \;/;\; n \in \mathbb{N}^+$$

06.34.06.0006.01

$$E_n(z) = -\frac{(-z)^n}{n!}\, {}_2F_2(1, 1; 2, n+1; -z) + \frac{(-z)^{n-1}}{(n-1)!}\,(\psi(n) - \log(z)) - \sum_{k=0}^{n-2} \frac{(-1)^k z^k}{(k-n+1)\, k!} \;/;\; n \in \mathbb{N}^+$$

06.34.06.0007.01

$$E_1(z) \propto -\log(z) + O(1) \;/;\; (z \to 0)$$

06.34.06.0008.01

$$E_2(z) \propto 1 + O(z \log(z)) \;/;\; (z \to 0)$$

06.34.06.0009.01

$$E_3(z) \propto \frac{1}{2} + O(z) \;/;\; (z \to 0)$$

06.34.06.0010.01

$$E_n(z) \propto \frac{1}{n-1}\,(1 + O(z)) + \frac{(-z)^{n-1}}{(n-1)!}\,(\psi(n) - \log(z)) \;/;\; (z \to 0) \bigwedge n - 2 \in \mathbb{N}^+$$

06.34.06.0011.01

$$E_{-n}(z) == n!\, z^{-n-1}\, e^{-z} \sum_{k=0}^{n} \frac{z^k}{k!} \;/;\; n \in \mathbb{N}$$

06.34.06.0012.01

$$E_{-n}(z) \propto n!\, z^{-n-1}\,(1 + O(z)) \;/;\; (z \to 0) \land n \in \mathbb{N}$$

Asymptotic series expansions

06.34.06.0013.01

$$E_\nu(z) \propto \frac{1}{z}\, e^{-z}\, {}_2F_0\!\left(1, \nu;\; ; -\frac{1}{z}\right) /;\; (|z| \to \infty)$$

06.34.06.0014.01

$$E_\nu(z) \propto \frac{1}{z}\, e^{-z} \left(1 + O\!\left(\frac{1}{z}\right)\right) /;\; (|z| \to \infty)$$

06.34.06.0035.01

$$E_\nu(z) \propto \frac{e^{-z}}{z} \;/;\; (|z| \to \infty)$$

Residue representations

06.34.06.0015.02

$$E_\nu(z) == z^{\nu-1}\, \Gamma(1-\nu) - \sum_{j=0}^{\infty} \operatorname{res}_s\!\left(\frac{z^{-s}}{1-\nu-s}\, \Gamma(s)\right)(-j)$$

06.34.06.0016.02

$$E_\nu(z) == \operatorname{res}_s\!\left(\Gamma(s)\, z^{-s}\, \frac{1}{s+\nu-1}\right)(1-\nu) + \sum_{j=0}^{\infty} \operatorname{res}_s\!\left(\frac{z^{-s}}{s+\nu-1}\, \Gamma(s)\right)(-j)$$

Integral representations

On the real axis

Of the direct function

06.34.07.0001.01

$$E_\nu(z) == z^{\nu-1} \int_z^{\infty} t^{-\nu}\, e^{-t}\, dt \;/;\; |\arg(z)| < \pi$$

Contour integral representations

06.34.07.0002.01

$$E_\nu(z) == z^{\nu-1}\, \Gamma(1-\nu) - \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\Gamma(s)\, \Gamma(1-\nu-s)}{\Gamma(2-\nu-s)}\, z^{-s}\, ds$$

06.34.07.0003.01

$$E_\nu(z) == z^{\nu-1}\, \Gamma(1-\nu) - \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\, \Gamma(1-\nu-s)}{\Gamma(2-\nu-s)}\, z^{-s}\, ds \;/;\; 0 < \gamma < 1 - \operatorname{Re}(\nu) \bigwedge |\arg(z)| < \frac{\pi}{2}$$

06.34.07.0004.01

$$E_\nu(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(s+\nu-1)\,\Gamma(s)\,z^{-s}}{\Gamma(s+\nu)}\,ds$$

06.34.07.0005.01

$$E_\nu(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s+\nu-1)\,\Gamma(s)\,z^{-s}}{\Gamma(s+\nu)}\,ds \;/;\; \max(1-\mathrm{Re}(\nu),\,0) < \gamma \bigwedge |\arg(z)| < \frac{\pi}{2}$$

