GCD

Notations

Traditional name

Greatest common divisor

Traditional notation

$\gcd(n_1, n_2, \ldots, n_m)$

Mathematica StandardForm notation

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GCD[n1, n2, .., nm]
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Primary definition

04.08.02.0001.01

$$\gcd(n_1, n_2, \ldots, n_m) = p \,/;\, p \in \mathbb{Z} \bigwedge \frac{n_k}{p} \in \mathbb{Z} \bigwedge 1 \le k \le m \bigwedge \left(\neg \exists_q\, (q \in \mathbb{Z} \land q > p) \bigwedge \frac{n_k}{q} \in \mathbb{Z} \bigwedge 1 \le k \le m\right)$$

04.08.02.0002.01

$$\gcd(n_1, n_2, \ldots, n_m) = p \,/;\, \mathrm{Re}(p) \in \mathbb{Z} \bigwedge \mathrm{Im}(p) \in \mathbb{Z} \bigwedge \mathrm{Re}\left(\frac{n_k}{p}\right) \in \mathbb{Z} \bigwedge \mathrm{Im}\left(\frac{n_k}{p}\right) \in \mathbb{Z} \bigwedge 1 \le k \le m \bigwedge \left(\neg \exists_q\, (|q| > |p| \land \mathrm{Re}(q) \in \mathbb{Z} \land \mathrm{Im}(q) \in \mathbb{Z}) \bigwedge \mathrm{Re}\left(\frac{n_k}{q}\right) \in \mathbb{Z} \bigwedge \mathrm{Im}\left(\frac{n_k}{q}\right) \in \mathbb{Z} \bigwedge 1 \le k \le m\right)$$

$\gcd(n_1, n_2, \ldots, n_m)$ is the greatest common divisor of the integers (or rational) n_k. It is a greatest integer factor common to all n_k, $1 \le k \le m$.

For complex values $n_k$with rational $\mathrm{Re}(n_k)$and $\mathrm{Im}(n_k)$ the function $\gcd(n_1, n_2, \ldots, n_m)$ is also defined as shown above.

Examples: The greatest common divisor $\gcd(21, 48)$ is 3; similar, other examples are $\gcd(27, 48, 36) = 3$, $\gcd(27 + 3\,i, 48 - 6\,i) = 3 + 3\,i$, $\gcd\left(\frac{2}{3}, \frac{3}{4}\right) = \frac{1}{12}$.

Specific values

Specialized values

04.08.03.0001.01

$$\gcd(n) = |n|$$

04.08.03.0002.01

$$\gcd(0, n) = n$$

04.08.03.0003.01

$$\gcd(n, n) = |n|$$

04.08.03.0004.01

$$\gcd(n, -n) = |n|$$

04.08.03.0005.01

$$\gcd(n_1, n_1, \ldots, n_1) = |n_1|$$

04.08.03.0006.01

$$\gcd(p_1, p_2) = 1 \;/;\; p_1 \neq p_2 \bigwedge p_1 \in \mathbb{P} \bigwedge p_2 \in \mathbb{P}$$

04.08.03.0007.01

$$\gcd(2^m - 1, 2^n - 1) = 2^{\gcd(m,n)} - 1 \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$$

04.08.03.0008.01

$$\gcd\left(2^{2^m} + 1, 2^{2^n} + 1\right) = 1 \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge m \neq n$$

04.08.03.0009.01

$$\gcd\left(\sum_{k=0}^{m-1} 10^k, \sum_{k=0}^{n-1} 10^k\right) = \sum_{k=0}^{\gcd(m,n)-1} 10^k \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$$

04.08.03.0010.01

$$\gcd(n, \operatorname{lcm}(m, n)) = n \;/;\; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$$

04.08.03.0011.01

$$\gcd(n, \operatorname{lcm}(p, q)) = \operatorname{lcm}(\gcd(n, p), \gcd(n, q)) \;/;\; n \in \mathbb{N}^+ \bigwedge p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+$$

04.08.03.0033.01

$$\gcd(\operatorname{lcm}(n, p), \operatorname{lcm}(n, q)) = \operatorname{lcm}(n, \gcd(p, q)) \;/;\; n \in \mathbb{N}^+ \bigwedge p \in \mathbb{N}^+ \bigwedge q \in \mathbb{N}^+$$

