# JacobiCS



## Notations

### Traditional name

Jacobi elliptic function sc

### Traditional notation

$\operatorname{cs}(z \mid m)$

### *Mathematica* StandardForm notation

`JacobiCS[z, m]`

## Primary definition

09.27.02.0001.01

$$\operatorname{cs}(z \mid m) == \frac{\operatorname{cn}(z \mid m)}{\operatorname{sn}(z \mid m)}$$

## Specific values

### Specialized values

**For fixed *z***

#### Case *m* = 0

09.27.03.0001.01

$$\operatorname{cs}(z \mid 0) == \cot(z)$$

09.27.03.0002.01

$$\operatorname{cs}\left(z + \frac{\pi}{2} \,\middle|\, 0\right) == -\tan(z)$$

09.27.03.0029.01

$$\operatorname{cs}\left(z + \frac{\pi k}{2} \,\middle|\, 0\right) == \cot\left(z + \frac{\pi k}{2}\right) /; k \in \mathbb{Z}$$

#### Case *m* = 1

09.27.03.0003.01

$$\operatorname{cs}(z \mid 1) == \operatorname{csch}(z)$$

09.27.03.0030.01

$$\text{cs}\left(z+\frac{\pi i}{2}\,\middle|\,1\right)=-i\,\text{sech}(z)$$

09.27.03.0031.01

$$\text{cs}\left(z+\frac{i\pi k}{2}\,\middle|\,1\right)=\text{csch}\left(z+\frac{i\pi k}{2}\right)/;\,k\in\mathbb{Z}$$

**For fixed *m***

## Values at quarter-period points in the fundamental period parallelogram

09.27.03.0004.01

$$\text{cs}(0\mid m)=\tilde{\infty}$$

09.27.03.0005.01

$$\text{cs}(K(m)\mid m)=0$$

09.27.03.0006.01

$$\text{cs}(2K(m)\mid m)=\tilde{\infty}$$

09.27.03.0007.01

$$\text{cs}(3K(m)\mid m)=0$$

09.27.03.0008.01

$$\text{cs}(4K(m)\mid m)=\tilde{\infty}$$

09.27.03.0009.01

$$\text{cs}(i\,K(1-m)\mid m)=-i$$

09.27.03.0010.01

$$\text{cs}(2\,i\,K(1-m)\mid m)=\tilde{\infty}$$

09.27.03.0011.01

$$\text{cs}(3\,i\,K(1-m)\mid m)=i$$

09.27.03.0012.01

$$\text{cs}(4\,i\,K(1-m)\mid m)=\tilde{\infty}$$

09.27.03.0013.01

$$\text{cs}(K(m)+i\,K(1-m)\mid m)=-i\sqrt{1-m}$$

09.27.03.0014.01

$$\text{cs}(2K(m)+i\,K(1-m)\mid m)=-i$$

09.27.03.0015.01

$$\text{cs}(3K(m)+i\,K(1-m)\mid m)=-i\sqrt{1-m}$$

09.27.03.0016.01

$$\text{cs}(4K(m)+i\,K(1-m)\mid m)=-i$$

09.27.03.0017.01

$$\text{cs}(K(m)+2\,i\,K(1-m)\mid m)=0$$

09.27.03.0018.01

$$\text{cs}(2K(m)+2\,i\,K(1-m)\mid m)=\tilde{\infty}$$

09.27.03.0019.01

$$\text{cs}(3K(m)+2\,i\,K(1-m)\mid m)=0$$

09.27.03.0020.01

$$\operatorname{cs}(4\,K(m) + 2\,i\,K(1-m) \mid m) = \tilde{\infty}$$

09.27.03.0021.01

$$\operatorname{cs}(K(m) + 3\,i\,K(1-m) \mid m) = i\,\sqrt{1-m}$$

09.27.03.0022.01

$$\operatorname{cs}(2\,K(m) + 3\,i\,K(1-m) \mid m) = i$$

09.27.03.0023.01

$$\operatorname{cs}(K(m) + 4\,i\,K(1-m) \mid m) = 0$$

09.27.03.0024.01

$$\operatorname{cs}(2\,K(m) + 4\,i\,K(1-m) \mid m) = \tilde{\infty}$$

09.27.03.0025.01

$$\operatorname{cs}(2\,r\,K(m) + 2\,s\,i\,K(1-m) \mid m) = \tilde{\infty} \;/;\; \{r, s\} \in \mathbb{Z}$$

## Values at half-quarter-period points

09.27.03.0026.01

$$\operatorname{cs}\left(\frac{K(m)}{2} \,\middle|\, m\right) = \sqrt[4]{1-m}$$

09.27.03.0027.01

$$\operatorname{cs}\left(\frac{i\,K(1-m)}{2} \,\middle|\, m\right) = -i\,\sqrt{1+\sqrt{m}}$$

09.27.03.0028.01

$$\operatorname{cs}\left(\frac{K(m)}{2} + \frac{i\,K(1-m)}{2} \,\middle|\, m\right) = (1-i)\,\frac{\sqrt[4]{1-m}}{i\,\sqrt{1-\sqrt{m}} + \sqrt{1+\sqrt{m}}}$$

# General characteristics

## Domain and analyticity

$\operatorname{cs}(z \mid m)$ is a meromorphic function of $z$ and $m$ which is defined over $\mathbb{C}^2$.

09.27.04.0001.01

$$(z * m) \longrightarrow \operatorname{cs}(z \mid m) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

## Symmetries and periodicities

### Parity

$\operatorname{cs}(z \mid m)$ is an odd function with respect to $z$.

09.27.04.0002.01

$$\operatorname{cs}(-z \mid m) = -\operatorname{cs}(z \mid m)$$

### Mirror symmetry

09.27.04.0003.01

$$\operatorname{cs}(\bar{z} \mid \bar{m}) = \overline{\operatorname{cs}(z \mid m)}$$

### Periodicity

$\text{cs}(z \mid m)$ is a doubly periodic function with respect to $z$ with periods $4\, i\, K(1-m)$ and $2\, K(m)$.

09.27.04.0004.01

$$\text{cs}(z + 2\, K(m) \mid m) = \text{cs}(z \mid m)$$

09.27.04.0005.01

$$\text{cs}(z + 2\, i\, K(1-m) \mid m) = -\text{cs}(z \mid m)$$

09.27.04.0006.01

$$\text{cs}(z + 4\, i\, K(1-m) \mid m) = \text{cs}(z \mid m)$$

09.27.04.0007.01

$$\text{cs}(z + 2\, K(m) + 2\, i\, K(1-m) \mid m) = -\text{cs}(z \mid m)$$

09.27.04.0008.01

$$\text{cs}(z + 2\, i\, s\, K(1-m) + 2\, r\, K(m) \mid m) = (-1)^s\, \text{cs}(z \mid m) \,/;\, \{r, s\} \in \mathbb{Z}$$

## Poles and essential singularities

### With respect to *z*

For fixed $m$, the function $\text{cs}(z \mid m)$ has an infinite set of singular points:

a) $z = 2\, r\, K(m) + 2\, s\, i\, K(1-m)$, $\{r, s\} \in \mathbb{Z}$, are the simple poles with residues $(-1)^s$;

b) $z = \tilde{\infty}$ is an essential singular point.

09.27.04.0009.01

$$\mathcal{S}ing_z(\text{cs}(z \mid m)) = \{\{\{2\, s\, i\, K(1-m) + 2\, r\, K(m),\, 1\} \,/;\, \{r, s\} \in \mathbb{Z}\},\, \{\tilde{\infty},\, \infty\}\}$$

09.27.04.0010.01

$$\text{res}_z(\text{cs}(z \mid m))\,(2\, s\, i\, K(1-m) + 2\, r\, K(m)) = (-1)^s \,/;\, \{r, s\} \in \mathbb{Z}$$

## Branch points

### With respect to *m*

For fixed $z$, the function $\text{cs}(z \mid m)$ is a meromorphic function in $m$ that has no branch points.

09.27.04.0013.01

$$\mathcal{BP}_m(\text{cs}(z \mid m)) = \{\}$$

P. Walker

### With respect to *z*

For fixed $m$, the function $\text{cs}(z \mid m)$ does not have branch points.

09.27.04.0011.01

$$\mathcal{BP}_z(\text{cs}(z \mid m)) = \{\}$$

## Branch cuts

### With respect to *m*

For fixed $z$, the function $\text{cs}(z \mid m)$ is a meromorphic function in $m$ that has no branch cuts.

09.27.04.0014.01

$\mathcal{BC}_m(\mathrm{cs}(z \mid m)) = \{\}$

P. Walker

### With respect to *z*

For fixed $m$, the function $\mathrm{cs}(z \mid m)$ does not have branch cuts.

09.27.04.0012.01

$\mathcal{BC}_z(\mathrm{cs}(z \mid m)) = \{\}$

# Series representations

## Generalized power series

### Expansions at $z = 0$

09.27.06.0005.01

$$\mathrm{cs}(z \mid m) \propto \frac{1}{z} + \frac{1}{6}(m-2)\,z + \frac{1}{360}\left(7\,m^2 + 8\,m - 8\right)z^3 + \ldots \;/;\; (z \to 0)$$

09.27.06.0001.02

$$\mathrm{cs}(z \mid m) \propto \frac{1}{z} + \frac{1}{6}(-2+m)\,z + \frac{1}{360}\left(-8 + 8\,m + 7\,m^2\right)z^3 + \frac{\left(-32 + 48\,m - 78\,m^2 + 31\,m^3\right)z^5}{15\,120} + \frac{\left(-128 + 256\,m + 96\,m^2 - 224\,m^3 + 127\,m^4\right)z^7}{604\,800} + \frac{\left(-512 + 1280\,m - 1568\,m^2 + 1072\,m^3 - 1294\,m^4 + 511\,m^5\right)z^9}{23\,950\,080} + O\left(z^{11}\right)$$

09.27.06.0006.01

$$\mathrm{cs}(z \mid m) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(j+1)\,(-1)^{k-j}\,\mathrm{cn}_{k-j}(m)}{(2\,k - 2\,j)!} \sum_{r=0}^{j} \frac{(-1)^r}{r+1} \binom{j}{r} q_{r,j}\, z^{2\,k-1} \;/;\; q_{j,0} = 1 \bigwedge q_{j,k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(j\,i + i - k)\,(-1)^i\,\mathrm{sn}_i(m)\,q_{j,k-i}}{(2\,i+1)!} \bigwedge k \in \mathbb{N}^+ \bigwedge \mathrm{sn}_0(m) = 1 \bigwedge \mathrm{sn}_n(m) = \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{2\,n}{2\,j} \mathrm{cn}_j(m)\,\mathrm{dn}_k(m)\,\delta_{j+k-n} \bigwedge \mathrm{cn}_0(m) = 1 \bigwedge \mathrm{cn}_n(m) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2\,n-1}{2\,j+1} \mathrm{sn}_j(m)\,\mathrm{dn}_k(m)\,\delta_{j+k-n+1} \bigwedge \mathrm{dn}_0(m) = 1 \bigwedge \mathrm{dn}_n(m) = m \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2\,n-1}{2\,j+1} \mathrm{sn}_j(m)\,\mathrm{cn}_k(m)\,\delta_{j+k-n+1}$$

09.27.06.0007.01

$$\mathrm{cs}(z \mid m) \propto \frac{1}{z}\left(1 + O\left(z^2\right)\right)$$

### Expansions at $z = 2\,r\,K(m) + 2\,i\,s\,K(1-m)$

09.27.06.0008.01

$$\mathrm{cs}(z \mid m) \propto (-1)^s \left(\frac{1}{z - z_0} + \frac{1}{6}(m-2)\,(z - z_0) + \frac{1}{360}\left(7\,m^2 + 8\,m - 8\right)(z - z_0)^3 + \ldots\right) /; (z \to z_0) \bigwedge z_0 = 2\,r\,K(m) + 2\,i\,s\,K(1-m) \bigwedge r \in \mathbb{Z} \bigwedge s \in \mathbb{Z}$$

