InverseJacobiNS

Notations

Traditional name

Inverse of the Jacobi elliptic function ns

Traditional notation

$\text{ns}^{-1}(z \mid m)$

Mathematica StandardForm notation

`InverseJacobiNS[z, m]`

Primary definition

09.45.02.0001.01

$$z = \text{ns}(w \mid m) \;/;\; w = \text{ns}^{-1}(z \mid m)$$

09.45.02.0002.01

$$\text{ns}^{-1}(z \mid m) = \int_z^{\infty} \frac{1}{\sqrt{t^2-1}\,\sqrt{t^2-m}}\, dt \;/;\; z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge z^2 > m$$

Specific values

Specialized values

For fixed *z*

09.45.03.0001.01

$$\text{ns}^{-1}(z \mid 0) = \csc^{-1}(z)$$

09.45.03.0002.01

$$\text{ns}^{-1}\left(z \,\middle|\, \frac{1}{2}\right) = \sqrt{2}\, F\left(\sin^{-1}(z) \,\middle|\, 2\right) + \frac{i\,\pi^{3/2}}{2\,\Gamma\left(\frac{3}{4}\right)^2}$$

09.45.03.0003.01

$$\text{ns}^{-1}(z \mid 1) = \coth^{-1}(z)$$

For fixed *m*

09.45.03.0004.01

$$\text{ns}^{-1}(-1 \mid m) = -K(m)$$

09.45.03.0005.01

$$\mathrm{ns}^{-1}\left(-\frac{1}{2} \,\middle|\, m\right) = K(m) - \frac{1}{\sqrt{m}} \left(F\left(\frac{\pi}{6} \,\middle|\, \frac{1}{m}\right) + K\left(\frac{1}{m}\right)\right)$$

09.45.03.0006.01

$$\mathrm{ns}^{-1}(0 \mid m) = K(m) - \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right)$$

09.45.03.0007.01

$$\mathrm{ns}^{-1}\left(\frac{1}{2} \,\middle|\, m\right) = \frac{1}{\sqrt{m}} \left(F\left(\frac{\pi}{6} \,\middle|\, \frac{1}{m}\right) - K\left(\frac{1}{m}\right)\right) + K(m)$$

09.45.03.0008.01

$$\mathrm{ns}^{-1}(1 \mid m) = K(m)$$

09.45.03.0009.01

$$\mathrm{ns}^{-1}(i \mid m) = K(m) - \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) - F\left(i \sinh^{-1}(1) \,\middle|\, \frac{1}{m}\right)\right)$$

09.45.03.0010.01

$$\mathrm{ns}^{-1}(-i \mid m) = K(m) - \frac{1}{\sqrt{m}} \left(K\left(\frac{1}{m}\right) + F\left(i \sinh^{-1}(1) \,\middle|\, \frac{1}{m}\right)\right)$$

Values at infinities

09.45.03.0011.01

$$\mathrm{ns}^{-1}(z \mid \infty) = 0$$

09.45.03.0012.01

$$\mathrm{ns}^{-1}(z \mid -\infty) = 0$$

09.45.03.0013.01

$$\mathrm{ns}^{-1}(\infty \mid m) = 0$$

09.45.03.0014.01

$$\mathrm{ns}^{-1}(-\infty \mid m) = 2\,K(m) - \frac{2}{\sqrt{m}} K\left(\frac{1}{m}\right)$$

General characteristics

Domain and analyticity

$\mathrm{ns}^{-1}(z \mid m)$ is an analytical function of z and m which is defined over $\mathbb{C}^2$.