04.08.03.0012.01

$$\gcd(\operatorname{lcm}(k, m), \operatorname{lcm}(k, n), \operatorname{lcm}(m, n)) = \operatorname{lcm}(\gcd(k, m), \gcd(k, n), \gcd(m, n)) \;/;\; k \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+$$

04.08.03.0013.01

$$\gcd(F_m, F_n) = F_{\gcd(m,n)} \;/;\; m \in \mathbb{Z} \land n \in \mathbb{Z}$$

04.08.03.0014.01

$$\gcd(n \bmod m, m) = \gcd(n, m) \;/;\; m > 0$$

Values at fixed points

04.08.03.0034.01

$$\gcd(0, 0) = 0$$

04.08.03.0015.01

$$\gcd(1, 1) = 1$$

04.08.03.0016.01

$$\gcd(1, 2) = 1$$

04.08.03.0017.01

$$\gcd(2, 2) = 2$$

04.08.03.0018.01

$$\gcd(3, 2) = 1$$

04.08.03.0019.01

$\gcd(4, 2) = 2$

04.08.03.0020.01

$\gcd(1, 3) = 1$

04.08.03.0021.01

$\gcd(2, 3) = 1$

04.08.03.0022.01

$\gcd(3, 3) = 3$

04.08.03.0023.01

$\gcd(4, 3) = 1$

04.08.03.0024.01

$\gcd(5, 3) = 1$

04.08.03.0025.01

$\gcd(6, 3) = 3$

04.08.03.0026.01

$\gcd(4, 6) = 2$

04.08.03.0027.01

$\gcd(36, 45) = 9$

04.08.03.0028.01

$\gcd(-36, 45) = 9$

04.08.03.0029.01

$\gcd(36, -45) = 9$

04.08.03.0030.01

$\gcd(-36, -45) = 9$

04.08.03.0031.01

$\gcd(-45, -36) = 9$

04.08.03.0032.01

$\gcd(30, 15, 5) = 5$

General characteristics

Domain and analyticity

$\gcd(n_1, n_2, \ldots, n_m)$ is nonanalytical function defined on $\mathbb{Z}^m$with values in $\mathbb{Z}$.

04.08.04.0001.01

$(n_1 * n_2 * \ldots * n_m) \longrightarrow \gcd(n_1, n_2, \ldots, n_m) :: \mathbb{Z}^m \longrightarrow \mathbb{Z}$

Symmetries and periodicities

Parity

$\gcd(n_1, n_2, \ldots, n_m)$ is an even function.

04.08.04.0002.01

$$\gcd(-n_1, -n_2, \ldots, -n_m) = \gcd(n_1, n_2, \ldots, n_m)$$

04.08.04.0003.01

$$\gcd(-n_1, n_2, \ldots, n_m) = \gcd(n_1, n_2, \ldots, n_m)$$

Permutation symmetry

04.08.04.0004.01

$$\gcd(m, n) = \gcd(n, m)$$

04.08.04.0005.01

$$\gcd(n_1, n_2, \ldots, n_k, \ldots, n_j, \ldots, n_m) = \gcd(n_1, n_2, \ldots, n_j, \ldots, n_k, \ldots, n_m) \;/;\; n_k \neq n_j \land k \neq j$$

Periodicity

No periodicity

Series representations

Generalized power series

04.08.06.0001.01

$$\gcd(m, n) = -n\,m + m + n + 2\sum_{k=1}^{m-1} \left\lfloor \frac{k\,n}{m} \right\rfloor$$

04.08.06.0002.01

$$\gcd(m, n) = 1 - 2\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor - \delta_{\frac{m}{2} - \left\lfloor \frac{m}{2} \right\rfloor}\, \delta_{\frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor} + 2\sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \left\lfloor \frac{k\,n}{m} \right\rfloor + 2\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left\lfloor \frac{k\,m}{n} \right\rfloor$$

Product representations

04.08.08.0001.01

$$\gcd(n_1, n_2) = \prod_{j=1}^{j_k} p_{i,j}^{\min(\alpha_{1,j}, \alpha_{2,j})} \;/;$$

$$n_1 \in \mathbb{N}^+ \bigwedge n_2 \in \mathbb{N}^+ \bigwedge \text{factors}(n_k) = \{\{p_{k,1}, \alpha_{k,1}\}, \ldots, \{p_{k,j_k}, \alpha_{k,j_k}\}\} \bigwedge p_{k,j} \in \mathbb{P} \bigwedge \alpha_{k,j} \in \mathbb{N}^+ \bigwedge 1 \leq k \leq 2$$