09.27.06.0009.01

$$\operatorname{cs}(z \mid m) = (-1)^s \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(j+1)\,(-1)^{k-j}\,\operatorname{cn}_{k-j}(m)}{(2k-2j)!} \sum_{r=0}^{j} \frac{(-1)^r}{r+1} \binom{j}{r} q_{r,j}\,(z-z_0)^{2k-1} \,/;$$

$$z_0 = 2\,r\,K(m) + 2\,i\,s\,K(1-m) \bigwedge r \in \mathbb{Z} \bigwedge s \in \mathbb{Z} \bigwedge q_{j,0} = 1 \bigwedge q_{j,k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(j\,i+i-k)\,(-1)^i\,\operatorname{sn}_i(m)\,q_{j,k-i}}{(2\,i+1)!} \bigwedge$$

$$k \in \mathbb{N}^+ \bigwedge \operatorname{sn}_0(m) = 1 \bigwedge \operatorname{sn}_n(m) = \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{2n}{2j} \operatorname{cn}_j(m)\,\operatorname{dn}_k(m)\,\delta_{j+k-n} \bigwedge \operatorname{cn}_0(m) = 1 \bigwedge$$

$$\operatorname{cn}_n(m) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2n-1}{2j+1} \operatorname{sn}_j(m)\,\operatorname{dn}_k(m)\,\delta_{j+k-n+1} \bigwedge \operatorname{dn}_0(m) = 1 \bigwedge \operatorname{dn}_n(m) = m \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \binom{2n-1}{2j+1} \operatorname{sn}_j(m)\,\operatorname{cn}_k(m)\,\delta_{j+k-n+1}$$

09.27.06.0010.01

$$\operatorname{cs}(z \mid m) \propto \frac{(-1)^s}{z-z_0} \left(1 + O\!\left((z-z_0)^2\right)\right) /;\ z_0 = 2\,r\,K(m) + 2\,i\,s\,K(1-m) \land r \in \mathbb{Z} \land s \in \mathbb{Z}$$

## Expansions at *m* == 0

09.27.06.0011.01

$$\operatorname{cs}(z \mid m) \propto \cot(z) + \frac{1}{4}\left(z \csc^2(z) - \cot(z)\right) m + \frac{1}{512}\left(4\left(8z^2-3\right)\cos(z) + 13\cos(3z) - \cos(5z) + 8\,z\sin(z)\right)\csc^3(z)\,m^2 + \ldots \,/;\ (m \to 0)$$

09.27.06.0012.01

$$\begin{aligned}
\operatorname{cs}(z \mid m) \propto{}& \cot(z) + \frac{1}{4}\left(z \csc^2(z) - \cot(z)\right) m + \frac{1}{512}\left(4\left(8z^2-3\right)\cos(z) + 13\cos(3z) - \cos(5z) + 8\,z\sin(z)\right)\csc^3(z)\,m^2 + \\
& \frac{1}{12\,288}\left(128\,z^3 + 2\left(32\,z^2+9\right)\cos(2z)\,z - 24\cos(4z)\,z + 6\cos(6z)\,z + 3\left(48\,z^2-59\right)\sin(2z) + 102\sin(4z) - 9\sin(6z)\right) \\
& \csc^4(z)\,m^3 + \frac{1}{1\,572\,864}\left(\left(5632\,z^4 + 5568\,z^2 - 7821\right)\cos(z) + \left(512\,z^4 - 5952\,z^2 + 12\,381\right)\cos(3z) + \right. \\
& 3\left(160\,z^2-1663\right)\cos(5z) - 6\left(16\,z^2-71\right)\cos(7z) + 3\cos(9z) + 96\,z\left(88\,z^2-51\right)\sin(z) + \\
& \left. 16\,z\left(176\,z^2+255\right)\sin(3z) - 2040\,z\sin(5z) + 408\,z\sin(7z)\right)\csc^5(z)\,m^4 + \\
& \frac{1}{62\,914\,560}\left(33\,792\,z^5 + 126\,720\,z^3 + 4\left(6656\,z^4 - 22\,400\,z^2 + 48\,585\right)\cos(2z)\,z + 8\left(128\,z^4 - 4440\,z^2 - 14\,895\right)\cos(4z)\,z - \right. \\
& 120\left(16\,z^2-369\right)\cos(6z)\,z + 10\left(32\,z^2-717\right)\cos(8z)\,z - 60\cos(10z)\,z - 112\,230\,z + \\
& 10\left(10\,240\,z^4 + 16\,896\,z^2 - 30\,387\right)\sin(2z) + 10\left(1024\,z^4 - 10\,296\,z^2 + 26\,187\right)\sin(4z) + \\
& \left. 165\left(96\,z^2-497\right)\sin(6z) - 15\left(176\,z^2-427\right)\sin(8z) + 105\sin(10z)\right)\csc^6(z)\,m^5 + \\
& \frac{1}{24\,159\,191\,040}\left(4\left(1\,236\,992\,z^6 + 4\,312\,320\,z^4 + 6\,395\,040\,z^2 - 11\,557\,035\right)\cos(z) + \right. \\
& 3\left(311\,296\,z^6 - 5\,135\,360\,z^4 - 14\,041\,920\,z^2 + 28\,700\,655\right)\cos(3z) + \\
& \left(16\,384\,z^6 - 1\,889\,280\,z^4 + 19\,742\,400\,z^2 - 52\,368\,975\right)\cos(5z) + 30\left(1792\,z^4 - 123\,648\,z^2 + 448\,491\right)\cos(7z) - \\
& 30\left(256\,z^4 - 16\,896\,z^2 + 31\,137\right)\cos(9z) + 45\left(128\,z^2-565\right)\cos(11z) - 45\cos(13z) + 92\,160\,z\left(112\,z^4 + 579\,z^2 - 850\right) \\
& \sin(z) + 1440\,z\left(4480\,z^4 - 7152\,z^2 + 39\,465\right)\sin(3z) + 144\,z\left(1792\,z^4 - 25\,440\,z^2 - 197\,015\right)\sin(5z) - \\
& \left. 5040\,z\left(144\,z^2-1705\right)\sin(7z) + 720\,z\left(144\,z^2-1573\right)\sin(9z) - 23\,760\,z\sin(11z)\right)\csc^7(z)\,m^6 + \frac{1}{1\,352\,914\,698\,240} \\
& \left(39\,583\,744\,z^7 + 299\,851\,776\,z^5 + 1\,256\,478\,720\,z^3 + 6\left(6\,504\,448\,z^6 - 18\,464\,768\,z^4 - 256\,231\,360\,z^2 + 586\,516\,665\right)\right. \\
& \cos(2z)\,z + 96\left(40\,960\,z^6 - 1\,895\,936\,z^4 + 2\,352\,560\,z^2 - 22\,632\,855\right)\cos(4z)\,z +
\end{aligned}$$

$$2\left(16\,384\,z^6-3\,451\,392\,z^4+15\,536\,640\,z^2+458\,842\,545\right)\cos(6\,z)\,z-672\left(256\,z^4-40\,480\,z^2+348\,465\right)\cos(8\,z)\,z+$$
$$42\left(512\,z^4-75\,200\,z^2+595\,815\right)\cos(10\,z)\,z-3360\left(16\,z^2-279\right)\cos(12\,z)\,z+1890\cos(14\,z)\,z-$$
$$2\,055\,826\,080\,z+35\left(5\,218\,304\,z^6+22\,364\,160\,z^4+44\,972\,928\,z^2-86\,951\,799\right)\sin(2\,z)+$$
$$28\left(1\,490\,944\,z^6-11\,304\,960\,z^4-52\,434\,000\,z^2+113\,726\,745\right)\sin(4\,z)+$$
$$7\left(106\,496\,z^6-7\,680\,000\,z^4+83\,145\,600\,z^2-216\,095\,715\right)\sin(6\,z)+840\left(4096\,z^4-133\,152\,z^2+391\,941\right)\sin(8\,z)-$$
$$105\left(4096\,z^4-116\,736\,z^2+189\,585\right)\sin(10\,z)+23\,940\left(16\,z^2-33\right)\sin(12\,z)-3465\sin(14\,z)\big)\csc^8(z)\,m^7+$$
$$\frac{1}{173\,173\,081\,374\,720}\big(2\left(1\,023\,606\,784\,z^8+7\,012\,999\,168\,z^6+21\,516\,284\,160\,z^4+42\,858\,547\,200\,z^2-82\,105\,338\,105\right)$$
$$\cos(z)+9\left(62\,521\,344\,z^8-1\,202\,962\,432\,z^6-6\,670\,164\,480\,z^4-18\,532\,442\,880\,z^2+37\,330\,928\,005\right)\cos(3\,z)+$$
$$2\left(16\,187\,392\,z^8-1\,569\,505\,280\,z^6+6\,832\,358\,400\,z^4+57\,050\,022\,960\,z^2-126\,632\,390\,535\right)\cos(5\,z)+$$
$$\left(131\,072\,z^8-61\,243\,392\,z^6+3\,616\,757\,760\,z^4-39\,545\,624\,160\,z^2+99\,139\,214\,475\right)\cos(7\,z)+$$
$$63\left(16\,384\,z^6-4\,984\,320\,z^4+113\,688\,000\,z^2-296\,165\,235\right)\cos(9\,z)-$$
$$224\left(512\,z^6-136\,560\,z^4+2\,695\,050\,z^2-4\,300\,785\right)\cos(11\,z)+105\left(8192\,z^4-362\,784\,z^2+500\,313\right)\cos(13\,z)-$$
$$315\left(288\,z^2-1243\right)\cos(15\,z)+315\cos(17\,z)+1120\,z\left(4\,444\,160\,z^6+39\,086\,208\,z^4+201\,035\,160\,z^2-402\,688\,125\right)$$
$$\sin(z)+2016\,z\left(2\,158\,592\,z^6+2\,344\,320\,z^4-50\,634\,360\,z^2+165\,406\,515\right)\sin(3\,z)+$$
$$56\,z\left(8\,634\,368\,z^6-197\,434\,368\,z^4+275\,753\,280\,z^2-3\,182\,522\,625\right)\sin(5\,z)+$$
$$8\,z\left(507\,904\,z^6-43\,760\,640\,z^4-208\,071\,360\,z^2+8\,131\,828\,635\right)\sin(7\,z)-$$
$$6048\,z\left(4736\,z^4-321\,640\,z^2+2\,350\,155\right)\sin(9\,z)+336\,z\left(9472\,z^4-505\,680\,z^2+3\,608\,835\right)\sin(11\,z)-$$
$$840\,z\left(11\,008\,z^2-85\,701\right)\sin(13\,z)+370\,440\,z\sin(15\,z)\big)\csc^9(z)\,m^8+$$
$$\frac{1}{12\,468\,461\,858\,979\,840}\big(20\,472\,135\,680\,z^9+243\,194\,757\,120\,z^7+1\,476\,800\,640\,000\,z^5+7\,154\,466\,480\,000\,z^3+$$
$$4\left(5\,782\,503\,424\,z^8-7\,668\,744\,192\,z^6-326\,379\,594\,240\,z^4-2\,624\,672\,272\,320\,z^2+6\,486\,583\,627\,755\right)\cos(2\,z)\,z+$$
$$2\left(1\,914\,699\,776\,z^8-94\,436\,868\,096\,z^6-276\,692\,516\,352\,z^4+1\,973\,108\,138\,400\,z^2-8\,390\,490\,202\,395\right)\cos(4\,z)\,z+$$
$$8\left(16\,449\,536\,z^8-2\,932\,310\,016\,z^6+46\,527\,409\,152\,z^4-87\,038\,990\,640\,z^2+984\,126\,412\,395\right)\cos(6\,z)\,z+$$
$$4\left(65\,536\,z^8-46\,227\,456\,z^6+2\,127\,299\,328\,z^4+39\,792\,679\,920\,z^2-629\,565\,305\,985\right)\cos(8\,z)\,z-$$
$$1440\left(2048\,z^6-1\,038\,912\,z^4+48\,056\,400\,z^2-330\,015\,105\right)\cos(10\,z)\,z+$$
$$18\left(16\,384\,z^6-6\,418\,944\,z^4+210\,184\,800\,z^2-1\,738\,766\,925\right)\cos(12\,z)\,z-756\left(8192\,z^4-745\,760\,z^2+3\,766\,335\right)$$
$$\cos(14\,z)\,z+22\,680\left(72\,z^2-1127\right)\cos(16\,z)\,z-22\,680\cos(18\,z)\,z-14\,961\,155\,195\,700\,z+$$
$$63\left(1\,908\,670\,464\,z^8+15\,278\,620\,672\,z^6+50\,914\,314\,240\,z^4+125\,354\,655\,360\,z^2-245\,794\,818\,375\right)\sin(2\,z)+$$
$$63\left(681\,836\,544\,z^8-4\,738\,736\,128\,z^6-36\,029\,829\,120\,z^4-140\,221\,976\,400\,z^2+289\,447\,017\,795\right)\sin(4\,z)+$$
$$27\left(96\,731\,136\,z^8-4\,384\,980\,992\,z^6+10\,427\,791\,360\,z^4+180\,593\,525\,280\,z^2-403\,520\,915\,805\right)\sin(6\,z)+$$
$$72\left(147\,456\,z^8-37\,381\,120\,z^6+1\,891\,236\,480\,z^4-20\,761\,489\,350\,z^2+50\,445\,670\,905\right)\sin(8\,z)+$$
$$945\left(114\,688\,z^6-13\,168\,640\,z^4+258\,456\,000\,z^2-634\,211\,787\right)\sin(10\,z)-$$
$$189\left(57\,344\,z^6-3\,880\,960\,z^4+76\,647\,120\,z^2-136\,557\,165\right)\sin(12\,z)+$$
$$22\,680\left(4096\,z^4-77\,724\,z^2+83\,281\right)\sin(14\,z)-5670\left(1944\,z^2-3595\right)\sin(16\,z)+42\,525\sin(18\,z)\big)\csc^{10}(z)\,m^9+$$
$$\frac{1}{7\,979\,815\,589\,747\,097\,600}\big(2\left(2\,748\,011\,511\,808\,z^{10}+29\,125\,921\,996\,800\,z^8+157\,454\,330\,019\,840\,z^6+\right.$$
$$\left.448\,907\,881\,766\,400\,z^4+1\,111\,733\,772\,559\,200\,z^2-2\,164\,413\,964\,533\,975\right)\cos(z)+$$
$$2\left(954\,606\,813\,184\,z^{10}-18\,720\,609\,730\,560\,z^8-206\,405\,127\,290\,880\,z^6-764\,016\,547\,795\,200\,z^4-\right.$$
$$\left.2\,359\,871\,063\,071\,200\,z^2+4\,712\,117\,404\,911\,525\right)\cos(3\,z)+10\left(20\,065\,550\,336\,z^{10}-1\,954\,603\,008\,000\,z^8+\right.$$
$$\left.5\,939\,132\,645\,376\,z^6+70\,263\,594\,858\,240\,z^4+388\,125\,336\,886\,560\,z^2-813\,651\,319\,331\,235\right)\cos(5\,z)+$$
$$4\left(1\,062\,207\,488\,z^{10}-314\,797\,916\,160\,z^8+9\,391\,436\,328\,960\,z^6-7\,984\,617\,984\,000\,z^4-\right.$$
$$\left.452\,547\,712\,184\,400\,z^2+1\,013\,014\,530\,506\,475\right)\cos(7\,z)+4\left(1\,048\,576\,z^{10}-1\,365\,442\,560\,z^8+\right.$$