09.45.04.0001.01

$$(z * m) \longrightarrow \mathrm{ns}^{-1}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Mirror symmetry

09.45.04.0002.01

$$\mathrm{ns}^{-1}(\bar{z} \mid \bar{m}) = \overline{\mathrm{ns}^{-1}(z \mid m)}$$

Quasi-reflection symmetry

09.45.04.0003.01

$$\text{ns}^{-1}(-z \mid m) = \text{ns}^{-1}(z \mid m) - \frac{2}{\sqrt{m}} F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{m}\right)$$

09.45.04.0004.01

$$\text{ns}^{-1}(-z \mid m) = \text{ns}^{-1}(z \mid m) - \frac{2}{\sqrt{m}} \text{ns}^{-1}\left(\frac{1}{z} \,\middle|\, \frac{1}{m}\right)$$

Poles and essential singularities

With respect to *m*

The function $\text{ns}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to m.

09.45.04.0005.01

$$\mathcal{S}ing_m\left(\text{ns}^{-1}(z \mid m)\right) = \{\}$$

With respect to *z*

The function $\text{ns}^{-1}(z \mid m)$ does not have poles and essential singularities with respect to z.

09.45.04.0006.01

$$\mathcal{S}ing_z\left(\text{ns}^{-1}(z \mid m)\right) = \{\}$$

Branch points

With respect to *m*

For fixed z, the function $\text{ns}^{-1}(z \mid m)$ has two branch points: $m = z^2$, $m = \tilde{\infty}$.

09.45.04.0007.01

$$\mathcal{BP}_m\left(\text{ns}^{-1}(z \mid m)\right) = \left\{z^2, \tilde{\infty}\right\}$$

09.45.04.0008.01

$$\mathcal{R}_m\left(\text{ns}^{-1}(z \mid m), z^2\right) = 2$$

09.45.04.0009.01

$$\mathcal{R}_m\left(\text{ns}^{-1}(z \mid m), \tilde{\infty}\right) = 2$$

With respect to *z*

For fixed m, the function $\text{ns}^{-1}(z \mid m)$ has five branch points: $z = \pm 1$, $z = \pm\sqrt{m}$, $z = \tilde{\infty}$.

09.45.04.0010.01

$$\mathcal{BP}_z\left(\text{ns}^{-1}(z \mid m)\right) = \left\{1, -1, \sqrt{m}, -\sqrt{m}, \tilde{\infty}\right\}$$

09.45.04.0011.01

$$\mathcal{R}_z\left(\text{ns}^{-1}(z \mid m), 1\right) = 2$$

09.45.04.0012.01

$$\mathcal{R}_z\left(\text{ns}^{-1}(z \mid m), -1\right) = 2$$

09.45.04.0013.01

$$\mathcal{R}_z\left(\text{ns}^{-1}(z \mid m), \sqrt{m}\right) = 2$$

09.45.04.0014.01

$$\mathcal{R}_z\left(\text{ns}^{-1}(z \mid m), -\sqrt{m}\right) = 2$$

09.45.04.0015.01

$$\mathcal{R}_z\left(\text{ns}^{-1}(z \mid m), \tilde{\infty}\right) = \log$$

Branch cuts

Branch cut locations: complicated

Series representations

Generalized power series

Expansions at $z = 0$

09.45.06.0001.02

$$\text{ns}^{-1}(z \mid m) \propto \frac{1}{\sqrt{-m}}\left(\left(-\frac{1}{m}\right)^{-1/2} K(m) + i\, K\left(\frac{1}{m}\right)\right) + \frac{1}{\sqrt{m}}\left(z + \frac{1+m}{6\,m} z^3 + \frac{3+2\,m+3\,m^2}{40\,m^2} z^5 + \ldots\right) /; (z \to 0)$$

09.45.06.0002.01

$$\text{ns}^{-1}(z \mid m) = \frac{1}{\sqrt{-m}}\left(\left(-\frac{1}{m}\right)^{-1/2} K(m) + i\, K\left(\frac{1}{m}\right)\right) + \sum_{k=0}^{\infty} \frac{m^{-k-\frac{1}{2}} \left(\frac{1}{2}\right)_k}{(2\,k+1)\,k!} \, {}_2F_1\left(\frac{1}{2}, -k; \frac{1}{2} - k; m\right) z^{2\,k+1}$$

