

KleinInvariantJ

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Notations

Traditional name

Klein invariant modular function

Traditional notation

$J(z)$

Mathematica StandardForm notation

KleinInvariantJ[z]

Primary definition

09.50.02.0001.01

$$J(z) = \frac{(\vartheta_2(0, e^{\pi iz})^8 + \vartheta_3(0, e^{\pi iz})^8 + \vartheta_4(0, e^{\pi iz})^8)^3}{54 (\vartheta_2(0, e^{\pi iz}) \vartheta_3(0, e^{\pi iz}) \vartheta_4(0, e^{\pi iz}))^8} \quad ; \operatorname{Im}(z) > 0$$

This definition can be continued to selected points on the real axis.

Specific values

Specialized values

09.50.03.0001.01

$$J(i + m) = 1 \quad ; m \in \mathbb{Z}$$

Values at fixed points

Values at complex z

09.50.03.0002.01

$$J(i) = 1$$

Values at quadratic irrationalities (Heegner Numbers)

09.50.03.0013.01

$$J\left(\frac{-1 + i\sqrt{3}}{2}\right) = 1$$

09.50.03.0004.01

$$J\left(\frac{1+i\sqrt{3}}{2}\right) = 0$$

09.50.03.0005.01

$$J\left(\frac{1+i\sqrt{7}}{2}\right) = -\frac{125}{64}$$

09.50.03.0006.01

$$J\left(\frac{1+i\sqrt{11}}{2}\right) = -\frac{512}{27}$$

09.50.03.0007.01

$$J\left(\frac{1+i\sqrt{19}}{2}\right) = -512$$

09.50.03.0008.01

$$J\left(\frac{1+i3\sqrt{3}}{2}\right) = -\frac{64000}{9}$$

09.50.03.0009.01

$$J\left(\frac{1+i\sqrt{43}}{2}\right) = -512000$$

09.50.03.0010.01

$$J\left(\frac{1+i\sqrt{67}}{2}\right) = -85184000$$

09.50.03.0011.01

$$J\left(\frac{1+i\sqrt{163}}{2}\right) = -151931373056000$$

Values at infinities

09.50.03.0012.01

$$J(i\infty) = \infty$$

General characteristics

Domain and analyticity

$J(z)$ is an analytical function of z which is defined over the upper half of the complex z -plane.

09.50.04.0001.01

$$z \rightarrow J(z) :: \mathbb{C} \rightarrow \mathbb{C}$$

Symmetries and periodicities

Periodicity

09.50.04.0002.01

$$J(m+z) = J(z) \ ; \ m \in \mathbb{Z}$$

Poles and essential singularities

On the boundary of analyticity the function $J(z)$ has a dense set of poles.

09.50.04.0003.01

$$\text{Sing}_z(J(z)) = \{ \} \ ; \ \text{Im}(z) > 0$$

Branch points

The function $J(z)$ does not have branch points.

09.50.04.0004.01

$$\mathcal{BP}_z(J(z)) = \{ \}$$

Branch cuts

The function $J(z)$ does not have branch cuts.

09.50.04.0005.01

$$\mathcal{BC}_z(J(z)) = \{ \}$$

Natural boundary of analyticity

The real axis $\text{Im}(z) = 0$ is the natural boundary of the region of analyticity.

09.50.04.0006.01

$$\mathcal{AB}_z(J(z)) = \{(-\infty, \infty)\}$$

Series representations

Generalized power series

Expansions at generic point $z = z_0$

For the function itself

09.50.06.0003.01

$$J(z) \propto J(z_0) + \frac{8 i K(w)^2 (w-1) w (\lambda(z_0)^2 - \lambda(z_0) + 1)^2 (2 \lambda(z_0)^3 - 3 \lambda(z_0)^2 - 3 \lambda(z_0) + 2)}{27 (K(1-w) (K(w) - E(w)) - E(1-w) K(w)) (\lambda(z_0) - 1)^3 \lambda(z_0)^3} (z - z_0) -$$