$$247\,200\,952\,320\,z^6 - 10\,978\,948\,915\,200\,z^4 + 123\,177\,730\,158\,000\,z^2 - 294\,044\,394\,487\,725\big)\cos(9\,z) +$$
$$45\left(1\,441\,792\,z^8 - 1\,042\,055\,168\,z^6 + 77\,228\,659\,200\,z^4 - 1\,566\,052\,488\,000\,z^2 + 3\,817\,653\,876\,225\right)\cos(11\,z) -$$
$$45\left(131\,072\,z^8 - 50\,692\,096\,z^6 - 329\,710\,080\,z^4 - 51\,470\,354\,880\,z^2 + 132\,716\,507\,175\right)\cos(13\,z) +$$
$$315\left(1\,048\,576\,z^6 - 172\,316\,160\,z^4 + 2\,063\,619\,360\,z^2 - 1\,866\,547\,935\right)\cos(15\,z) -$$
$$14\,175\left(13\,824\,z^4 - 514\,208\,z^2 + 593\,979\right)\cos(17\,z) + 14\,175\left(512\,z^2 - 2177\right)\cos(19\,z) -$$
$$14\,175\cos(21\,z) + 240\,z\big(60\,865\,380\,352\,z^8 + 813\,022\,838\,784\,z^6 +$$
$$5\,255\,225\,644\,032\,z^4 + 29\,643\,273\,777\,600\,z^2 - 68\,653\,370\,969\,505\big)\sin(z) + 480\,z$$
$$\left(32\,971\,882\,496\,z^8 + 124\,073\,791\,488\,z^6 - 446\,811\,757\,056\,z^4 - 8\,746\,254\,697\,320\,z^2 + 26\,144\,882\,285\,445\right)\sin(3\,z) +$$
$$960\,z\left(3\,158\,540\,288\,z^8 - 64\,913\,602\,560\,z^6 - 294\,922\,236\,672\,z^4 + 1\,463\,761\,178\,460\,z^2 - 7\,482\,523\,932\,135\right)\sin(5\,z) +$$
$$480\,z\left(224\,362\,496\,z^8 - 18\,346\,008\,576\,z^6 + 231\,438\,340\,224\,z^4 - 591\,295\,420\,800\,z^2 + 6\,232\,106\,981\,835\right)\sin(7\,z) +$$
$$160\,z\left(1\,343\,488\,z^8 - 370\,778\,112\,z^6 + 10\,596\,942\,720\,z^4 + 450\,710\,870\,400\,z^2 - 5\,291\,217\,495\,585\right)\sin(9\,z) -$$
$$63\,360\,z\left(48\,128\,z^6 - 7\,194\,432\,z^4 + 308\,621\,250\,z^2 - 2\,186\,282\,385\right)\sin(11\,z) +$$
$$2880\,z\left(96\,256\,z^6 - 712\,320\,z^4 + 87\,346\,980\,z^2 - 2\,285\,491\,635\right)\sin(13\,z) -$$
$$7560\,z\left(868\,352\,z^4 - 32\,539\,680\,z^2 + 125\,224\,455\right)\sin(15\,z) +$$
$$6\,690\,600\,z\left(288\,z^2 - 1885\right)\sin(17\,z) - 29\,484\,000\,z\sin(19\,z)\big)\csc^{11}(z)\,m^{10} + O\left(m^{11}\right)$$

09.27.06.0013.01

$$\operatorname{cs}(z\mid m) \propto \cot(z)\,(1 + O(m))$$

## Expansions at $m == 1$

09.27.06.0014.01

$$\operatorname{cs}(z\mid m) \propto \operatorname{csch}(z) + \frac{1}{4}\coth(z)\,(\cosh(z) - z\operatorname{csch}(z))\,(m-1) +$$
$$\frac{1}{512}\left(8\cosh(2\,z)\,z^2 + 24\,z^2 - 4\sinh(2\,z)\,z + 4\sinh(4\,z)\,z - 5\cosh(4\,z) + 5\right)\operatorname{csch}^3(z)\,(m-1)^2 - \ldots\ /;\ (m \to 1)$$

09.27.06.0015.01

$$\operatorname{cs}(z\mid m) \propto \operatorname{csch}(z) + \frac{1}{4}\coth(z)\,(\cosh(z) - z\operatorname{csch}(z))\,(m-1) +$$
$$\frac{1}{512}\left(8\cosh(2\,z)\,z^2 + 24\,z^2 - 4\sinh(2\,z)\,z + 4\sinh(4\,z)\,z - 5\cosh(4\,z) + 5\right)\operatorname{csch}^3(z)\,(m-1)^2 -$$
$$\frac{1}{49\,152}\big(32\,z\left(23\,z^2 + 6\right)\cosh(z) + 8\,z\left(4\,z^2 - 39\right)\cosh(3\,z) + 120\,z\cosh(5\,z) + 3\left(160\,z^2 + 83\right)\sinh(z) +$$
$$3\left(88\,z^2 + 43\right)\sinh(3\,z) - 3\left(8\,z^2 + 41\right)\sinh(5\,z) - 3\sinh(7\,z)\big)\operatorname{csch}^4(z)\,(m-1)^3 +$$
$$\frac{1}{1\,572\,864}\big(2\left(1840\,z^4 - 1380\,z^2 - 1257\right) + \left(2432\,z^4 - 312\,z^2 + 1167\right)\cosh(2\,z) + 2\left(16\,z^4 + 1716\,z^2 + 1287\right)\cosh(4\,z) -$$
$$3\left(120\,z^2 + 389\right)\cosh(6\,z) - 60\cosh(8\,z) + 4\,z\left(1976\,z^2 + 1527\right)\sinh(2\,z) +$$
$$4\,z\left(56\,z^2 - 1227\right)\sinh(4\,z) + 4\,z\left(8\,z^2 + 297\right)\sinh(6\,z) + 36\,z\sinh(8\,z)\big)\operatorname{csch}^5(z)\,(m-1)^4 +$$
$$\frac{1}{251\,658\,240}\big(-16\,z\left(13\,456\,z^4 - 28\,940\,z^2 - 23\,745\right)\cosh(z) - 24\,z\left(1264\,z^4 + 18\,980\,z^2 + 27\,975\right)\cosh(3\,z) -$$
$$8\,z\left(16\,z^4 + 540\,z^2 - 44\,145\right)\cosh(5\,z) - 20\,z\left(160\,z^2 + 2859\right)\cosh(7\,z) - 4500\,z\cosh(9\,z) - 840\left(476\,z^4 - 87\,z^2 - 411\right)$$
$$\sinh(z) - 30\left(6288\,z^4 - 10\,116\,z^2 - 2755\right)\sinh(3\,z) - 5\left(736\,z^4 + 44\,904\,z^2 + 41\,811\right)\sinh(5\,z) +$$
$$10\left(16\,z^4 + 1848\,z^2 + 5835\right)\sinh(7\,z) + 30\left(36\,z^2 + 161\right)\sinh(9\,z) + 15\sinh(11\,z)\big)\operatorname{csch}^6(z)\,(m-1)^5 + \frac{1}{24\,159\,191\,040}$$
$$\big(2\left(1\,507\,072\,z^6 - 8\,562\,960\,z^4 - 1\,386\,000\,z^2 + 6\,169\,995\right) + 2\left(1\,349\,504\,z^6 + 4\,620\,960\,z^4 + 7\,884\,360\,z^2 - 4\,624\,785\right)$$
$$\cosh(2\,z) + 4\left(46\,208\,z^6 + 1\,922\,880\,z^4 - 5\,262\,840\,z^2 - 2\,644\,965\right)\cosh(4\,z) +$$