09.45.06.0007.01

$$\text{ns}^{-1}(z \mid m) \propto \frac{1}{\sqrt{-m}}\left(K(m) \frac{1}{\sqrt{-\frac{1}{m}}} + i\, K\left(\frac{1}{m}\right)\right) + \frac{z}{\sqrt{m}}\left(1 + O\left(z^2\right)\right)$$

Expansions at $m = 0$

09.45.06.0003.02

$$\text{ns}^{-1}(z \mid m) \propto \csc^{-1}(z) - \frac{1}{4\,z}\left(\sqrt{1-\frac{1}{z^2}} - z\csc^{-1}(z)\right) m - \frac{3}{64\,z^3}\left(\sqrt{1-\frac{1}{z^2}}\left(3\,z^2+2\right) - 3\,z^3 \csc^{-1}(z)\right) m^2 - \ldots /; (m \to 0)$$

09.45.06.0004.01

$$\text{ns}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{z^{-2\,k-1} \left(\frac{1}{2}\right)_k}{k!\,(2\,k+1)} \, {}_2F_1\left(\frac{1}{2}, k + \frac{1}{2}; k + \frac{3}{2}; \frac{1}{z^2}\right) m^k /; |m| < 1$$

09.45.06.0008.01

$$\text{ns}^{-1}(z \mid m) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}\left(\csc^{-1}(z) - \frac{1}{2} z \sqrt{1-\frac{1}{z^2}} \sum_{j=1}^{k} \frac{(j-1)!\, z^{-2\,j}}{\left(\frac{1}{2}\right)_j}\right) m^k /; |m| < 1$$

09.45.06.0005.01

$$\operatorname{ns}^{-1}(z \mid m) \propto \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{m^k z^{-2j-2k-1} \left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_k}{(2j+2k+1)\, j!\, k!}$$

09.45.06.0006.01

$$\operatorname{ns}^{-1}(z \mid m) \propto \frac{1}{z} F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix} \frac{m}{z^2}, \frac{1}{z^2} \right)$$

09.45.06.0009.01

$$\operatorname{ns}^{-1}(z \mid m) \propto \csc^{-1}(z)\,(1 + O(m))$$

Integral representations

On the real axis

Of the direct function

09.45.07.0001.01

$$\operatorname{ns}^{-1}(z \mid m) \propto \int_z^{\infty} \frac{1}{\sqrt{t^2-1}\,\sqrt{t^2-m}}\, dt \,/;\, z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge z^2 > m$$

09.45.07.0002.01

$$\operatorname{ns}^{-1}(z \mid m) = \operatorname{ns}^{-1}(z_0 \mid m) - \frac{\sqrt{z^2-m}\, \operatorname{cd}\left(\operatorname{ns}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2-1}} \int_{z_0}^{z} \frac{1}{\sqrt{t^2-1}\,\sqrt{t^2-m}}\, dt \,/;$$

$$\neg \exists_{\tau, \{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\operatorname{Im}\left((\tau(z-z_0)+z_0)^2 - 1\right) = 0 \bigwedge (\tau(z-z_0)+z_0)^2 - 1 < 0 \bigwedge \operatorname{Im}\left((\tau(z-z_0)+z_0)^2 - m\right) = 0 \bigwedge (\tau(z-z_0)+z_0)^2 - m < 0\right)$$