Continued fraction representations

06.34.10.0001.01

$$E_\nu(z) = \cfrac{e^{-z}}{z + \cfrac{\nu}{1 + \cfrac{1}{z + \cfrac{\nu+1}{1 + \cfrac{2}{z + \cfrac{\nu+2}{1 + \cfrac{3}{z + \ldots}}}}}}} \;/;\; z \notin (-\infty,\,0)$$

06.34.10.0002.01

$$E_\nu(z) = \frac{e^{-z}}{z + \mathrm{K}_k\left(2^{\frac{1}{2}\left(-1-(-1)^k\right)}\,k^{\frac{1}{2}\left(1+(-1)^k\right)}\left(\frac{k-1}{2}+\nu\right)^{\frac{1}{2}\left(1-(-1)^k\right)},\, z^{\frac{1}{2}\left((-1)^k+1\right)}\right)_1^\infty} \;/;\; z \notin (-\infty,\,0)$$

06.34.10.0003.01

$$E_\nu(z) = e^{-z} \Bigg/ \left(\nu + z - \nu \Bigg/ \left(2 + \nu + z - \cfrac{2(\nu+1)}{4 + \nu + z - \cfrac{3(\nu+2)}{6 + \nu + z - \cfrac{4(\nu+3)}{8 + \nu + z - \cfrac{5(\nu+4)}{10 + \nu + z + \ldots}}}}\right)\right) \;/;\; z \notin (-\infty,\,0)$$

06.34.10.0004.01

$$E_\nu(z) = \frac{e^{-z}}{z + \nu + \mathrm{K}_k(-k\,(k+\nu-1),\, \nu + 2\,k + z)_1^\infty} \;/;\; z \notin (-\infty,\,0)$$

06.34.10.0005.01

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) - \cfrac{e^{-z}}{1 - \nu - \cfrac{(1-\nu)\,z}{2 - \nu + \cfrac{z}{3 - \nu - \cfrac{(2-\nu)\,z}{4 - \nu + \cfrac{2\,z}{5 - \nu - \cfrac{(3-\nu)\,z}{6 - \nu + \ldots}}}}}} \;/;\; z \notin (-\infty,\,0)$$

06.34.10.0006.01

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) - \frac{e^{-z}}{1-\nu+\mathrm{K}_k\left((-1)^k\left((1-\nu)^{\frac{1}{2}(1-(-1)^k)} + \left\lfloor \frac{k-1}{2} \right\rfloor\right) z,\, 1-\nu+k\right)_1^\infty} \;/;\; z \notin (-\infty, 0)$$

06.34.10.0007.01

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) - e^{-z} \Bigg/ \left(1-\nu-\cfrac{(1-\nu)\,z}{2-\nu+z-\cfrac{(2-\nu)\,z}{3-\nu+z-\cfrac{(3-\nu)\,z}{4-\nu+z-\cfrac{(4-\nu)\,z}{5-\nu+z-\cfrac{(5-\nu)\,z}{6-\nu+z+\ldots}}}}}\right)$$

06.34.10.0008.01

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) - \frac{e^{-z}}{1-\nu+\mathrm{K}_k((\nu-k)\,z,\, 1-\nu+k+z)_1^\infty}$$

06.34.10.0009.01

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) - \cfrac{e^{-z}}{1-\nu-\cfrac{(1-\nu)\,z}{2-\nu+\cfrac{z}{3-\nu-\cfrac{(2-\nu)\,z}{4-\nu+\cfrac{2\,z}{5-\nu-\cfrac{(3-\nu)\,z}{6-\nu+\ldots}}}}}}$$

06.34.10.0010.01

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) - \frac{e^{-z}}{1-\nu+\mathrm{K}_k\left((-1)^k\left(\frac{k}{2}\right)^{\frac{1}{2}(1+(-1)^k)}\left(\frac{k+1}{2}-\nu\right)^{\frac{1}{2}(1-(-1)^k)} z,\, 1-\nu+k\right)_1^\infty}$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

06.34.13.0001.01

$$z\,w''(z) + (z-\nu+2)\,w'(z) + (1-\nu)\,w(z) = 0 \;/;\; w(z) = c_1\,E_\nu(z) + c_2\,z^{\nu-1}$$