Generating functions

04.08.11.0001.01

$$\sum_{k=1}^{\infty} \gcd(k, n)\, x^k = \sum_{d|n} \phi(d)\, \frac{x^d}{1 - x^d}$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

04.08.16.0001.01

$$\gcd(-n_1, -n_2, \ldots, -n_m) = \gcd(n_1, n_2, \ldots, n_m)$$

04.08.16.0002.01

$$\gcd(-n_1, n_2, \ldots, n_m) = \gcd(n_1, n_2, \ldots, n_m)$$

Multiple arguments

04.08.16.0003.02

$$\gcd(p\, n_1, p\, n_2, \ldots, p\, n_m) = p \gcd(n_1, n_2, \ldots, n_m) \;/;\; p \in \mathbb{N}$$

04.08.16.0004.01

$$\gcd(\mu\, m, \nu\, n) = \gcd(m, n) \gcd(\mu, \nu) \gcd\left(\frac{m}{\gcd(m, n)}, \frac{\nu}{\gcd(\mu, \nu)}\right) \gcd\left(\frac{n}{\gcd(m, n)}, \frac{\mu}{\gcd(\mu, \nu)}\right) /;$$
$$m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N}^+ \bigwedge \mu \in \mathbb{N}^+ \bigwedge \nu \in \mathbb{N}^+$$

Identities

Functional identities

04.08.17.0001.01

$$\gcd(\gcd(m, n), p) = \gcd(m, \gcd(n, p))$$

04.08.17.0002.01

$$\gcd(n_1, \gcd(n_2, n_3, \ldots, n_m)) = \gcd(n_1, n_2, n_3, \ldots, n_m)$$

04.08.17.0003.01

$$\gcd(m, n, p) = \gcd(m, \gcd(n, p))$$

04.08.17.0004.01

$$\gcd(n_1, n_2, n_3, \ldots, n_m) = \gcd(n_1, \gcd(n_2, n_3, \ldots, n_m))$$

Summation

Finite summation

04.08.23.0001.01

$$\sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \ldots \sum_{k_m=1}^{n} F(\gcd(k_1, k_2, \ldots, k_m)) = \sum_{k=1}^{n} f(d) \left\lfloor \frac{n}{d} \right\rfloor^m /;\; F(n) = \sum_{d|n} f(d)$$

Infinite summation

04.08.23.0002.01

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\delta_{1,\gcd(k,n)}}{k\, n\, (k+n)} = \frac{5}{4}$$

04.08.23.0003.01

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\delta_{1,\gcd(k,n)}}{n^2\, (k+n)} = \frac{3}{4}$$

04.08.23.0004.01

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\delta_{1,\gcd(k,n])}}{k\,n\,(k+n)^2}=\frac{3}{8}$$

04.08.23.0005.01

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\delta_{1,\gcd(k,n)}}{(k\,n\,(k+n))^2}=\frac{7}{24}$$

04.08.23.0006.01

$$\sum_{b=1}^{\infty}\sum_{d=1}^{\infty}\frac{\delta_{1,\gcd(b,d)}}{(b\,d\,(b+d))^2}=\frac{1}{3}$$

Operations

Limit operation

04.08.25.0001.01

$$\lim_{n\to\infty}\frac{1}{n^r}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\dots\sum_{k_r=1}^{n}\gcd(k_1,\,k_2,\,\dots,\,k_r)^k=\frac{\zeta(r-k)}{\zeta(r)}$$

04.08.25.0002.01

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\delta_{1,\gcd(k,l)}=\frac{6}{\pi^2}$$

Representations through more general functions

Through other functions

04.08.26.0001.01

$$\mathrm{egcd}(m,\,n)=\{\gcd(m,\,n),\,\{r,\,s\}\}$$

Representations through equivalent functions

With related functions

04.08.27.0001.01

$$\gcd(m,\,n)=\frac{m\,n}{\mathrm{lcm}(m,\,n)}\;/;\;m\in\mathbb{N}^+\bigwedge n\in\mathbb{N}^+$$