09.45.07.0003.01

$$\operatorname{ns}^{-1}(z \mid m) = \frac{\sqrt{z^2-m}\, \operatorname{cd}\left(\operatorname{ns}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2-1}} \int_{z}^{\infty} \frac{1}{\sqrt{t^2-1}\,\sqrt{t^2-m}}\, dt \,/;$$

$$\neg \exists_{\tau, \{\tau \in \mathbb{R}, 0<\tau<1\}} \left(\operatorname{Im}\left(\left(z + \tan\left(\frac{\pi\tau}{2}\right)\right)^2 - 1\right) = 0 \bigwedge \left(z + \tan\left(\frac{\pi\tau}{2}\right)\right)^2 - 1 < 0 \bigwedge \operatorname{Im}\left(\left(z + \tan\left(\frac{\pi\tau}{2}\right)\right)^2 - m\right) = 0 \bigwedge \left(z + \tan\left(\frac{\pi\tau}{2}\right)\right)^2 - m < 0\right)$$

Differential equations

Ordinary nonlinear differential equations

09.45.13.0001.01

$$w''(z) + \left(2z^2 - m - 1\right) z\, w'(z)^3 \propto 0 \,/;\, w(z) \propto \operatorname{ns}^{-1}(z \mid m)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.45.16.0001.01

$$\mathrm{ns}^{-1}(-z \mid m) = \mathrm{ns}^{-1}(z \mid m) - \frac{2}{\sqrt{m}} F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{m}\right)$$

09.45.16.0002.01

$$\mathrm{ns}^{-1}(-z \mid m) = \mathrm{ns}^{-1}(z \mid m) - \frac{2}{\sqrt{m}} \mathrm{ns}^{-1}\left(\frac{1}{z} \,\middle|\, \frac{1}{m}\right)$$

Identities

Functional identities

09.45.17.0001.01

$$\left(z_1^2 - z_2^2\right)^2 \mathrm{ns}(w(z_1) + w(z_2) \mid m)^4 + \left(-2 z_2^2 z_1^4 + \left(-2 z_2^4 + 4 (m+1) z_2^2 - 2m\right) z_1^2 - 2 m z_2^2\right) \mathrm{ns}(w(z_1) + w(z_2) \mid m)^2 + \left(m - z_1^2 z_2^2\right)^2 = 0 \;/;\; w(z) = \mathrm{ns}^{-1}(z \mid m)$$

Differentiation

Low-order differentiation

With respect to *z*

09.45.20.0001.01

$$\frac{\partial \mathrm{ns}^{-1}(z \mid m)}{\partial z} = -\frac{1}{\mathrm{cs}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right) \mathrm{ds}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)}$$

09.45.20.0011.01

$$\frac{\partial \mathrm{ns}^{-1}(z \mid m)}{\partial z} = \frac{\mathrm{cs}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right) \mathrm{ds}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)}{\left(m - z^2\right)\left(z^2 - 1\right)}$$

09.45.20.0002.01

$$\frac{\partial \mathrm{ns}^{-1}(z \mid m)}{\partial z} = -\frac{1}{\sqrt{z^2 - 1}\,\sqrt{z^2 - m}} \;/;\; z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge z^2 > m$$

09.45.20.0003.02

$$\frac{\partial^2 \mathrm{ns}^{-1}(z \mid m)}{\partial z^2} = \frac{z\left(2 z^2 - m - 1\right) \mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)}{\left(z^2 - 1\right)^2 \left(z^2 - m\right)}$$

09.45.20.0012.01

$$\frac{\partial^2 \mathrm{ns}^{-1}(z \mid m)}{\partial z^2} = -\frac{\sqrt{z^2 - m}\; \mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2 - 1}} \frac{\partial \frac{1}{\sqrt{z^2 - 1}\,\sqrt{z^2 - m}}}{\partial z}$$

With respect to *m*

09.45.20.0004.01

$$\frac{\partial \mathrm{ns}^{-1}(z \mid m)}{\partial m} = \frac{1}{2(m-1)m}\left(\frac{m\,\mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)}{z} - E\left(\mathrm{am}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right) \mid m\right) + (1-m)\,\mathrm{ns}^{-1}(z \mid m)\right)$$