$$\left(256 z^6 + 204\,480 z^4 + 8\,508\,960 z^2 + 9\,467\,235\right)\cosh(6 z) - 30\left(400 z^4 + 11\,904 z^2 + 58\,617\right)\cosh(8 z) -$$
$$45\left(2160 z^2 + 4837\right)\cosh(10 z) - 1620\cosh(12 z) + 24 z\left(485\,296 z^4 - 1\,407\,720 z^2 - 1\,672\,185\right)\sinh(2 z) +$$
$$60 z\left(31\,808 z^4 + 276\,960 z^2 + 638\,391\right)\sinh(4 z) + 180 z\left(32 z^4 + 408 z^2 - 80\,953\right)\sinh(6 z) +$$
$$24 z\left(16 z^4 + 2760 z^2 + 62\,415\right)\sinh(8 z) + 180 z\left(72 z^2 + 1381\right)\sinh(10 z) + 900 z\sinh(12 z)\Big)\operatorname{csch}^7(z)(m-1)^6 -$$
$$\frac{1}{5\,411\,658\,792\,960}\Big(8 z\left(33\,244\,544 z^6 - 169\,790\,880 z^4 + 373\,682\,400 z^2 + 460\,109\,475\right)\cosh(z) +$$
$$12 z\left(5\,176\,064 z^6 + 94\,597\,440 z^4 - 370\,032\,320 z^2 - 594\,245\,715\right)\cosh(3 z) +$$
$$16 z\left(139\,456 z^6 + 13\,918\,800 z^4 + 88\,750\,830 z^2 + 293\,798\,925\right)\cosh(5 z) +$$
$$4 z\left(256 z^6 + 94\,080 z^4 + 9\,042\,600 z^2 - 336\,907\,305\right)\cosh(7 z) +$$
$$420 z\left(192 z^4 - 20\,072 z^2 + 162\,363\right)\cosh(9 z) + 2520 z\left(1260 z^2 + 11\,297\right)\cosh(11 z) +$$
$$258\,300 z\cosh(13 z) + 63\left(9\,614\,336 z^6 - 67\,529\,280 z^4 - 48\,535\,200 z^2 + 46\,854\,555\right)\sinh(z) +$$
$$140\left(3\,269\,504 z^6 + 1\,174\,224 z^4 + 27\,202\,662 z^2 + 992\,079\right)\sinh(3 z) +$$
$$70\left(478\,976 z^6 + 10\,396\,272 z^4 - 37\,828\,116 z^2 - 25\,073\,235\right)\sinh(5 z) +$$
$$35\left(1792 z^6 + 490\,272 z^4 + 20\,050\,128 z^2 + 27\,450\,513\right)\sinh(7 z) -$$
$$7\left(256 z^6 - 84\,960 z^4 - 1\,830\,960 z^2 + 16\,994\,475\right)\sinh(9 z) - 630\left(432 z^4 + 22\,276 z^2 + 35\,213\right)\sinh(11 z) -$$
$$2520\left(25 z^2 + 109\right)\sinh(13 z) - 315\sinh(15 z)\Big)\operatorname{csch}^8(z)(m-1)^7 +$$
$$\frac{1}{173\,173\,081\,374\,720}\Big(18\left(66\,489\,088 z^8 - 645\,626\,240 z^6 + 3\,292\,539\,600 z^4 + 2\,870\,555\,940 z^2 - 2\,108\,262\,975\right) +$$
$$2\left(636\,233\,728 z^8 + 1\,416\,993\,536 z^6 - 30\,708\,972\,000 z^4 - 53\,216\,291\,520 z^2 + 17\,945\,797\,275\right)\cosh(2 z) +$$
$$4\left(42\,446\,336 z^8 + 2\,036\,204\,800 z^6 - 2\,051\,632\,800 z^4 + 22\,241\,422\,350 z^2 + 6\,187\,089\,195\right)\cosh(4 z) +$$
$$63\left(53\,248 z^8 + 10\,174\,720 z^6 + 163\,077\,600 z^4 - 668\,248\,200 z^2 - 553\,255\,345\right)\cosh(6 z) +$$
$$2\left(256 z^8 + 760\,704 z^6 + 12\,390\,000 z^4 + 3\,620\,465\,100 z^2 + 6\,758\,870\,265\right)\cosh(8 z) -$$
$$35\left(1792 z^6 - 2\,028\,000 z^4 - 25\,849\,368 z^2 + 29\,453\,985\right)\cosh(10 z) -$$
$$1260\left(8640 z^4 + 195\,426 z^2 + 251\,863\right)\cosh(12 z) - 1260\left(2300 z^2 + 4459\right)\cosh(14 z) -$$
$$16\,380\cosh(16 z) + 28 z\left(225\,685\,504 z^6 - 1\,385\,289\,216 z^4 + 3\,419\,854\,200 z^2 + 5\,152\,437\,585\right)\sinh(2 z) +$$
$$4 z\left(464\,069\,120 z^6 + 3\,558\,439\,584 z^4 - 18\,625\,005\,840 z^2 - 39\,050\,472\,195\right)\sinh(4 z) +$$
$$144 z\left(480\,320 z^6 + 23\,703\,624 z^4 + 113\,082\,060 z^2 + 540\,952\,755\right)\sinh(6 z) +$$
$$4 z\left(5888 z^6 + 6\,093\,024 z^4 + 375\,391\,800 z^2 - 4\,243\,004\,955\right)\sinh(8 z) +$$
$$32 z\left(32 z^6 - 129\,948 z^4 - 12\,802\,230 z^2 - 2\,131\,605\right)\sinh(10 z) + 756 z\left(864 z^4 + 95\,200 z^2 + 578\,675\right)\sinh(12 z) +$$
$$420 z\left(1000 z^2 + 16\,371\right)\sinh(14 z) + 8820 z\sinh(16 z)\Big)\operatorname{csch}^9(z)(m-1)^8 - \frac{1}{49\,873\,847\,435\,919\,360}$$
$$\Big(8 z\left(17\,769\,803\,264 z^8 - 154\,337\,008\,896 z^6 + 662\,044\,192\,992 z^4 - 1\,413\,285\,787\,620 z^2 - 2\,207\,834\,702\,955\right)\cosh(z) +$$
$$4 z\left(11\,114\,481\,664 z^8 + 218\,002\,922\,496 z^6 - 1\,813\,216\,327\,776 z^4 + 4\,986\,817\,337\,880 z^2 + 9\,150\,568\,206\,255\right)$$
$$\cosh(3 z) + 20 z\left(179\,849\,216 z^8 + 17\,458\,283\,520 z^6 + 75\,235\,589\,856 z^4 - 516\,684\,231\,000 z^2 - 1\,408\,419\,633\,915\right)$$
$$\cosh(5 z) + 64 z\left(629\,536 z^8 + 211\,179\,024 z^6 + 6\,915\,622\,266 z^4 + 21\,606\,985\,575 z^2 + 173\,396\,382\,075\right)\cosh(7 z) +$$
$$128 z\left(16 z^8 + 22\,536 z^6 + 88\,216\,317 z^4 + 3\,070\,844\,595 z^2 - 14\,032\,657\,485\right)\cosh(9 z) +$$
$$36 z\left(10\,240 z^6 - 65\,446\,752 z^4 - 2\,648\,877\,000 z^2 - 3\,662\,620\,605\right)\cosh(11 z) + 11\,340 z$$
$$\left(23\,328 z^4 + 1\,054\,072 z^2 + 5\,123\,503\right)\cosh(13 z) + 56\,700 z\left(3400 z^2 + 23\,959\right)\cosh(15 z) + 4\,524\,660 z\cosh(17 z) +$$
$$63\left(5\,765\,895\,168 z^8 - 64\,101\,931\,520 z^6 + 358\,754\,198\,880 z^4 + 461\,814\,328\,800 z^2 - 210\,208\,477\,185\right)\sinh(z) +$$
$$216\left(1\,654\,016\,128 z^8 - 3\,795\,293\,152 z^6 - 30\,335\,368\,280 z^4 - 117\,788\,351\,835 z^2 + 5\,787\,713\,820\right)\sinh(3 z) +$$
$$720\left(74\,894\,880 z^8 + 1\,610\,434\,336 z^6 - 3\,479\,640\,570 z^4 + 22\,032\,817\,380 z^2 + 9\,879\,190\,965\right)\sinh(5 z) +$$
$$63\left(17\,245\,696 z^8 + 1\,590\,476\,032 z^6 + 22\,333\,139\,520 z^4 - 86\,714\,291\,880 z^2 - 90\,423\,595\,035\right)\sinh(7 z) +$$
$$9\left(24\,064 z^8 - 26\,967\,808 z^6 - 5\,753\,905\,920 z^4 + 53\,348\,533\,560 z^2 + 184\,980\,199\,635\right)\sinh(9 z) -$$

$$36\left(128 z^8-2892288 z^6-583461480 z^4-5821526340 z^2+1616170815\right)\sinh(11 z)-189\left(62208 z^6+12659840 z^4+187133400 z^2+213575595\right)\sinh(13 z)-945\left(20000 z^4+800040 z^2+997737\right)\sinh(15 z)-5670\left(196 z^2+839\right)\sinh(17 z)-2835\sinh(19 z)\big)$$

$$\operatorname{csch}^{10}(z)(m-1)^9+\frac{1}{7979815589747097600}\Big(3127485374464 z^{10}-43994001300480 z^8+330560432294400 z^6-1585067158382400 z^4-2197145541729600 z^2+1080\left(18545000960 z^8-183880478592 z^6+874548867248 z^4-1915061306760 z^2-3460899395415\right)\sinh(2 z)\, z+120\left(69804123904 z^8+410456968704 z^6-5520309884208 z^4+16449365164680 z^2+36940206271815\right)\sinh(4 z)\, z+160\left(4557007168 z^8+197971975584 z^6+530033745528 z^4-4517746166430 z^2-16561877387265\right)\sinh(6 z)\, z+40\left(206488576 z^8+31450871808 z^6+739283503392 z^4+653416076880 z^2+21042320833485\right)\sinh(8 z)\, z+320\left(992 z^8+17632080 z^6+6037860528 z^4+148398278175 z^2-293396228055\right)\sinh(10 z)\, z+20\left(512 z^8-53001216 z^6-18679375008 z^4-500341983120 z^2-880724436795\right)\sinh(12 z)\, z+540\left(186624 z^6+62952736 z^4+1786255800 z^2+7697400375\right)\sinh(14 z)\, z+18900\left(20000 z^4+1596816 z^2+7072143\right)\sinh(16 z)\, z+18900\left(2744 z^2+41535\right)\sinh(18 z)\, z+510300\sinh(20 z)\, z+\left(3714757763072 z^{10}+725097369600 z^8-264064801674240 z^6+2031769019901600 z^4+4031463722718600 z^2-969296803112925\right)\cosh(2 z)+2\left(365160251392 z^{10}+18368995703040 z^8-80431555034880 z^6-126842760399600 z^4-1454744457558000 z^2-200711093812425\right)\cosh(4 z)+\left(37339713536 z^{10}+6399354193920 z^8+85954169468160 z^6-302198583304800 z^4+1412940934326600 z^2+861620257661625\right)\cosh(6 z)+32\left(7556864 z^{10}+4110213600 z^8+266226090480 z^6+3641863424850 z^4-10988271798525 z^2-15273874335825\right)\cosh(8 z)+\left(4096 z^{10}+32532480 z^8-133951507200 z^6-9643178109600 z^4-4911089891400 z^2+110501743056075\right)\cosh(10 z)-90\left(11520 z^8-344065792 z^6-27964693680 z^4-238785870960 z^2-6743301075\right)\cosh(12 z)-4725\left(622080 z^6+47775520 z^4+557948568 z^2+597795471\right)\cosh(14 z)-4725\left(1120000 z^4+18727968 z^2+17561343\right)\cosh(16 z)-2778300\left(124 z^2+221\right)\cosh(18 z)-963900\cosh(20 z)+889662247584075\Big)\operatorname{csch}^{11}(z)(m-1)^{10}+O\left((m-1)^{11}\right)$$

09.27.06.0016.01

$$\operatorname{cs}(z \mid m) \propto \operatorname{csch}(z)\,(1+O(m-1))$$

## $q$-series

09.27.06.0002.01

$$\operatorname{cs}(z \mid m) = \frac{\pi}{2 K(m)} \cot\left(\frac{\pi z}{2 K(m)}\right) - \frac{2\pi}{K(m)} \sum_{k=1}^{\infty} \frac{q(m)^{2k}}{q(m)^{2k}+1} \sin\left(\frac{k \pi z}{K(m)}\right)$$

## Other series representations

09.27.06.0003.01

$$\operatorname{cs}(z \mid m) = \frac{\pi}{2 K(1-m)} \sum_{k=-\infty}^{\infty} \operatorname{csch}\left(\pi \frac{K(m)}{K(1-m)}\left(k+\frac{z}{2 K(m)}\right)\right)$$

09.27.06.0004.01

$$\operatorname{cd}(z \mid m) \propto \frac{(-1)^s}{z-2 s i K(1-m)-2 r K(m)}+O(1) \;/;\; (z \to 2 s i K(1-m)+2 r K(m)) \bigwedge \{r, s\} \in \mathbb{Z}$$

# Product representations

09.27.08.0001.01

$$\operatorname{cs}(z \mid m) = \sqrt[4]{1-m}\,\cot\left(\frac{\pi z}{2K(m)}\right)\prod_{k=1}^{\infty}\frac{1+2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right)+q(m)^{4k}}{1-2q(m)^{2k}\cos\left(\frac{\pi z}{K(m)}\right)+q(m)^{4k}}$$

# Differential equations

## Ordinary nonlinear differential equations

09.27.13.0001.01

$$w''(z) - w(z)\left(2w(z)^2 - m + 2\right) = 0 \;/;\; w(z) = \operatorname{cs}(z \mid m)$$

# Transformations

## Transformations and argument simplifications

### Argument involving basic arithmetic operations

09.27.16.0001.01

$$\operatorname{cs}(i z, m) = -i \operatorname{ns}(z, 1-m)$$

09.27.16.0002.01

$$\operatorname{cs}(z \mid 1-m) = i \operatorname{ns}(i z \mid m)$$

09.27.16.0003.01

$$\operatorname{cs}(i z \mid 1-m) = -i \operatorname{ns}(z \mid m)$$

09.27.16.0007.01

$$\operatorname{cs}(x + i y \mid m) = (\operatorname{cn}(x \mid m)\operatorname{cn}(y \mid 1-m) - i \operatorname{sn}(x \mid m)\operatorname{dn}(x \mid m)\operatorname{sn}(y \mid 1-m)\operatorname{dn}(y \mid 1-m)) / \\ (\operatorname{dn}(y \mid 1-m)\operatorname{sn}(x \mid m) + i \operatorname{cn}(x \mid m)\operatorname{cn}(y \mid 1-m)\operatorname{dn}(x \mid m)\operatorname{sn}(y \mid 1-m)) \;/;\; \{x, y\} \in \mathbb{R}$$