06.34.13.0002.01

$$W_z\left(z^{\nu-1}, E_\nu(z)\right) = -e^{-z}\,z^{\nu-2}$$

06.34.13.0003.01

$$w''(z)+\left(\frac{(-\nu+g(z)+2)\,g'(z)}{g(z)}-\frac{g''(z)}{g'(z)}\right)w'(z)+\frac{(1-\nu)\,g'(z)^2}{g(z)}\,w(z)=0\;/;\;w(z)=c_1\,E_\nu(g(z))+c_2\,g(z)^{\nu-1}$$

06.34.13.0004.01

$$W_z\left(E_\nu(g(z)),\,g(z)^{\nu-1}\right)=e^{-g(z)}\,g(z)^{\nu-2}\,g'(z)$$

06.34.13.0005.01

$$w''(z)+\left(\frac{(-\nu+g(z)+2)\,g'(z)}{g(z)}-\frac{2\,h'(z)}{h(z)}-\frac{g''(z)}{g'(z)}\right)w'(z)+\left(\frac{(1-\nu)\,g'(z)^2}{g(z)}+\frac{(\nu-2)\,h'(z)\,g'(z)}{g(z)\,h(z)}+\frac{2\,h'(z)^2}{h(z)^2}+\frac{h'(z)\,g''(z)}{h(z)\,g'(z)}-\frac{g'(z)\,h'(z)+h''(z)}{h(z)}\right)w(z)=0\;/;\;w(z)=c_1\,h(z)\,E_\nu(g(z))+c_2\,h(z)\;g(z)^{\nu-1}$$

06.34.13.0006.01

$$W_z\left(h(z)\,E_\nu(g(z)),\,h(z)\,g(z)^{\nu-1}\right)=e^{-g(z)}\,g(z)^{\nu-2}\,h(z)^2\,g'(z)$$

06.34.13.0007.01

$$z^2\,w''(z)+(a\,r\,z^r+r-2\,s-r\,\nu+1)\,z\,w'(z)+(s-a\,r\,z^r)\,(s+r\,(\nu-1))\,w(z)=0\;/;\;w(z)=c_2\,z^s\,(a\,z^r)^{\nu-1}+c_1\,z^s\,E_\nu(a\,z^r)$$

06.34.13.0008.01

$$W_z\left(z^s\,E_\nu(a\,z^r),\,z^s\,(a\,z^r)^{\nu-1}\right)=e^{-a\,z^r}\,r\,z^{2\,s-1}\,(a\,z^r)^{\nu-1}$$

06.34.13.0009.01

$$w''(z)+((a\,r^z-\nu+1)\log(r)-2\log(s))\,w'(z)-(a\,r^z\log(r)-\log(s))\,((\nu-1)\log(r)+\log(s))\,w(z)=0\;/;\;w(z)=c_1\,s^z\,E_\nu(a\,r^z)+c_2\,s^z\,(a\,r^z)^{\nu-1}$$

06.34.13.0010.01

$$W_z\left(s^z\,E_\nu(a\,r^z),\,s^z\,(a\,r^z)^{\nu-1}\right)=e^{-a\,r^z}\,(a\,r^z)^{\nu-1}\,s^{2\,z}\log(r)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.34.16.0001.01

$$E_{\nu+1}(z)=\frac{1}{\nu}\left(e^{-z}-z\,E_\nu(z)\right)$$

06.34.16.0002.01

$$E_{\nu-1}(z)=\frac{1}{z}\left(e^{-z}-(\nu-1)\,E_\nu(z)\right)$$

06.34.16.0003.01

$$E_{n+\nu}(z)=\frac{(-z)^n\,E_\nu(z)}{(\nu)_n}-e^{-z}\sum_{k=0}^{n-1}\frac{z^k}{(1-n-\nu)_{k+1}}\;/;\;n\in\mathbb{N}$$

06.34.16.0004.01

$$E_{\nu-n}(z)=(1-\nu)_n\,z^{-n}\,E_\nu(z)-e^{-z}\sum_{k=0}^{n-1}(\nu-n)_k\,(-z)^{-k-1}\;/;\;n\in\mathbb{N}$$