09.45.20.0005.01

$$\frac{\partial \mathrm{ns}^{-1}(z \mid m)}{\partial m} = \frac{1}{2m(m-1)}\left(\frac{\sqrt{z^2-1}\,z}{\sqrt{z^2-m}} + \sqrt{m}\,E\left(\frac{1}{m}\right) - E(m) - \sqrt{m}\,E\left(\sin^{-1}(z) \,\middle|\, \frac{1}{m}\right) + (1-m)K(m)\right) /;$$

$$z \in \mathbb{R} \bigwedge z^2 > 1 \bigwedge z^2 > m$$

09.45.20.0006.02

$$\frac{\partial^2 \mathrm{ns}^{-1}(z \mid m)}{\partial m^2} = \frac{1}{4(m-1)^2 m^2 z^2}\Bigg(\left((4m-2)E\left(\mathrm{am}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right) \mid m\right) + (m-1)F\left(\mathrm{am}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right) \mid m\right)\right)z^2 +$$
$$3(m-1)^2\,\mathrm{ns}^{-1}(z \mid m)\,z^2 + m\left(-3mz + \frac{(m-1)mz}{z^2-m} + z\right)\mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)\Bigg)$$

09.45.20.0013.01

$$\frac{\partial^3 \mathrm{ns}^{-1}(z \mid m)}{\partial m^3} =$$
$$\frac{1}{8(m-1)^3 m^3}\Bigg((-23(m-1)m-8)E\left(\mathrm{am}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right) \mid m\right) - (m-1)(11m-7)F\left(\mathrm{am}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right) \mid m\right) + \frac{1}{\left(m-z^2\right)^3}$$
$$\Bigg(mz\Bigg(\frac{1}{z^2}\left(m-z^2\right)\left((3m(5m-4)+5)z^4 - m(5m(7m-6)+11)z^2 + m^2(m(23m-24)+9)\right)\mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right) -$$
$$(m-1)\sqrt{1-\frac{m}{z^2}}\left(m-z^2\right)^2\mathrm{cn}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)\Bigg) - 15(m-1)^3\left(m-z^2\right)^3\mathrm{ns}^{-1}(z \mid m)\Bigg)\Bigg)$$

Symbolic differentiation

With respect to *z*

09.45.20.0014.01

$$\frac{\partial^n \mathrm{ns}^{-1}(z \mid m)}{\partial z^n} = \mathrm{ns}^{-1}(z \mid m)\,\delta_n +$$
$$\frac{\sqrt{z^2-m}\,\mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2-1}}\sum_{j=0}^{n-1}\frac{(-1)^{j-1}(1-n)_{2(n-j)-2}}{(n-j-1)!\,(2z)^{n-2j-1}}\sum_{k=0}^{j}\binom{j}{k}\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{j-k}\left(z^2-1\right)^{-k-\frac{1}{2}}\left(z^2-m\right)^{k-j-\frac{1}{2}} /; n \in \mathbb{N}$$

09.45.20.0015.01

$$\frac{\partial^n \mathrm{ns}^{-1}(z \mid m)}{\partial z^n} = \mathrm{ns}^{-1}(z \mid m)\,\delta_n +$$
$$\frac{\mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)}{z^2-1}\sum_{j=0}^{n-1}\frac{(-1)^{j-1}\,2^{2j-n+1}\,z^{2j-n+1}\left(z^2-m\right)^{-j}(1-n)_{2(n-j)-2}\left(\frac{1}{2}\right)_j}{(n-j-1)!}\,{}_2F_1\left(\frac{1}{2}, -j; \frac{1}{2}-j; \frac{z^2-m}{z^2-1}\right) /; n \in \mathbb{N}$$

09.45.20.0016.01

$$\frac{\partial^n \mathrm{ns}^{-1}(z \mid m)}{\partial z^n} = \delta_n \, \mathrm{ns}^{-1}(z \mid m) - \frac{\sqrt{z^2-m}\;\mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \,\middle|\, m\right)}{\sqrt{z^2-1}} \frac{\partial^{n-1} \frac{1}{\sqrt{z^2-1}\,\sqrt{z^2-m}}}{\partial z^{n-1}} \;/;\; n \in \mathbb{N}^+$$