09.27.16.0008.01

$$\operatorname{cs}\left(\sqrt{1-m}\, z \,\middle|\, \frac{m}{m-1}\right) = \frac{1}{\sqrt{1-m}}\operatorname{cs}(z \mid m)$$

09.27.16.0009.01

$$\operatorname{cs}\left(\sqrt{m}\, z \,\middle|\, \frac{1}{m}\right) = \frac{1}{\sqrt{m}}\operatorname{ds}(z \mid m)$$

09.27.16.0010.01

$$\operatorname{sc}\left(i\sqrt{1-m}\, z \,\middle|\, \frac{1}{1-m}\right) = -\frac{i}{\sqrt{1-m}}\operatorname{ds}(z \mid m)$$

09.27.16.0011.01

$$\operatorname{cs}\left(i\sqrt{m}\, z \,\middle|\, \frac{m-1}{m}\right) = -\frac{i}{\sqrt{m}}\operatorname{sn}(z \mid m)$$

Landen's transformation:

09.27.16.0012.01

$$\operatorname{cs}\left(\left(1+\sqrt{1-m}\right)z \,\middle|\, \left(\frac{1-\sqrt{1-m}}{1+\sqrt{1-m}}\right)^2\right) = \frac{1-\left(1+\sqrt{1-m}\right)\operatorname{sn}(z\mid m)^2}{\left(1+\sqrt{1-m}\right)\operatorname{sn}(z\mid m)\operatorname{cn}(z\mid m)}$$

Gauss' transformation:

09.27.16.0013.01

$$\operatorname{cs}\left(\left(1+\sqrt{m}\right)z \,\middle|\, \frac{4\sqrt{m}}{\left(1+\sqrt{m}\right)^2}\right) = \frac{\operatorname{cn}(z\mid m)\operatorname{dn}(z\mid m)}{\left(1+\sqrt{m}\right)\operatorname{sn}(z\mid m)}$$

$n$ th degree transformations:

09.27.16.0014.01

$$\operatorname{cs}\left(\frac{z}{M} \,\middle|\, l\right) = M \operatorname{cs}(z\mid m) \prod_{r=1}^{\frac{n-1}{2}} \frac{1-\operatorname{sn}(z\mid m)^2 \operatorname{ns}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^2}{1-\operatorname{sn}(z\mid m)^2 \operatorname{ns}\left(\frac{2rK(m)}{n} \,\middle|\, m\right)^2} \;/;$$

$$\frac{n+1}{2} \in \mathbb{Z}^+ \bigwedge l = m^n \prod_{r=1}^{\frac{n-1}{2}} \operatorname{sn}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^8 \bigwedge M = \prod_{r=1}^{\frac{n-1}{2}} \frac{\operatorname{sn}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^2}{\operatorname{sn}\left(\frac{2rK(m)}{n} \,\middle|\, m\right)^2}$$

09.27.16.0015.01

$$\operatorname{cs}\left(\frac{z}{M}+\frac{K(m)}{nM} \,\middle|\, l\right) = -\frac{\sqrt{1-l}}{M} \operatorname{sc}(z\mid m) \prod_{r=1}^{\frac{n}{2}} \frac{1-\operatorname{sn}(z\mid m)^2 \operatorname{ns}\left(\frac{2rK(m)}{n} \,\middle|\, m\right)^2}{1-\operatorname{sn}(z\mid m)^2 \operatorname{ns}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^2} \;/;$$

$$\frac{n}{2} \in \mathbb{Z}^+ \bigwedge l = m^n \prod_{r=1}^{\frac{n}{2}} \operatorname{sn}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^8 \bigwedge M = \prod_{r=1}^{\frac{n}{2}} \frac{\operatorname{sn}\left(\frac{(2r-1)K(m)}{n} \,\middle|\, m\right)^2}{\operatorname{sn}\left(\frac{2rK(m)}{n} \,\middle|\, m\right)^2}$$

### Argument involving half-periods

09.27.16.0004.01

$$\operatorname{cs}(z+K(m)\mid m) = -\sqrt{1-m}\,\operatorname{sc}(z\mid m)$$

09.27.16.0020.01

$$\operatorname{cs}(z-K(m)\mid m) = -\sqrt{1-m}\,\operatorname{sc}(z\mid m)$$

09.27.16.0021.01

$$\operatorname{cs}(z+3K(m)\mid m) = -\sqrt{1-m}\,\operatorname{sc}(z\mid m)$$

09.27.16.0022.01

$$\operatorname{cs}(z+(2r+1)K(m)\mid m) = -\sqrt{1-m}\,\operatorname{sc}(z\mid m) \;/;\; r \in \mathbb{Z}$$

09.27.16.0005.01

$$\operatorname{cs}(z+iK(1-m)\mid m) = -i\operatorname{dn}(z\mid m)$$

09.27.16.0023.01

$$\operatorname{cs}(z-iK(1-m)\mid m) = i\operatorname{dn}(z\mid m)$$

09.27.16.0024.01

$$\operatorname{cs}(z+3iK(1-m)\mid m) = i\operatorname{dn}(z\mid m) \;/;\; s \in \mathbb{Z}$$

09.27.16.0025.01

$$\operatorname{cs}(z + (2s+1)\, i\, K(1-m) \mid m) = (-1)^{s-1}\, i\, \operatorname{dn}(z \mid m) \;/;\; s \in \mathbb{Z}$$

09.27.16.0006.01

$$\operatorname{cs}(z + i\, K(1-m) + K(m) \mid m) = -i \sqrt{1-m}\, \operatorname{nd}(z \mid m)$$

09.27.16.0026.01

$$\operatorname{cs}(z - i\, K(1-m) + K(m) \mid m) = i \sqrt{1-m}\, \operatorname{nd}(z \mid m)$$

09.27.16.0027.01

$$\operatorname{cs}(z + i\, K(1-m) - K(m) \mid m) = -i \sqrt{1-m}\, \operatorname{nd}(z \mid m)$$

09.27.16.0028.01

$$\operatorname{cs}(z - i\, K(1-m) - K(m) \mid m) = i \sqrt{1-m}\, \operatorname{nd}(z \mid m)$$

09.27.16.0029.01

$$\operatorname{cs}(z + i\, K(1-m) + 3\, K(m) \mid m) = -i \sqrt{1-m}\, \operatorname{nd}(z \mid m)$$

09.27.16.0030.01

$$\operatorname{cs}(z + (4s+1)\, i\, K(1-m) + (2r+1)\, K(m) \mid m) = -i \sqrt{1-m}\, \operatorname{nd}(z \mid m) \;/;\; \{r, s\} \in \mathbb{Z}$$

09.27.16.0031.01

$$\operatorname{cs}(z + (4s-1)\, i\, K(1-m) + (2r+1)\, K(m) \mid m) = i \sqrt{1-m}\, \operatorname{nd}(z \mid m) \;/;\; \{r, s\} \in \mathbb{Z}$$

09.27.16.0032.01

$$\operatorname{cs}(z + (2s+1)\, i\, K(1-m) + (2r+1)\, K(m) \mid m) = (-1)^{s-1}\, i \sqrt{1-m}\, \operatorname{nd}(z \mid m) \;/;\; \{r, s\} \in \mathbb{Z}$$

## Argument involving inverse Jacobi functions

09.27.16.0033.01

$$\operatorname{cs}\left(\operatorname{cd}^{-1}(z \mid m) \mid m\right)^2 = \frac{(m-1)\, z^2}{z^2 - 1}$$

09.27.16.0034.01

$$\operatorname{cs}\left(\operatorname{cn}^{-1}(z \mid m) \mid m\right)^2 = \frac{z^2}{1 - z^2}$$

09.27.16.0035.01

$$\operatorname{cs}\left(\operatorname{dc}^{-1}(z \mid m) \mid m\right)^2 = \frac{m-1}{1 - z^2}$$

09.27.16.0036.01

$$\operatorname{cs}\left(\operatorname{dn}^{-1}(z \mid m) \mid m\right)^2 = \frac{z^2 + m - 1}{1 - z^2}$$

09.27.16.0037.01

$$\operatorname{cs}\left(\operatorname{ds}^{-1}(z \mid m) \mid m\right)^2 = z^2 + m - 1$$

09.27.16.0038.01

$$\operatorname{cs}\left(\operatorname{nc}^{-1}(z \mid m) \mid m\right)^2 = \frac{1}{z^2 - 1}$$

09.27.16.0039.01

$$\operatorname{cs}\left(\operatorname{nd}^{-1}(z \mid m) \mid m\right)^2 = \frac{(m-1)\, z^2 + 1}{z^2 - 1}$$

09.27.16.0040.01

$$\operatorname{cs}\left(\operatorname{ns}^{-1}(z \mid m) \mid m\right)^2 = z^2 - 1$$

09.27.16.0041.01

$$\operatorname{cs}\left(\operatorname{sc}^{-1}(z \mid m) \mid m\right) = \frac{1}{z}$$

09.27.16.0042.01

$$\operatorname{cs}\left(\operatorname{sd}^{-1}(z \mid m) \mid m\right)^2 = \frac{(m-1) z^2 + 1}{z^2}$$

09.27.16.0043.01

$$\operatorname{cs}\left(\operatorname{sn}^{-1}(z \mid m) \mid m\right)^2 = \frac{1 - z^2}{z^2}$$

### Addition formulas

09.27.16.0016.01

$$\operatorname{cs}(u + v \mid m) == \frac{\operatorname{cn}(u \mid m) \operatorname{cn}(v \mid m) - \operatorname{sn}(u \mid m) \operatorname{dn}(u \mid m) \operatorname{sn}(v \mid m) \operatorname{dn}(v \mid m)}{\operatorname{cn}(v \mid m) \operatorname{dn}(v \mid m) \operatorname{sn}(u \mid m) + \operatorname{cn}(u \mid m) \operatorname{dn}(u \mid m) \operatorname{sn}(v \mid m)}$$

09.27.16.0017.01

$$\operatorname{cs}(u + v \mid m) \operatorname{cs}(u - v \mid m) == \frac{\operatorname{cn}(v \mid m)^2 - \operatorname{dn}(v \mid m)^2 \operatorname{sn}(u \mid m)^2}{\operatorname{sn}(u \mid m)^2 - \operatorname{sn}(v \mid m)^2}$$

### Half-angle formulas

09.27.16.0018.01

$$\operatorname{cs}\left(\frac{z}{2} \,\middle|\, m\right)^2 == \frac{\operatorname{cn}(z \mid m) + \operatorname{dn}(z \mid m)}{1 - \operatorname{cn}(z \mid m)}$$

### Multiple arguments

#### Double angle formulas

09.27.16.0019.01

$$\operatorname{cs}(2 z \mid m) == \frac{\operatorname{cn}(z \mid m)^2 - \operatorname{sn}(z \mid m)^2 \operatorname{dn}(z \mid m)^2}{2 \operatorname{sn}(z \mid m) \operatorname{cn}(z \mid m) \operatorname{dn}(z \mid m)}$$

## Identities

### Functional identities

09.27.17.0001.01

$$4 w(z)^2 \left(w(z)^2 + 1\right) \left(w(z)^2 - m + 1\right) w(2 z)^2 - \left(w(z)^4 + m - 1\right)^2 == 0 \,/;\, w(z) == \operatorname{cs}(z \mid m)$$

## Complex characteristics

### Real part

09.27.19.0001.01

$$\operatorname{Re}(\operatorname{cs}(x+i\,y\mid m)) = \frac{\operatorname{cn}(x\mid m)\operatorname{cn}(y\mid 1-m)\operatorname{dn}(y\mid 1-m)\operatorname{sn}(x\mid m)\left(1-\operatorname{dn}(x\mid m)^2\operatorname{sn}(y\mid 1-m)^2\right)}{\operatorname{dn}(y\mid 1-m)^2\operatorname{sn}(x\mid m)^2+\operatorname{cn}(x\mid m)^2\operatorname{cn}(y\mid 1-m)^2\operatorname{dn}(x\mid m)^2\operatorname{sn}(y\mid 1-m)^2} \;/;\; \{x, y, m\}\in\mathbb{R}$$