Identities

Recurrence identities

Consecutive neighbors

06.34.17.0001.01

$$E_\nu(z) = \frac{1}{z}\left(e^{-z} - \nu\, E_{\nu+1}(z)\right)$$

06.34.17.0002.01

$$E_\nu(z) = \frac{1}{\nu-1}\left(e^{-z} - z\, E_{\nu-1}(z)\right)$$

Distant neighbors

06.34.17.0003.01

$$E_\nu(z) = (-1)^n\, (\nu)_n\, z^{-n}\, E_{n+\nu}(z) - e^{-z} \sum_{k=0}^{n-1} (\nu)_k\, (-z)^{-k-1} \;/;\; n \in \mathbb{N}$$

06.34.17.0004.01

$$E_\nu(z) = \frac{z^n}{(1-\nu)_n}\, E_{\nu-n}(z) - e^{-z} \sum_{k=0}^{n-1} \frac{z^k}{(1-\nu)_{k+1}} \;/;\; n \in \mathbb{N}$$

Functional identities

Relations of special kind

06.34.17.0005.01

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}\, E_1(z) - e^{-z} \sum_{k=1}^{n-1} \frac{z^{k-1}}{(1-n)_k} \;/;\; n \in \mathbb{N}^+$$

Differentiation

Low-order differentiation

With respect to ν

06.34.20.0001.01

$$\frac{\partial E_\nu(z)}{\partial \nu} = z^{\nu-1}\, \Gamma(1-\nu)\, (\log(z) - \psi(1-\nu)) - \Gamma(1-\nu)^2\, {}_2\tilde{F}_2(1-\nu, 1-\nu; 2-\nu, 2-\nu; -z)$$

06.34.20.0002.01

$$\frac{\partial^2 E_\nu(z)}{\partial \nu^2} = \Gamma(1-\nu)\, z^{\nu-1} \left(\pi^2 \cot^2(\pi\nu) - 2\pi \log(z) \cot(\pi\nu) + \pi^2 \csc^2(\pi\nu) + \log^2(z) + \psi(\nu)^2 + 2\, (\pi \cot(\pi\nu) - \log(z))\, \psi(\nu) - \psi^{(1)}(\nu)\right) + \frac{2}{(\nu-1)^3}\, {}_3F_3(1-\nu, 1-\nu, 1-\nu; 2-\nu, 2-\nu, 2-\nu; -z)$$

With respect to z

06.34.20.0003.01

$$\frac{\partial E_\nu(z)}{\partial z} == -E_{\nu-1}(z)$$

06.34.20.0004.01

$$\frac{\partial^2 E_\nu(z)}{\partial z^2} == E_{\nu-2}(z)$$

Symbolic differentiation

With respect to ν

06.34.20.0005.02

$$\frac{\partial^n E_\nu(z)}{\partial \nu^n} == \sum_{k=0}^{n} z^{\nu-1} \binom{n}{k} \log^k(z) \frac{\partial^{n-k}\Gamma(1-\nu)}{\partial \nu^{n-k}} + n! \sum_{k=0}^{\infty} \frac{(-1)^{n-k} z^k}{(\nu-k-1)^{n+1} k!} /; \nu \notin \mathbb{N}^+ \bigwedge n \in \mathbb{N}$$

06.34.20.0006.02

$$\frac{\partial^n E_\nu(z)}{\partial \nu^n} == z^{\nu-1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \log^{n-k}(z) \left(\Gamma^{(k)}(1-\nu) - z^{1-\nu} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (k-j)!\, \Gamma(1-\nu)^{k-j+1} \log^j(z)\, {}_{k-j+1}\tilde{F}_{k-j+1}\big(a_1, a_2, \ldots, a_{k-j+1}; a_1+1, a_2+1, \ldots, a_{k-j+1}+1; -z\big)\right) /; a_1 == a_2 == \ldots == a_{n+1} == 1-\nu \bigwedge n \in \mathbb{N}$$

With respect to z

06.34.20.0012.01

$$\frac{\partial^n E_\nu(z)}{\partial z^n} == (-1)^n (1-\nu)_n \left(z^{-n} E_\nu(z) + e^{-z} \sum_{k=0}^{n-1} \frac{z^{k-n}}{(1-\nu)_{k+1}}\right) /; n \in \mathbb{N}$$