09.45.20.0007.02

$$\frac{\partial^n \mathrm{ns}^{-1}(z \mid m)}{\partial z^n} = \delta_n \, \mathrm{ns}^{-1}(z \mid m) - \frac{2^{n-1} \pi z^{n-1} (n-1)! \, \mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \,\middle|\, m\right)}{z^2-1} \sum_{j=0}^{n-1} \frac{\left(z^2-1\right)^{-j} \left(z^2-m\right)^{j-n+1}}{j! \, (n-j-1)! \, \Gamma\left(\frac{1}{2}-j\right) \Gamma\left(j-n+\frac{3}{2}\right)}$$
$$ {}_2F_1\left(\frac{1-j}{2}, -\frac{j}{2}; \frac{1}{2}-j; 1-\frac{1}{z^2}\right) {}_2F_1\left(\frac{j-n+2}{2}, \frac{j-n+1}{2}; j-n+\frac{3}{2}; 1-\frac{m}{z^2}\right) /;\; n \in \mathbb{N}$$

With respect to m

09.45.20.0008.02

$$\frac{\partial^n \mathrm{ns}^{-1}(z \mid m)}{\partial m^n} = \frac{(-1)^n \sqrt{\pi}\, z^{-2n-1}}{(2n+1)\,\Gamma\left(\frac{1}{2}-n\right)} F_1\left(n+\frac{1}{2}; \frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; \frac{1}{z^2}, \frac{m}{z^2}\right) /;\; |z| > 1 \land n \in \mathbb{N}$$

Fractional integro-differentiation

With respect to z

09.45.20.0009.01

$$\frac{\partial^\alpha \mathrm{ns}^{-1}(z \mid m)}{\partial z^\alpha} = \frac{\sqrt{\pi}\, z^{1-\alpha}}{\sqrt{m}} \tilde{F}_{2\times 0\times 0}^{2\times 1\times 1}\left(\begin{matrix} \frac{1}{2}, 1; \frac{1}{2}; \frac{1}{2}; \\ \frac{3-\alpha}{2}, 1-\frac{\alpha}{2};;; \end{matrix}\; z^2, \frac{z^2}{m}\right) + \frac{z^{-\alpha}}{\Gamma(1-\alpha)} \left(K(m) - \frac{1}{\sqrt{m}} K\left(\frac{1}{m}\right)\right) /;\; -1 < z < 1 \land m < 0$$

With respect to m

09.45.20.0010.01

$$\frac{\partial^\alpha \mathrm{ns}^{-1}(z \mid m)}{\partial m^\alpha} = \frac{m^{-\alpha} \sqrt{\pi}}{2z} \tilde{F}_{1\times 0\times 1}^{1\times 1\times 2}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}, 1; \\ \frac{3}{2};; 1-\alpha; \end{matrix}\; \frac{1}{z^2}, \frac{m}{z^2}\right) /;\; (z > 1 \lor z < -1) \bigwedge z^2 > m$$

Integration

Indefinite integration

Involving only one direct function

09.45.21.0001.01

$$\int \mathrm{ns}^{-1}(z \mid m)\, dz = z\, \mathrm{ns}^{-1}(z \mid m) + \log\left(\mathrm{cs}\left(\mathrm{ns}^{-1}(z \mid m) \,\middle|\, m\right) + \mathrm{ds}\left(\mathrm{ns}^{-1}(z \mid m) \,\middle|\, m\right)\right)$$

Involving only one direct function with respect to m

09.45.21.0002.01

$$\int \mathrm{ns}^{-1}(z \mid m)\, dm = 2\left(-z - \sqrt{m}\, E\left(\frac{1}{m}\right) + E(m) + \sqrt{m}\, E\left(\sin^{-1}(z) \,\middle|\, \frac{1}{m}\right) + (m-1)\, K(m)\right) /;\; z > 1 \land m < 1$$