$$\frac{8 (1-w) w (\lambda(z_0)^2 - \lambda(z_0) + 1)}{27 (K(1-w) (E(w) - K(w)) + E(1-w) K(w))^2 (\lambda(z_0) - 1)^4 \lambda(z_0)^4}$$

$$(E(w) \lambda(z_0) (-2 \lambda(z_0)^6 + 7 \lambda(z_0)^5 - 7 \lambda(z_0)^4 + 7 \lambda(z_0)^2 - 7 \lambda(z_0) + 2) +$$

$$w K(w) (2 \lambda(z_0)^7 - 9 \lambda(z_0)^6 + 13 \lambda(z_0)^5 - 4 \lambda(z_0)^4 - 9 \lambda(z_0)^3 - 11 \lambda(z_0)^2 + 18 \lambda(z_0) + 2 w (\lambda(z_0)^6 - 3 \lambda(z_0)^5 +$$

$$2 \lambda(z_0)^4 + \lambda(z_0)^3 + 9 \lambda(z_0)^2 - 10 \lambda(z_0) + 3) - 6) K(w)^3 (z - z_0)^2 + \dots \ ; \ (z \rightarrow z_0) \wedge w = q^{-1} (e^{i\pi z_0})$$

09.50.06.0004.01

$$J(z) \propto J(z_0) (1 + O(z - z_0))$$

Exponential Fourier series

09.50.06.0001.01

$$J(z) = \frac{1}{1728} \left(e^{-2i\pi z} + 744 + \sum_{k=1}^{\infty} a_k e^{2ki\pi z} \right) /;$$

$$a_k = \frac{2\pi}{\sqrt{k}} \sum_{j=1}^{\infty} \frac{1}{j} A_j(k) I_1 \left(\frac{(4\pi) \sqrt{k}}{j} \right) \wedge \left(A_j(k) = \sum_{h=0}^{j-1} \delta_{1, \gcd[h,j]} \exp \left(-\frac{(2\pi i)(hk + H(j, h))}{j} \right) /; (hH(j, h)) \bmod j = -1 \right)$$

09.50.06.0002.01

$$J(z) = \frac{1}{1728} \left(e^{-2i\pi z} + 744 + \sum_{k=1}^{\infty} a_k e^{2ki\pi z} \right) /;$$

$$\left(a_1 = 196884 \wedge a_2 = 21493760 \wedge a_3 = 864299970 \wedge a_4 = 20245856256, a_5 = 333202640600 \wedge \right. \\ \left. \left(a_v = a_{2n+1} + \frac{1}{2} (a_n^2 - a_n) + \sum_{k=1}^{n-1} a_k a_{2n-k} /; n = \frac{v}{4} \wedge v \bmod 4 = 0 \right) \wedge \right. \\ \left. \left(a_v = a_{2n+3} - a_2 a_{2n} + \frac{1}{2} (a_{n+1}^2 - a_{n+1}) + \frac{1}{2} (a_{2n}^2 + a_{2n}) + \sum_{k=1}^n a_k a_{2n-k+2} - \sum_{k=1}^{2n-1} (-1)^{k-1} a_k a_{4n-k} + \sum_{k=1}^{n-1} a_k a_{4n-4k} /; n = \right. \right. \\ \left. \left. \frac{v-1}{4} \wedge v \bmod 4 = 1 \right) \wedge \right. \\ \left. \left(a_v = a_{2n+2} + \sum_{k=1}^n a_k a_{2n-k+1} /; n = \frac{v-2}{4} \wedge v \bmod 4 = 2 \right) \wedge \right. \\ \left. \left(a_v = a_{2n+4} - a_2 a_{2n+1} - \frac{1}{2} (a_{2n+1}^2 - a_{2n+1}) + \sum_{k=1}^{n+1} a_k a_{2n-k+3} - \sum_{k=1}^{2n} (-1)^{k-1} a_k a_{4n-k+2} + \sum_{k=1}^n a_k a_{4n-4k+2} /; n = \right. \right. \\ \left. \left. \frac{v-3}{4} \wedge v \bmod 4 = 3 \right) \right)$$