## Imaginary part

09.27.19.0002.01

$$\operatorname{Im}(\operatorname{cs}(x+i\,y\mid m)) = -\frac{\operatorname{dn}(x\mid m)\left(\operatorname{cn}(x\mid m)^2\operatorname{cn}(y\mid 1-m)^2+\operatorname{dn}(y\mid 1-m)^2\operatorname{sn}(x\mid m)^2\right)\operatorname{sn}(y\mid 1-m)}{\operatorname{dn}(y\mid 1-m)^2\operatorname{sn}(x\mid m)^2+\operatorname{cn}(x\mid m)^2\operatorname{cn}(y\mid 1-m)^2\operatorname{dn}(x\mid m)^2\operatorname{sn}(y\mid 1-m)^2} \;/;\; \{x, y, m\}\in\mathbb{R}$$

## Absolute value

09.27.19.0003.01

$$|\operatorname{cs}(x+i\,y\mid m)| = \sqrt{\frac{\operatorname{cn}(x\mid m)^2\operatorname{cn}(y\mid 1-m)^2+\operatorname{dn}(x\mid m)^2\operatorname{dn}(y\mid 1-m)^2\operatorname{sn}(x\mid m)^2\operatorname{sn}(y\mid 1-m)^2}{\operatorname{dn}(y\mid 1-m)^2\operatorname{sn}(x\mid m)^2+\operatorname{cn}(x\mid m)^2\operatorname{cn}(y\mid 1-m)^2\operatorname{dn}(x\mid m)^2\operatorname{sn}(y\mid 1-m)^2}} \;/;\; \{x, y, m\}\in\mathbb{R}$$

## Argument

09.27.19.0004.01

$$\arg(\operatorname{cs}(x+i\,y\mid m)) = \tan^{-1}\Big(\operatorname{cn}(x\mid m)\operatorname{cn}(y\mid 1-m)\operatorname{dn}(y\mid 1-m)\operatorname{sn}(x\mid m)\left(1-\operatorname{dn}(x\mid m)^2\operatorname{sn}(y\mid 1-m)^2\right),$$
$$-\operatorname{dn}(x\mid m)\left(\operatorname{cn}(x\mid m)^2\operatorname{cn}(y\mid 1-m)^2+\operatorname{dn}(y\mid 1-m)^2\operatorname{sn}(x\mid m)^2\right)\operatorname{sn}(y\mid 1-m)\Big) \;/;\; \{x, y, m\}\in\mathbb{R}$$

## Conjugate value

09.27.19.0005.01

$$\overline{\operatorname{cs}(x+i\,y\mid m)} = \frac{\operatorname{cn}(x\mid m)\operatorname{cn}(y\mid 1-m)+i\operatorname{dn}(x\mid m)\operatorname{dn}(y\mid 1-m)\operatorname{sn}(x\mid m)\operatorname{sn}(y\mid 1-m)}{\operatorname{sn}(x\mid m)\operatorname{dn}(y\mid 1-m)-i\operatorname{cn}(x\mid m)\operatorname{dn}(x\mid m)\operatorname{cn}(y\mid 1-m)\operatorname{sn}(y\mid 1-m)} \;/;\; \{x, y, m\}\in\mathbb{R}$$

# Differentiation

## Low-order differentiation

### With respect to *z*

09.27.20.0001.01

$$\frac{\partial\operatorname{cs}(z\mid m)}{\partial z} = -\operatorname{ds}(z\mid m)\operatorname{ns}(z\mid m)$$

09.27.20.0002.01

$$\frac{\partial^2\operatorname{cs}(z\mid m)}{\partial z^2} = \operatorname{cs}(z\mid m)\left(\operatorname{ds}(z\mid m)^2+\operatorname{ns}(z\mid m)^2\right)$$

### With respect to *m*

09.27.20.0003.01

$$\frac{\partial\operatorname{cs}(z\mid m)}{\partial m} = \frac{1}{2\,m\,(1-m)}\left(\operatorname{ns}(z\mid m)\operatorname{ds}(z\mid m)\left((m-1)\,z+E(\operatorname{am}(z\mid m)\mid m)-m\operatorname{sn}(z\mid m)\operatorname{cd}(z\mid m)\right)\right)$$

09.27.20.0004.01

$$\frac{\partial^2 \mathrm{cs}(z \mid m)}{\partial m^2} = \frac{1}{4\,(m-1)^2\, m^2} \Big(\mathrm{cs}(z \mid m)\,((m-1)\,z + E(\mathrm{am}(z \mid m) \mid m) - m\,\mathrm{dn}(z \mid m)\,\mathrm{sc}(z \mid m))\,((m-1)\,z + E(\mathrm{am}(z \mid m) \mid m) - m\,\mathrm{cd}(z \mid m)\,\mathrm{sn}(z \mid m))\,\mathrm{ns}(z \mid m)^2 + 2\,(m-1)\,\mathrm{ds}(z \mid m)\,((m-1)\,z + E(\mathrm{am}(z \mid m) \mid m) - m\,\mathrm{cd}(z \mid m)\,\mathrm{sn}(z \mid m))\,\mathrm{ns}(z \mid m) + 2\,m\,\mathrm{ds}(z \mid m)\,((m-1)\,z + E(\mathrm{am}(z \mid m) \mid m) - m\,\mathrm{cd}(z \mid m)\,\mathrm{sn}(z \mid m))\,\mathrm{ns}(z \mid m) + (1-m)\,m\,\mathrm{ds}(z \mid m) \Big(2\,z + \frac{E(\mathrm{am}(z \mid m) \mid m) - F(\mathrm{am}(z \mid m) \mid m)}{m} - 2\,\mathrm{cd}(z \mid m)\,\mathrm{sn}(z \mid m) - ((m-1)\,z + E(\mathrm{am}(z \mid m) \mid m))\,\mathrm{nd}(z \mid m)\,\mathrm{sd}(z \mid m)\,\mathrm{sn}(z \mid m) + \frac{1}{m-1}\,(\mathrm{cd}(z \mid m)\,\mathrm{cn}(z \mid m)\,\mathrm{dn}(z \mid m)\,(-m\,z + z - E(\mathrm{am}(z \mid m) \mid m) + m\,\mathrm{cd}(z \mid m)\,\mathrm{sn}(z \mid m))) + \frac{1}{(m-1)\,m}\Big(((((m-1)\,z + E(\mathrm{am}(z \mid m) \mid m))\,\mathrm{dn}(z \mid m) - m\,\mathrm{cn}(z \mid m)\,\mathrm{sn}(z \mid m))\,\sqrt{1 - m\,\mathrm{sn}(z \mid m)^2}\,\Big)\Big)\,\mathrm{ns}(z \mid m) + \mathrm{cs}(z \mid m)\,\mathrm{ds}(z \mid m)^2\,((m-1)\,z + E(\mathrm{am}(z \mid m) \mid m) - m\,\mathrm{cd}(z \mid m)\,\mathrm{sn}(z \mid m))^2\Big)$$

## Symbolic differentiation

### With respect to *z*

09.27.20.0007.01

$$\frac{\partial^n \mathrm{cs}(z \mid m)}{\partial z^n} = \mathrm{cs}(z \mid m)\,\delta_n - \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{\partial^j \mathrm{ns}(z \mid m)}{\partial z^j}\,\frac{\partial^{-j+n-1} \mathrm{ds}(z \mid m)}{\partial z^{-j+n-1}} \;/;\; n \in \mathbb{N}$$

09.27.20.0005.01

$$\frac{\partial^n \mathrm{cs}(z \mid m)}{\partial z^n} = (-1)^n\, n!\, z^{-n-1} + \frac{1}{2}\, z^{-n-1} \sum_{k=1}^{\infty} \frac{(-1)^k B_{2k}}{k\,(2k-n-1)!} \left(\frac{\pi z}{K(m)}\right)^{2k} - \frac{2\,\pi^{n+1}}{K(m)^{n+1}} \sum_{k=1}^{\infty} \frac{k^n\, q(m)^{2k}}{q(m)^{2k}+1} \sin\left(\frac{\pi n}{2} + \frac{k\,\pi\, z}{K(m)}\right) /;\; n \in \mathbb{N}^+$$

## Fractional integro-differentiation

### With respect to *z*

09.27.20.0006.01

$$\frac{\partial^\alpha \mathrm{cs}(z \mid m)}{\partial z^\alpha} = \mathcal{F}C_{\exp}^{(\alpha)}(z, -1)\, z^{-\alpha-1} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k\, \pi^{2k}\, B_{2k}\, K(m)^{-2k}\, z^{2k-\alpha-1}}{k\,\Gamma(2k-\alpha)} - \frac{2^\alpha\, \pi^{5/2}\, z^{1-\alpha}}{K(m)^2} \sum_{k=1}^{\infty} \frac{k\, q(m)^{2k}}{q(m)^{2k}+1}\, {}_1\tilde{F}_2\left(1;\, 1 - \frac{\alpha}{2},\, \frac{3-\alpha}{2};\, -\frac{k^2\,\pi^2\, z^2}{4\,K(m)^2}\right)$$

# Integration

## Indefinite integration

### Involving only one direct function

09.27.21.0001.01

$$\int \mathrm{cs}(z \mid m)\, dz = \log(\mathrm{ns}(z \mid m) - \mathrm{ds}(z \mid m))$$

# Representations through equivalent functions

## With inverse function

09.27.27.0001.01

$$\mathrm{cs}\left(\mathrm{cs}^{-1}(z \mid m) \,\middle|\, m\right) = z$$

## With related functions

### Involving am

09.27.27.0028.01

$$\mathrm{cs}(z \mid m) = \cot(\mathrm{am}(z \mid m))$$

### Involving one other Jacobi elliptic function

#### Involving cd

09.27.27.0004.01

$$\mathrm{cs}(z \mid m)^2 = \frac{(m-1)\,\mathrm{cd}(z \mid m)^2}{\mathrm{cd}(z \mid m)^2 - 1}$$

#### Involving cn

09.27.27.0007.01

$$\mathrm{cs}(z \mid m)^2 = \frac{\mathrm{cn}(z \mid m)^2}{1 - \mathrm{cn}(z \mid m)^2}$$

#### Involving dc

09.27.27.0010.01

$$\mathrm{cs}(z \mid m)^2 = \frac{m-1}{1 - \mathrm{dc}(z \mid m)^2}$$

#### Involving dn

09.27.27.0011.01

$$\mathrm{cs}(z \mid m)^2 = \frac{\mathrm{dn}(z \mid m)^2 + m - 1}{1 - \mathrm{dn}(z \mid m)^2}$$

#### Involving ds

09.27.27.0012.01

$$\mathrm{cs}(z \mid m)^2 = \mathrm{ds}(z \mid m)^2 + m - 1$$

#### Involving nc

09.27.27.0015.01

$$\mathrm{cs}(z \mid m)^2 = \frac{1}{\mathrm{nc}(z \mid m)^2 - 1}$$

## Involving nd

09.27.27.0016.01

$$\mathrm{cs}(z \mid m)^2 = \frac{(m-1)\,\mathrm{nd}(z \mid m)^2 + 1}{\mathrm{nd}(z \mid m)^2 - 1}$$

## Involving ns

09.27.27.0017.01

$$\mathrm{cs}(z \mid m) = i\,\mathrm{ns}(i\,z \mid 1-m)$$

09.27.27.0018.01

$$\mathrm{cs}(z \mid m)^2 = \mathrm{ns}(z \mid m)^2 - 1$$

## Involving sc

09.27.27.0019.01

$$\mathrm{cs}(z \mid m) = \frac{1}{\mathrm{sc}(z \mid m)}$$

## Involving sd

09.27.27.0020.01

$$\mathrm{cs}(z \mid m)^2 = \frac{(m-1)\,\mathrm{sd}(z \mid m)^2 + 1}{\mathrm{sd}(z \mid m)^2}$$

## Involving sn

09.27.27.0021.01

$$\mathrm{cs}(z \mid m) = \frac{i}{\mathrm{sn}(i\,z \mid 1-m)}$$

09.27.27.0022.01

$$\mathrm{cs}(z \mid m)^2 = \frac{1 - \mathrm{sn}(z \mid m)^2}{\mathrm{sn}(z \mid m)^2}$$