06.34.20.0007.02

$$\frac{\partial^n E_\nu(z)}{\partial z^n} == (-1)^n E_{\nu-n}(z) /; n \in \mathbb{N}$$

06.34.20.0008.02

$$\frac{\partial^n E_\nu(z)}{\partial z^n} == \frac{\pi \csc(\pi \nu) z^{\nu-n-1}}{\Gamma(\nu-n)} - z^{-n} \Gamma(1-\nu)\, {}_2\tilde{F}_2(1, 1-\nu; 1-n, 2-\nu; -z) /; n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to ν

06.34.20.0009.01

$$\frac{\partial^\alpha E_\nu(z)}{\partial \nu^\alpha} == \frac{\nu^{-\alpha}}{z} \sum_{k=0}^{\infty} \Gamma^{(k)}(1) (-\nu)^k\, {}_1\tilde{F}_1(k+1; k-\alpha+1; \nu \log(z)) - \nu^{-\alpha} \sum_{k=0}^{\infty} \frac{(-z)^k}{(k+1)!} \left(\frac{\sqrt{\nu}}{\sqrt{k+1}}\right)^\alpha I_{-\alpha}\left(\frac{2\sqrt{\nu}}{\sqrt{k+1}}\right) /; \nu \notin \mathbb{N}^+$$

06.34.20.0010.01

$$\frac{\partial^\alpha E_\nu(z)}{\partial \nu^\alpha} == \nu^{-\alpha} \int_1^\infty e^{-z t} t^{-\nu} (-\nu \log(t))^\alpha Q(-\alpha, 0, -\nu \log(t))\, dt /; \mathrm{Re}(z) > 0$$

With respect to z

06.34.20.0011.01

$$\frac{\partial^\alpha E_\nu(z)}{\partial z^\alpha} = \Gamma(1-\nu)\, \mathcal{FC}_{\exp}^{(\alpha)}(z, \nu-1)\, z^{\nu-\alpha-1} - z^{-\alpha}\, \Gamma(1-\nu)\, {}_2\tilde{F}_2(1, 1-\nu; 1-\alpha, 2-\nu; -z)$$

Integration

Indefinite integration

Involving only one direct function

06.34.21.0001.01

$$\int E_\nu(a\, z)\, dz = -\frac{E_{\nu+1}(a\, z)}{a}$$

06.34.21.0002.01

$$\int E_\nu(z)\, dz = -E_{\nu+1}(z)$$

Involving one direct function and elementary functions

Involving power function

06.34.21.0003.01

$$\int z^{\alpha-1}\, E_\nu(a\, z)\, dz \equiv \frac{z^\alpha}{\alpha+\nu-1}\,(E_\nu(a\, z) - E_{1-\alpha}(a\, z))$$

06.34.21.0004.01

$$\int z^{\alpha-1}\, E_\nu(z)\, dz \equiv \frac{z^\alpha}{\alpha+\nu-1}\,(E_\nu(z) - E_{1-\alpha}(z))$$

Involving only one direct function with respect to ν

06.34.21.0005.01

$$\int E_\nu(z)\, d\nu = \frac{\nu}{z} \sum_{k=0}^{\infty} \Gamma^{(k)}(1)\, (-\nu)^k\, {}_1\tilde{F}_1(k+1; k+2; \nu \log(z)) - \sqrt{\nu} \sum_{k=0}^{\infty} \frac{(-z)^k}{\sqrt{k+1}\; k!}\, I_1\left(\frac{2\sqrt{\nu}}{\sqrt{k+1}}\right) /; \nu \notin \mathbb{N}^+$$

Integral transforms

Fourier cos transforms

06.34.22.0001.01

$$\mathcal{F}c_t[E_\nu(t)](x) = \sqrt{\frac{\pi}{2}}\, \csc\left(\frac{\pi\,\nu}{2}\right) \left(x^2\right)^{-\frac{\nu}{2}} + \sqrt{\frac{2}{\pi}}\, \frac{1}{x^2\,(\nu-2)}\, {}_2F_1\left(1, 1-\frac{\nu}{2}; 2-\frac{\nu}{2}; -\frac{1}{x^2}\right) /; x \in \mathbb{R} \land \mathrm{Re}(\nu) > 0$$