04.08.27.0002.01

$$\gcd(n_1,\,n_2)=\prod_{j=1}^{j_k}p_{i,j}^{\min\left(\alpha_{1,j},\alpha_{2,j}\right)}\;/;$$

$$n_1\in\mathbb{N}^+\bigwedge n_2\in\mathbb{N}^+\bigwedge \mathrm{factors}(n_k)=\left\{\{p_{k,1},\,\alpha_{k,1}\},\,\dots,\,\left\{p_{k,j_k},\,\alpha_{k,j_k}\right\}\right\}\bigwedge p_{k,j}\in\mathbb{P}\bigwedge \alpha_{k,j}\in\mathbb{N}^+\bigwedge 1\le k\le 2$$

04.08.27.0003.01

$$\gcd(n_1, n_2, \ldots, n_m) = \left(\prod_{k_1=1}^{m} n_{k_1} \prod_{k_1=1}^{m} \prod_{k_2=k_1+1}^{m} \prod_{k_3=k_2+1}^{m} \operatorname{lcm}(n_{k_1}, n_{k_2}, n_{k_3}) \ldots \right) \Big/ \left(\prod_{k_1=1}^{m} \prod_{k_2=k_1+1}^{m} \operatorname{lcm}(n_{k_1}, n_{k_2}) \prod_{k_1=1}^{m} \prod_{k_2=k_1+1}^{m} \prod_{k_3=k_2+1}^{m} \prod_{k_4=k_3+1}^{m} \operatorname{lcm}(n_{k_1}, n_{k_2}, n_{k_3}, n_{k_4}) \ldots \right)$$

Inequalities

04.08.29.0001.01

$$\gcd(n_1, n_2, \ldots, n_k) \operatorname{lcm}(n_1, n_2, \ldots, n_k)^{k-1} \leq \prod_{j=1}^{k} n_j \leq \gcd(n_1, n_2, \ldots, n_k)^{k-1} \operatorname{lcm}(n_1, n_2, \ldots, n_k) \;/;\; n_1 \in \mathbb{N}^+ \bigwedge n_2 \in \mathbb{N}^+ \bigwedge \ldots \bigwedge n_k \in \mathbb{N}^+ \bigwedge k \in \mathbb{N}^+$$

04.08.29.0002.01

$$\left(\prod_{j_1=1}^{k} \prod_{j_2=j_1+1}^{k} \ldots \prod_{j_m=j_{m-1}+1}^{k} \gcd(n_{j_1}, n_{j_2}, \ldots, n_{j_m}) \right) \left(\prod_{j_1=1}^{k} \prod_{j_2=j_1+1}^{k} \ldots \prod_{j_m=j_{m-1}+1}^{k} \operatorname{lcm}(n_{j_1}, n_{j_2}, \ldots, n_{j_m})^{m-1} \right) \geq \left(\prod_{j=1}^{k} n_j \right)^{\frac{(k-1)!}{(k-m)!\,(m-1)!}} \geq \left(\prod_{j_1=1}^{k} \prod_{j_2=j_1+1}^{k} \ldots \prod_{j_m=j_{m-1}+1}^{k} \gcd(n_{j_1}, n_{j_2}, \ldots, n_{j_m})^{m-1} \right) \left(\prod_{j_1=1}^{k} \prod_{j_2=j_1+1}^{k} \ldots \prod_{j_m=j_{m-1}+1}^{k} \operatorname{lcm}(n_{j_1}, n_{j_2}, \ldots, n_{j_m}) \right) \;/;\; n_1 \in \mathbb{N}^+ \bigwedge n_2 \in \mathbb{N}^+ \bigwedge \ldots \bigwedge n_k \in \mathbb{N}^+ \bigwedge k \in \mathbb{N}^+ \bigwedge m \in \mathbb{N}^+ \bigwedge 1 \leq m \leq k$$

The products are over all subsets of m numbers from the k integers n_j.

Theorems

Distribution of digits in the decimal expansion of the rational number

The decimal expansion of the rational number $\frac{p}{q} \;/;\; \gcd(p, q) = 1$ and $b = 2^{\alpha} 5^{\beta} R \;/;\; \gcd(10, R) = 1$ has exactly $\max(\alpha, \beta)$ nonrepeating and $\nu \;/;\; 10^{\nu} \bmod R = 1$ repeating digits.

The human hearing system

The human auditory perception of the pitch of two frequencies is the gcd of their frequencies.

History

Euclid

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