### Involving two other Jacobi elliptic functions

## Involving cd and ds

09.27.27.0002.01

$$\mathrm{cs}(z \mid m) = \mathrm{cd}(z \mid m)\,\mathrm{ds}(z \mid m)$$

## Involving cd and sd

09.27.27.0003.01

$$\mathrm{cs}(z \mid m) = \frac{\mathrm{cd}(z \mid m)}{\mathrm{sd}(z \mid m)}$$

## Involving cn and ns

09.27.27.0005.01

$$\mathrm{cs}(z \mid m) = \mathrm{cn}(z \mid m)\, \mathrm{ns}(z \mid m)$$

## Involving cn and sc

09.27.27.0029.01

$$\mathrm{cs}(z \mid m) = -\frac{\mathrm{cn}(z \mid m)^2\, \mathrm{sc}(z \mid m)}{(\mathrm{cn}(z \mid m) - 1)\, (\mathrm{cn}(z \mid m) + 1)}$$

## Involving cn and sn

09.27.27.0006.01

$$\mathrm{cs}(z \mid m) = \frac{\mathrm{cn}(z \mid m)}{\mathrm{sn}(z \mid m)}$$

## Involving dc and ds

09.27.27.0008.01

$$\mathrm{cs}(z \mid m) = \frac{\mathrm{ds}(z \mid m)}{\mathrm{dc}(z \mid m)}$$

## Involving dc and sd

09.27.27.0009.01

$$\mathrm{cs}(z \mid m) = \frac{1}{\mathrm{dc}(z \mid m)\, \mathrm{sd}(z \mid m)}$$

## Involving dn and sc

09.27.27.0030.01

$$\mathrm{cs}(z \mid m) = -\frac{\left(\mathrm{dn}(z \mid m)^2 + m - 1\right) \mathrm{sc}(z \mid m)}{(\mathrm{dn}(z \mid m) - 1)\, (\mathrm{dn}(z \mid m) + 1)}$$

## Involving nc and ns

09.27.27.0013.01

$$\mathrm{cs}(z \mid m) = \frac{\mathrm{ns}(z \mid m)}{\mathrm{nc}(z \mid m)}$$

## Involving nc and sn

09.27.27.0014.01

$$\operatorname{cs}(z \mid m) == \frac{1}{\operatorname{nc}(z \mid m) \operatorname{sn}(z \mid m)}$$

09.27.27.0031.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \operatorname{sn}(z \mid m)}{(\operatorname{nc}(z \mid m) - 1)(\operatorname{nc}(z \mid m) + 1)}$$

## Involving sc and sn

09.27.27.0032.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{sc}(z \mid m)(\operatorname{sn}(z \mid m) - 1)(\operatorname{sn}(z \mid m) + 1)}{\operatorname{sn}(z \mid m)^2}$$

### Involving three other Jacobi elliptic functions

09.27.27.0033.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m)^2 \operatorname{dc}(z \mid m)}{(\operatorname{cn}(z \mid m) - 1)(\operatorname{cn}(z \mid m) + 1) \operatorname{ds}(z \mid m)}$$

09.27.27.0034.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dc}(z \mid m)(\operatorname{dn}(z \mid m) - \operatorname{ds}(z \mid m))(\operatorname{dn}(z \mid m) + \operatorname{ds}(z \mid m))}{\operatorname{dn}(z \mid m)^2 \operatorname{ds}(z \mid m)}$$

09.27.27.0035.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m)}{\operatorname{ds}(z \mid m)(\operatorname{nc}(z \mid m) - 1)(\operatorname{nc}(z \mid m) + 1)}$$

09.27.27.0036.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dn}(z \mid m) \operatorname{nc}(z \mid m)}{\operatorname{ds}(z \mid m)(\operatorname{nc}(z \mid m) - 1)(\operatorname{nc}(z \mid m) + 1)}$$

09.27.27.0037.01

$$\operatorname{cs}(z \mid m) = -\frac{m \operatorname{cn}(z \mid m)}{(\operatorname{dn}(z \mid m) - 1)(\operatorname{dn}(z \mid m) + 1) \operatorname{ns}(z \mid m)}$$

09.27.27.0038.01

$$\operatorname{cs}(z \mid m) = \frac{m \operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m)}{(\operatorname{nd}(z \mid m) - 1)(\operatorname{nd}(z \mid m) + 1) \operatorname{ns}(z \mid m)}$$

09.27.27.0039.01

$$\operatorname{cs}(z \mid m) = \frac{\left(\operatorname{dn}(z \mid m)^2 + m - 1\right) \operatorname{ns}(z \mid m)}{m \operatorname{cd}(z \mid m) \operatorname{dn}(z \mid m)}$$

09.27.27.0040.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dn}(z \mid m)^2 \operatorname{sc}(z \mid m)}{(\operatorname{dc}(z \mid m) - \operatorname{dn}(z \mid m))(\operatorname{dc}(z \mid m) + \operatorname{dn}(z \mid m))}$$

09.27.27.0041.01

$$\operatorname{cs}(z \mid m) = -\frac{(\operatorname{dn}(z \mid m) - \operatorname{ds}(z \mid m))(\operatorname{dn}(z \mid m) + \operatorname{ds}(z \mid m)) \operatorname{sc}(z \mid m)}{\operatorname{dn}(z \mid m)^2}$$

09.27.27.0042.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{cn}(z \mid m) - \operatorname{nc}(z \mid m)}$$

09.27.27.0043.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cd}(z \mid m)^2 \operatorname{sc}(z \mid m)}{(\operatorname{cd}(z \mid m) - \operatorname{nd}(z \mid m)) (\operatorname{cd}(z \mid m) + \operatorname{nd}(z \mid m))}$$

09.27.27.0044.01

$$\operatorname{cs}(z \mid m) = -\frac{(\operatorname{dn}(z \mid m) + m \operatorname{nd}(z \mid m) - \operatorname{nd}(z \mid m)) \operatorname{sc}(z \mid m)}{\operatorname{dn}(z \mid m) - \operatorname{nd}(z \mid m)}$$

09.27.27.0045.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{ns}(z \mid m)}{\operatorname{cn}(z \mid m)}$$

09.27.27.0046.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dn}(z \mid m) \operatorname{nc}(z \mid m)}{\left(m \operatorname{nc}(z \mid m)^2 - \operatorname{nc}(z \mid m)^2 - m\right) \operatorname{sd}(z \mid m)}$$

09.27.27.0047.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m)^2 \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)}{(\operatorname{cn}(z \mid m) - 1) (\operatorname{cn}(z \mid m) + 1)}$$

09.27.27.0048.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{dn}(z \mid m)^2 \operatorname{sd}(z \mid m)}{(\operatorname{dc}(z \mid m) - \operatorname{dn}(z \mid m)) (\operatorname{dc}(z \mid m) + \operatorname{dn}(z \mid m))}$$

09.27.27.0049.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)}{(\operatorname{nc}(z \mid m) - 1) (\operatorname{nc}(z \mid m) + 1)}$$

09.27.27.0050.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \operatorname{sd}(z \mid m)}{(\operatorname{nc}(z \mid m) - 1) (\operatorname{nc}(z \mid m) + 1) \operatorname{nd}(z \mid m)}$$

09.27.27.0051.01

$$\operatorname{cs}(z \mid m) = \frac{m \operatorname{nd}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{nc}(z \mid m) (\operatorname{nd}(z \mid m) - 1) (\operatorname{nd}(z \mid m) + 1)}$$

09.27.27.0052.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{sc}(z \mid m) (\operatorname{nd}(z \mid m) - \operatorname{sd}(z \mid m)) (\operatorname{nd}(z \mid m) + \operatorname{sd}(z \mid m))}{\operatorname{sd}(z \mid m)^2}$$

09.27.27.0053.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{nd}(z \mid m) - \operatorname{sd}(z \mid m)) (\operatorname{nd}(z \mid m) + \operatorname{sd}(z \mid m))}{\operatorname{sd}(z \mid m)}$$

09.27.27.0054.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{sc}(z \mid m) (\operatorname{ns}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{sn}(z \mid m)}$$

09.27.27.0055.01

$$\operatorname{cs}(z \mid m) = \frac{(m-1)\operatorname{cd}(z \mid m)\operatorname{sn}(z \mid m)}{(\operatorname{cd}(z \mid m)-1)(\operatorname{cd}(z \mid m)+1)\operatorname{dn}(z \mid m)}$$

09.27.27.0056.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\operatorname{dn}(z \mid m)\operatorname{sn}(z \mid m)}{(\operatorname{dc}(z \mid m)-\operatorname{dn}(z \mid m))(\operatorname{dc}(z \mid m)+\operatorname{dn}(z \mid m))}$$

09.27.27.0057.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m)\operatorname{sn}(z \mid m)}{(\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m)-1)(\operatorname{cd}(z \mid m)\operatorname{dn}(z \mid m)+1)}$$

09.27.27.0058.01

$$\operatorname{cs}(z \mid m) = -\frac{\left(\operatorname{dn}(z \mid m)^2+m-1\right)\operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m)(\operatorname{dn}(z \mid m)-1)\operatorname{dn}(z \mid m)(\operatorname{dn}(z \mid m)+1)}$$

09.27.27.0059.01

$$\operatorname{cs}(z \mid m) = -\frac{m\operatorname{sn}(z \mid m)}{(\operatorname{dn}(z \mid m)-1)(\operatorname{dn}(z \mid m)+1)\operatorname{nc}(z \mid m)}$$

09.27.27.0060.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cd}(z \mid m)\operatorname{nd}(z \mid m)\operatorname{sn}(z \mid m)}{(\operatorname{cd}(z \mid m)-\operatorname{nd}(z \mid m))(\operatorname{cd}(z \mid m)+\operatorname{nd}(z \mid m))}$$

09.27.27.0061.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m)\operatorname{nd}(z \mid m)\operatorname{sn}(z \mid m)}{(\operatorname{dc}(z \mid m)\operatorname{nd}(z \mid m)-1)(\operatorname{dc}(z \mid m)\operatorname{nd}(z \mid m)+1)}$$

09.27.27.0062.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dc}(z \mid m)(\operatorname{sn}(z \mid m)-1)(\operatorname{sn}(z \mid m)+1)}{\operatorname{ds}(z \mid m)\operatorname{sn}(z \mid m)^2}$$

09.27.27.0063.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dc}(z \mid m)\operatorname{sd}(z \mid m)(\operatorname{sn}(z \mid m)-1)(\operatorname{sn}(z \mid m)+1)}{\operatorname{sn}(z \mid m)^2}$$

09.27.27.0064.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dc}(z \mid m)(\operatorname{sn}(z \mid m)-1)(\operatorname{sn}(z \mid m)+1)}{\operatorname{dn}(z \mid m)\operatorname{sn}(z \mid m)}$$

09.27.27.0065.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{nd}(z \mid m)(\operatorname{sn}(z \mid m)-1)(\operatorname{sn}(z \mid m)+1)}{\operatorname{cd}(z \mid m)\operatorname{sn}(z \mid m)}$$

09.27.27.0066.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{sc}(z \mid m)\operatorname{sn}(z \mid m)-\operatorname{nc}(z \mid m)}{\operatorname{sn}(z \mid m)}$$

### Involving four other Jacobi elliptic functions

09.27.27.0067.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m)\operatorname{dc}(z \mid m)-\operatorname{ds}(z \mid m)^2\operatorname{nd}(z \mid m)}{\operatorname{cn}(z \mid m)\operatorname{ds}(z \mid m)}$$

09.27.27.0068.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dc}(z \mid m)\left(\operatorname{dn}(z \mid m) - \operatorname{ds}(z \mid m)^2 \operatorname{nd}(z \mid m)\right)}{\operatorname{dn}(z \mid m) \operatorname{ds}(z \mid m)}$$

09.27.27.0069.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{nc}(z \mid m)\left(\operatorname{ds}(z \mid m)^2 \operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m)\right)}{\operatorname{ds}(z \mid m)}$$

09.27.27.0070.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{ds}(z \mid m)^2 \operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{dc}(z \mid m)}{\operatorname{ds}(z \mid m)}$$

09.27.27.0071.01

$$\operatorname{cs}(z \mid m) = \frac{(\operatorname{dn}(z \mid m) + m \operatorname{nd}(z \mid m) - \operatorname{nd}(z \mid m)) \operatorname{ns}(z \mid m)}{m \operatorname{cd}(z \mid m)}$$

09.27.27.0072.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) \operatorname{dc}(z \mid m) - \operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m)}{\operatorname{cn}(z \mid m) \operatorname{ds}(z \mid m)}$$

09.27.27.0073.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dc}(z \mid m) (\operatorname{dn}(z \mid m) - \operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m))}{\operatorname{dn}(z \mid m) \operatorname{ds}(z \mid m)}$$

09.27.27.0074.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) \operatorname{dn}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{cn}(z \mid m) \operatorname{dn}(z \mid m) - \operatorname{dc}(z \mid m)}$$