Differential equations

Ordinary nonlinear differential equations

09.50.13.0001.01

$$(-36w(z)^2 + 41w(z) - 32)w'(z)^4 - 72(w(z) - 1)^2w(z)^2w^{(3)}(z)w'(z) + 108(w(z) - 1)^2w(z)^2w''(z)^2 = 0 /; w(z) = J(z)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

09.50.16.0001.01

$$J(z + 1) = J(z)$$

09.50.16.0002.01

$$J\left(-\frac{1}{z}\right) = J(z)$$

Identities

Functional identities

09.50.17.0001.01

$$J(z) = J(z + 1)$$

09.50.17.0002.01

$$J(z) = J\left(-\frac{1}{z}\right)$$

09.50.17.0003.01

$$64 J(z)^3 - 110 592 J(2z)^2 J(z)^2 + 95 232 J(2z) J(z)^2 - 6000 J(z)^2 + 95 232 J(2z)^2 J(z) + 1 510 125 J(2z) J(z) + 187 500 J(z) + 64 J(2z)^3 - 6000 J(2z)^2 + 187 500 J(2z) - 1 953 125 = 0$$

Differentiation

Low-order differentiation

09.50.20.0005.01

$$J'(i) = 0$$

09.50.20.0001.02

$$\frac{\partial J(z)}{\partial z} = -\frac{16 i K(\lambda(z))^2 (\lambda(z) - 2) (\lambda(z) + 1) (2 \lambda(z) - 1) (\lambda(z)^2 - \lambda(z) + 1)^2}{27 \pi (\lambda(z) - 1)^2 \lambda(z)^2}$$

09.50.20.0002.01

$$\frac{\partial J(z)}{\partial z} = \frac{\pi i}{864} \left(-e^{-2i\pi z} + \sum_{k=1}^{\infty} k a_k e^{2ki\pi z} \right) /;$$

$$a_k = \frac{2\pi}{\sqrt{k}} \sum_{j=1}^{\infty} \frac{1}{j} A_j(k) I_1\left(\frac{(4\pi)\sqrt{k}}{j}\right) \wedge \left(A_j(k) = \sum_{h=0}^{j-1} \delta_{1,\gcd[h,j]} \exp\left(-\frac{(2\pi i)(hk + H(j, h))}{j}\right) /; (h H(j, h)) \bmod j = -1 \right)$$

09.50.20.0003.01

$$\frac{\partial J(z)}{\partial z} = \frac{\pi i}{864} \left(-e^{-2i\pi z} + \sum_{k=1}^{\infty} k a_k e^{2ki\pi z} \right) /;$$

$$a_k = \frac{2\pi}{\sqrt{k}} \sum_{j=1}^{\infty} \frac{1}{j} A_j(k) I_1\left(\frac{(4\pi)\sqrt{k}}{j}\right) \wedge \left(A_j(k) = \sum_{h=0}^{j-1} \delta_{1,\gcd[h,j]} \exp\left(-\frac{(2\pi i)(hk + H(j, h))}{j}\right) /; (h H(j, h)) \bmod j = -1 \right)$$

09.50.20.0004.02

$$\frac{\partial^2 J(z)}{\partial z^2} = \frac{64 K(\lambda(z))^3}{27 \pi^2 (\lambda(z) - 1)^2 \lambda(z)^2} \left(E(\lambda(z)) (\lambda(z) - 2) (\lambda(z) + 1) (2 \lambda(z) - 1) ((\lambda(z) - 1) \lambda(z) + 1)^2 - K(\lambda(z)) (\lambda(z)^2 - \lambda(z) + 1) (4 \lambda(z)^6 - 11 \lambda(z)^5 + 6 \lambda(z)^4 + 4 \lambda(z)^3 + 13 \lambda(z)^2 - 18 \lambda(z) + 6) \right)$$