Representations through more general functions

Through hypergeometric functions of two variables

09.45.26.0001.01

$$\mathrm{ns}^{-1}(z \mid m) = \frac{1}{z} F_{1\times 0\times 0}^{1\times 1\times 1}\left(\begin{matrix} \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; \\ \frac{3}{2};;; \end{matrix} \frac{m}{z^2}, \frac{1}{z^2}\right)$$

Through other functions

Involving some hypergeometric-type functions

09.45.26.0002.01

$$\mathrm{ns}^{-1}(z \mid m) = \frac{1}{z} F_1\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{1}{z^2}, \frac{m}{z^2}\right) /; (z > 1 \lor z < -1) \bigwedge z^2 > m$$

Representations through equivalent functions

With inverse function

09.45.27.0001.01

$$\mathrm{ns}\left(\mathrm{ns}^{-1}(z \mid m) \,\middle|\, m\right) = z$$

With related functions

Involving cd^{-1}

09.45.27.0002.01

$$\mathrm{ns}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{m}} \mathrm{cd}^{-1}\left(z \,\middle|\, \frac{1}{m}\right) /; -1 < z < 1 \land m < 0$$

Involving cn^{-1}

09.45.27.0003.01

$$\mathrm{ns}^{-1}(z \mid m) = \mathrm{cn}^{-1}\left(\frac{\sqrt{z^2 - 1}}{z} \,\middle|\, m\right) /; z > 1 \land m < 1$$

Involving cs^{-1}

09.45.27.0004.01

$$\mathrm{ns}^{-1}(i z \mid m) = i \,\mathrm{cs}^{-1}(-z \mid 1 - m)$$

Involving dc^{-1}

09.45.27.0005.01

$$\mathrm{ns}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{m}} \mathrm{dc}^{-1}\left(\frac{1}{z} \,\middle|\, \frac{1}{m}\right) /; -1 < z < 1 \land m < 0$$

Involving dn^{-1}

09.45.27.0006.01

$$\mathrm{ns}^{-1}(z \mid m) = K(m) + \frac{i}{\sqrt{m}} \mathrm{dn}^{-1}\left(\frac{1}{z} \,\middle|\, 1 - \frac{1}{m}\right) /; z > -1 \land m > 1$$

Involving ds^{-1}

09.45.27.0007.01

$$\text{ns}^{-1}(z \mid m) = \frac{1}{\sqrt{1-m}} \text{ds}^{-1}\left(\frac{z}{\sqrt{1-m}} \,\middle|\, \frac{m}{m-1}\right) /; 0 < z < 1 \land m < 1$$

Involving nc^{-1}

09.45.27.0008.01

$$\text{ns}^{-1}(z \mid m) = K(m) + \frac{i}{\sqrt{1-m}} \text{nc}^{-1}\left(\frac{1}{z} \,\middle|\, \frac{1}{1-m}\right) /; z > 1 \land m \in \mathbb{R}$$

09.45.27.0009.01

$$\text{ns}^{-1}(z \mid m) = K(m) - \frac{1}{\sqrt{1-m}} \text{nc}^{-1}\left(z \,\middle|\, \frac{m}{m-1}\right) /; z > 1 \land m < 1$$

Involving nd^{-1}

09.45.27.0010.01

$$\text{ns}^{-1}(z \mid m) = K(m) + \frac{i}{\sqrt{m}} \text{nd}^{-1}\left(z \,\middle|\, 1 - \frac{1}{m}\right) /; -1 < z < 1 \land m < 0$$

Involving sc^{-1}

09.45.27.0011.01

$$\text{ns}^{-1}(z \mid m) = -i \, \text{sc}^{-1}\left(\frac{i}{z} \,\middle|\, 1-m\right) /; z > 1 \land m \in \mathbb{R}$$