09.27.27.0075.01

$$\operatorname{cs}(z \mid m) = -\frac{(\operatorname{dn}(z \mid m) + m \operatorname{cd}(z \mid m) \operatorname{nc}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{nc}(z \mid m)) \operatorname{sc}(z \mid m)}{\operatorname{dn}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{nc}(z \mid m)}$$

09.27.27.0076.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dn}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{dn}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{nc}(z \mid m)}$$

09.27.27.0077.01

$$\operatorname{cs}(z \mid m) = -\frac{(\operatorname{cn}(z \mid m) + m \operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m)) \operatorname{sc}(z \mid m)}{\operatorname{cn}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m)}$$

09.27.27.0078.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{cn}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}$$

09.27.27.0079.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dn}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{dn}(z \mid m) - \operatorname{dc}(z \mid m)^2 \operatorname{nd}(z \mid m)}$$

09.27.27.0080.01

$$\operatorname{cs}(z \mid m) = -\frac{\left(\operatorname{dn}(z \mid m) - \operatorname{ds}(z \mid m)^2 \operatorname{nd}(z \mid m)\right) \operatorname{sc}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

09.27.27.0081.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cd}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{cd}(z \mid m) - \operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m)}$$

09.27.27.0082.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cd}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{cd}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)^2}$$

09.27.27.0083.01

$$\operatorname{cs}(z \mid m) = -\frac{(\operatorname{dn}(z \mid m) - \operatorname{ds}(z \mid m) \operatorname{ns}(z \mid m)) \operatorname{sc}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

09.27.27.0084.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cd}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m)^2}{\operatorname{cd}(z \mid m)}$$

09.27.27.0085.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cd}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{nd}(z \mid m) \operatorname{ns}(z \mid m)}{\operatorname{cd}(z \mid m)}$$

09.27.27.0086.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dn}(z \mid m) \operatorname{ns}(z \mid m) + m \operatorname{cd}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{sc}(z \mid m)}{\operatorname{cd}(z \mid m)}$$

09.27.27.0087.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m)}{\operatorname{cn}(z \mid m)}$$

09.27.27.0088.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dn}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{ds}(z \mid m) \operatorname{nc}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

09.27.27.0089.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dn}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{ns}(z \mid m)}{\operatorname{dn}(z \mid m)}$$

09.27.27.0090.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) \operatorname{dn}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{ds}(z \mid m)}{\operatorname{cn}(z \mid m) \operatorname{dn}(z \mid m)}$$

09.27.27.0091.01

$$\operatorname{cs}(z \mid m) = -\frac{1}{\operatorname{dn}(z \mid m)} \left(m \operatorname{cd}(z \mid m) \operatorname{ns}(z \mid m) \operatorname{sc}(z \mid m)^2 - \operatorname{cd}(z \mid m) \operatorname{ns}(z \mid m) \operatorname{sc}(z \mid m)^2 + \operatorname{dn}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{ns}(z \mid m)\right)$$

09.27.27.0092.01

$$\operatorname{cs}(z \mid m) = \operatorname{dc}(z \mid m) \left(\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m)^2 - \operatorname{sd}(z \mid m)\right)$$

09.27.27.0093.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{nc}(z \mid m) \left(\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m)^2 - \operatorname{sd}(z \mid m)\right)}{\operatorname{nd}(z \mid m)}$$

09.27.27.0094.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{nc}(z \mid m) (\operatorname{dn}(z \mid m) + m \operatorname{nd}(z \mid m) - \operatorname{nd}(z \mid m))}{m \operatorname{sd}(z \mid m)}$$

09.27.27.0095.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{sc}(z \mid m) \left(\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m)^2 - \operatorname{sd}(z \mid m)\right)}{\operatorname{sd}(z \mid m)}$$

09.27.27.0096.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{cn}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}$$

09.27.27.0097.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{dn}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{dc}(z \mid m)^2 \operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m)}$$

09.27.27.0098.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cn}(z \mid m) \operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{ns}(z \mid m)}{\operatorname{cn}(z \mid m)}$$

09.27.27.0099.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dc}(z \mid m) (\operatorname{dn}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{ns}(z \mid m))}{\operatorname{dn}(z \mid m)}$$

09.27.27.0100.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{sc}(z \mid m) (\operatorname{dn}(z \mid m) + m \operatorname{ns}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{ns}(z \mid m) \operatorname{sd}(z \mid m))}{\operatorname{dn}(z \mid m) - \operatorname{ns}(z \mid m) \operatorname{sd}(z \mid m)}$$

09.27.27.0101.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m)}{\operatorname{sd}(z \mid m)}$$

09.27.27.0102.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)^2}{\operatorname{sd}(z \mid m)}$$

09.27.27.0103.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dn}(z \mid m) \operatorname{nc}(z \mid m) + m \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{sc}(z \mid m) \operatorname{sd}(z \mid m)}{\operatorname{sd}(z \mid m)}$$

09.27.27.0104.01

$$\operatorname{cs}(z \mid m) = \operatorname{nc}(z \mid m) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{sn}(z \mid m))$$

09.27.27.0105.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{nd}(z \mid m) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{cd}(z \mid m)}$$

09.27.27.0106.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{ns}(z \mid m) - \operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m) \operatorname{dn}(z \mid m)}$$

09.27.27.0107.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{ns}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{dn}(z \mid m)}$$

09.27.27.0108.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{nd}(z \mid m) (\operatorname{ns}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{cd}(z \mid m)}$$

09.27.27.0109.01

$$\operatorname{cs}(z \mid m) = \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m) (\operatorname{ns}(z \mid m) - \operatorname{sn}(z \mid m))$$

09.27.27.0110.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{ds}(z \mid m) \operatorname{sn}(z \mid m)}$$

09.27.27.0111.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{sc}(z \mid m) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{sn}(z \mid m)}$$

09.27.27.0112.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{ns}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{ds}(z \mid m) \operatorname{sn}(z \mid m)}$$

09.27.27.0113.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{sd}(z \mid m) (\operatorname{ns}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{sn}(z \mid m)}$$

09.27.27.0114.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dn}(z \mid m) \operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m) \operatorname{dn}(z \mid m)^2 - \operatorname{dc}(z \mid m)}$$

09.27.27.0115.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m) \operatorname{dn}(z \mid m) - \operatorname{nc}(z \mid m)}$$

09.27.27.0116.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{sn}(z \mid m)}{\operatorname{dc}(z \mid m) \operatorname{nc}(z \mid m) - \operatorname{dn}(z \mid m)}$$

09.27.27.0117.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{cd}(z \mid m) \operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m)^2 \operatorname{dn}(z \mid m) - \operatorname{nd}(z \mid m)}$$

09.27.27.0118.01

$$\operatorname{cs}(z \mid m) = -\frac{(\operatorname{dn}(z \mid m) + m \operatorname{nd}(z \mid m) - \operatorname{nd}(z \mid m)) \operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m) (\operatorname{dn}(z \mid m) - 1) (\operatorname{dn}(z \mid m) + 1)}$$

09.27.27.0119.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) \operatorname{sn}(z \mid m)}{\operatorname{dc}(z \mid m)^2 \operatorname{nd}(z \mid m) - \operatorname{dn}(z \mid m)}$$

09.27.27.0120.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{nd}(z \mid m) \operatorname{sn}(z \mid m)}{\operatorname{nc}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{cd}(z \mid m)}$$

09.27.27.0121.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{nd}(z \mid m) \operatorname{sn}(z \mid m)}{\operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)^2 - \operatorname{cd}(z \mid m)}$$

09.27.27.0122.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{sc}(z \mid m) (m \operatorname{nd}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{nd}(z \mid m) \operatorname{sd}(z \mid m) + \operatorname{sn}(z \mid m))}{\operatorname{nd}(z \mid m) \operatorname{sd}(z \mid m) - \operatorname{sn}(z \mid m)}$$

09.27.27.0123.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dn}(z \mid m) \operatorname{sn}(z \mid m) - \operatorname{ds}(z \mid m)}{\operatorname{cd}(z \mid m) \operatorname{dn}(z \mid m)^2}$$

09.27.27.0124.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dc}(z \mid m) (\operatorname{dn}(z \mid m) \operatorname{sn}(z \mid m) - \operatorname{ds}(z \mid m))}{\operatorname{dn}(z \mid m)^2}$$

09.27.27.0125.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{sc}(z \mid m) (\operatorname{dn}(z \mid m) \operatorname{sn}(z \mid m) - \operatorname{ds}(z \mid m))}{\operatorname{dn}(z \mid m) \operatorname{sn}(z \mid m)}$$

09.27.27.0126.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}{\operatorname{sn}(z \mid m)}$$

09.27.27.0127.01

$$\operatorname{cs}(z \mid m) = -\frac{1}{\operatorname{sn}(z \mid m)} \left(m \operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m)^2 - \operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m)^2 + \operatorname{sn}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m)\right)$$

09.27.27.0128.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dn}(z \mid m) \operatorname{sc}(z \mid m) \operatorname{sn}(z \mid m) - \operatorname{dc}(z \mid m)}{\operatorname{dn}(z \mid m) \operatorname{sn}(z \mid m)}$$

09.27.27.0129.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{dc}(z \mid m) (\operatorname{sd}(z \mid m) \operatorname{sn}(z \mid m) - \operatorname{nd}(z \mid m))}{\operatorname{sn}(z \mid m)}$$

### Involving five other Jacobi elliptic functions

09.27.27.0130.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m) \operatorname{dn}(z \mid m)}$$

09.27.27.0131.01

$$\operatorname{cs}(z \mid m) = \frac{\operatorname{dc}(z \mid m) (\operatorname{ds}(z \mid m) \operatorname{nd}(z \mid m) - \operatorname{sn}(z \mid m))}{\operatorname{dn}(z \mid m)}$$

09.27.27.0132.01

$$\operatorname{cs}(z \mid m) = -\frac{\operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m) \operatorname{dn}(z \mid m) - \operatorname{dc}(z \mid m) \operatorname{nd}(z \mid m)}$$

09.27.27.0133.01

$$\operatorname{cs}(z \mid m) = -\frac{m \operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m) - \operatorname{cd}(z \mid m) \operatorname{nd}(z \mid m) \operatorname{sc}(z \mid m) + \operatorname{sn}(z \mid m)}{\operatorname{cd}(z \mid m) (\operatorname{dn}(z \mid m) - \operatorname{nd}(z \mid m))}$$

### Involving Weierstrass functions

09.27.27.0023.01

$$\operatorname{cs}(z \mid m) = \frac{1}{\sqrt{e_1 - e_3}} \frac{\sigma_1\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)}{\sigma\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right)} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

09.27.27.0024.01

$$\operatorname{cs}(z \mid m)^2 = \frac{\wp\left(\frac{z}{\sqrt{e_1-e_3}}; g_2, g_3\right) - e_1}{e_1 - e_3} /;$$

$$\{\omega_1, \omega_2, \omega_3\} = \{\omega_1(g_2, g_3), -\omega_1(g_2, g_3) - \omega_3(g_2, g_3), \omega_3(g_2, g_3)\} \bigwedge m = \lambda\left(\frac{\omega_3}{\omega_1}\right) \bigwedge e_n = \wp(\omega_n; g_2, g_3) \bigwedge n \in \{1, 2, 3\}$$

### Involving theta functions

09.27.27.0025.01

$$\operatorname{cs}(z \mid m) = \sqrt[4]{1 - m} \frac{\vartheta_2\left(\frac{\pi z}{2 K(m)}, q(m)\right)}{\vartheta_1\left(\frac{\pi z}{2 K(m)}, q(m)\right)}$$

09.27.27.0026.01

$$\operatorname{cs}(z \mid m) = \frac{\vartheta_4(0, q(m))\, \vartheta_2\left(\frac{z}{\vartheta_3(0,q(m))^2}, q(m)\right)}{\vartheta_3(0, q(m))\, \vartheta_1\left(\frac{z}{\vartheta_3(0,q(m))^2}, q(m)\right)}$$

09.27.27.0027.01

$$\operatorname{cs}(z \mid m) = \frac{\vartheta_c(z \mid m)}{\vartheta_s(z \mid m)}$$

## Zeros

09.27.30.0001.01

$$\operatorname{cs}((2 r + 1) K(m) + 2 s i K(1 - m) \mid m) = 0 /; \{r, s\} \in \mathbb{Z}$$

## History

– C. G. J. Jacobi (1827)
– N.H. Abel (1827)
– J. Glaisher (1882) introduced the notations cs

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