Fourier sin transforms

06.34.22.0002.01

$$\mathcal{F}s_t[E_\nu(t)](x) = \sqrt{\frac{\pi}{2}}\, \left(x^2\right)^{-\frac{\nu}{2}} \mathrm{sgn}(x) \sec\left(\frac{\pi\,\nu}{2}\right) + \sqrt{\frac{2}{\pi}}\, \frac{1}{x\,(\nu-1)}\, {}_2F_1\left(1, \frac{1-\nu}{2}; \frac{3-\nu}{2}; -\frac{1}{x^2}\right) /; x \in \mathbb{R} \land \mathrm{Re}(\nu) > -1$$

Laplace transforms

06.34.22.0003.01

$$\mathcal{L}_t[E_\nu(t)](z) = \frac{z^{-\nu-1}}{\nu-1}\left(z^\nu\,{}_2F_1\left(1,\,1-\nu;\,2-\nu;\,-\frac{1}{z}\right)+(\nu-1)\,\pi\csc(\pi\,\nu)\,z\right)/;\ \text{Re}(z)>0\land\text{Re}(\nu)>0$$

Representations through more general functions

Through hypergeometric functions

Involving ${}_1\tilde{F}_1$

06.34.26.0001.01

$$E_\nu(z) = \Gamma(1-\nu)\left(z^{\nu-1} - {}_1\tilde{F}_1(1-\nu;\,2-\nu;\,-z)\right)/;\ \nu\notin\mathbb{N}^+$$

Involving ${}_1F_1$

06.34.26.0002.01

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) - \frac{1}{1-\nu}\,{}_1F_1(1-\nu;\,2-\nu;\,-z)$$

Involving hypergeometric U

06.34.26.0003.01

$$E_\nu(z) = z^{\nu-1}\,e^{-z}\,U(\nu,\,\nu,\,z)$$

Through Meijer G

Classical cases for the direct function itself

06.34.26.0004.01

$$E_\nu(z) = z^{\nu-1}\,\Gamma(1-\nu) - G_{1,2}^{1,1}\left(z\,\middle|\,\begin{array}{c}\nu\\0,\,\nu-1\end{array}\right)$$

06.34.26.0005.01

$$E_\nu(z) = G_{1,2}^{2,0}\left(z\,\middle|\,\begin{array}{c}\nu\\\nu-1,\,0\end{array}\right)$$

Classical cases involving exp

06.34.26.0006.01

$$e^z\,E_\nu(z) = \frac{1}{\Gamma(\nu)}\,G_{1,2}^{2,1}\left(z\,\middle|\,\begin{array}{c}0\\0,\,\nu-1\end{array}\right)$$

Classical cases for products of exponential integrals E

06.34.26.0007.01

$$E_\nu(-z)\,E_\nu(z) = \frac{2^{\nu-2}}{\sqrt{\pi}\,\Gamma(\nu)}\,G_{2,4}^{4,1}\left(-\frac{z^2}{4}\,\middle|\,\begin{array}{c}0,\,\nu\\0,\,\frac{\nu}{2},\,\frac{\nu-1}{2},\,\nu-1\end{array}\right)$$

06.34.26.0008.01

$$E_\nu\left(-i\sqrt{z}\right)E_\nu\left(i\sqrt{z}\right) = \frac{2^{\nu-2}}{\sqrt{\pi}\,\Gamma(\nu)}\,G_{2,4}^{4,1}\left(\frac{z}{4}\,\middle|\,\begin{array}{c}0,\,\nu\\0,\,\frac{\nu}{2},\,\frac{\nu-1}{2},\,\nu-1\end{array}\right)$$

Representations through equivalent functions

With inverse function

06.34.27.0001.01

$$E_{\nu}\left(Q^{-1}(1-\nu, z)\right) = Q^{-1}(1-\nu, z)^{\nu-1}\,\Gamma(1-\nu)\,z$$

With related functions

06.34.27.0002.01

$$E_{\nu}(z) = z^{\nu-1}\,\Gamma(1-\nu, z)$$

06.34.27.0003.01

$$E_{\nu}(z) = z^{\nu-1}\,\Gamma(1-\nu)\,Q(1-\nu, z)$$

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