Involving sd^{-1}

09.45.27.0012.01

$$\text{ns}^{-1}(z \mid m) = \frac{i}{\sqrt{1-m}} \text{sd}^{-1}\left(\frac{\sqrt{m-1}}{z} \,\middle|\, \frac{1}{1-m}\right) /; z > 0 \land m > 1$$

Involving sn^{-1}

09.45.27.0013.01

$$\text{ns}^{-1}(z \mid m) = \text{sn}^{-1}\left(\frac{1}{z} \,\middle|\, m\right) /; z \in \mathbb{R} \land m < 1$$

Involving elliptic integrals

09.45.27.0014.01

$$\text{ns}^{-1}(z \mid m) = F\left(\sin^{-1}\left(\frac{1}{z}\right) \,\middle|\, m\right) /; (z < -1 \lor z > 1) \land m < 1$$

09.45.27.0015.01

$$\text{ns}^{-1}(z \mid m) = \frac{1}{\sqrt{m}} F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{m}\right) + \frac{1}{\sqrt{-m}} \left(\left(-\frac{1}{m}\right)^{-1/2} K(m) + i \, K\left(\frac{1}{m}\right)\right)$$

09.45.27.0017.01

$$\mathrm{ns}^{-1}(z \mid m) = \mathrm{ns}^{-1}(z_0 \mid m) - \frac{\sqrt{z^2-m}\ \mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)}{\sqrt{z^2-1}} \left(\frac{\sqrt{1-z^2}\ \sqrt{\frac{m-z^2}{m}}}{\sqrt{z^2-1}\ \sqrt{z^2-m}} F\left(\sin^{-1}(z) \,\middle|\, \frac{1}{m}\right) - \frac{\sqrt{1-z_0^2}\ \sqrt{\frac{m-z_0^2}{m}}}{\sqrt{z_0^2-1}\ \sqrt{z_0^2-m}} F\left(\sin^{-1}(z_0) \,\middle|\, \frac{1}{m}\right) \right) /;$$

$$\neg\, \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\mathrm{Im}\left((\tau\,(z-z_0)+z_0)^2-1\right)=0 \bigwedge (\tau\,(z-z_0)+z_0)^2-1<0 \bigwedge \mathrm{Im}\left((\tau\,(z-z_0)+z_0)^2-m\right)=0 \bigwedge (\tau\,(z-z_0)+z_0)^2-m<0\right)$$

09.45.27.0018.01

$$\mathrm{ns}^{-1}(z \mid m) = \frac{z^2\ \mathrm{cd}\left(\mathrm{ns}^{-1}(z \mid m) \mid m\right)}{z^2-1} \sqrt{1-\frac{1}{z^2}}\ \sqrt{1-\frac{m}{z^2}}\ F\left(\csc^{-1}(z) \mid m\right) /;$$

$$\neg\, \exists_{\tau,\{\tau\in\mathbb{R},0<\tau<1\}} \left(\mathrm{Im}\left(\left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2-1\right)=0 \bigwedge \left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2-1<0 \bigwedge \mathrm{Im}\left(\left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2-m\right)=0 \bigwedge \left(z+\tan\left(\frac{\pi\,\tau}{2}\right)\right)^2-m<0\right)$$

Involving other related functions

09.45.27.0016.01

$$\mathrm{ns}^{-1}(z \mid m) == -\frac{\sqrt{z_2^2}}{z_2}\ \mathrm{elog}(z_1,\, z_2;\, a,\, b) /; \{a,\, b,\, z_1\} == \left\{-m-1,\, m,\, z^2\right\} \bigwedge z_1^3 + a\, z_1^2 + b\, z_1 - z_2^2 == 0 \bigwedge z > 0 \bigwedge m < 1$$

History

– N. H. Abel (1826)
– A. G. Greenhill (1892)
– L. M. Milne–Thompson (1948)

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