09.50.20.0006.01

$$\frac{\partial^3 J(z)}{\partial z^3} = \frac{128 i K(\lambda(z))^4}{27 \pi^3 (\lambda(z) - 1)^2 \lambda(z)^2} \left((\lambda(z) - 2) (16 \lambda(z)^8 - 28 \lambda(z)^7 + 25 \lambda(z)^6 + 8 \lambda(z)^5 - 32 \lambda(z)^4 + 98 \lambda(z)^3 - 140 \lambda(z)^2 + 92 \lambda(z) - 23) K(\lambda(z))^2 - 6 E(\lambda(z)) (\lambda(z)^2 - \lambda(z) + 1) (4 \lambda(z)^6 - 11 \lambda(z)^5 + 6 \lambda(z)^4 + 4 \lambda(z)^3 + 13 \lambda(z)^2 - 18 \lambda(z) + 6) K(\lambda(z)) + 3 E(\lambda(z))^2 (\lambda(z) - 2) (\lambda(z) + 1) (2 \lambda(z) - 1) (\lambda(z)^2 - \lambda(z) + 1)^2 \right)$$

09.50.20.0007.01

$$\frac{\partial^4 J(z)}{\partial z^4} = \frac{512 K(\lambda(z))^5}{27 \pi^4 (\lambda(z) - 1)^2 \lambda(z)^2} \left(-6 (\lambda(z) - 2) (\lambda(z) + 1) (2 \lambda(z) - 1) (\lambda(z)^2 - \lambda(z) + 1)^2 E(\lambda(z))^3 + 18 K(\lambda(z)) (\lambda(z)^2 - \lambda(z) + 1) (4 \lambda(z)^6 - 11 \lambda(z)^5 + 6 \lambda(z)^4 + 4 \lambda(z)^3 + 13 \lambda(z)^2 - 18 \lambda(z) + 6) E(\lambda(z))^2 - 6 K(\lambda(z))^2 (\lambda(z) - 2) (16 \lambda(z)^8 - 28 \lambda(z)^7 + 25 \lambda(z)^6 + 8 \lambda(z)^5 - 32 \lambda(z)^4 + 98 \lambda(z)^3 - 140 \lambda(z)^2 + 92 \lambda(z) - 23) E(\lambda(z)) + K(\lambda(z))^3 (32 \lambda(z)^{10} - 112 \lambda(z)^9 + 123 \lambda(z)^8 - 12 \lambda(z)^7 + 36 \lambda(z)^6 - 402 \lambda(z)^5 + 1380 \lambda(z)^4 - 2340 \lambda(z)^3 + 2175 \lambda(z)^2 - 1060 \lambda(z) + 212) \right)$$

09.50.20.0008.01

$$\frac{\partial^5 J(z)}{\partial z^5} = \frac{2048 i K(\lambda(z))^6}{27 \pi^5 (\lambda(z) - 1)^2 \lambda(z)^2} \left(-15 (\lambda(z) - 2) (\lambda(z) + 1) (2 \lambda(z) - 1) (\lambda(z)^2 - \lambda(z) + 1)^2 E(\lambda(z))^4 + 60 K(\lambda(z)) (\lambda(z)^2 - \lambda(z) + 1) (4 \lambda(z)^6 - 11 \lambda(z)^5 + 6 \lambda(z)^4 + 4 \lambda(z)^3 + 13 \lambda(z)^2 - 18 \lambda(z) + 6) E(\lambda(z))^3 - 30 K(\lambda(z))^2 (\lambda(z) - 2) (16 \lambda(z)^8 - 28 \lambda(z)^7 + 25 \lambda(z)^6 + 8 \lambda(z)^5 - 32 \lambda(z)^4 + 98 \lambda(z)^3 - 140 \lambda(z)^2 + 92 \lambda(z) - 23) E(\lambda(z))^2 + 10 K(\lambda(z))^3 (32 \lambda(z)^{10} - 112 \lambda(z)^9 + 123 \lambda(z)^8 - 12 \lambda(z)^7 + 36 \lambda(z)^6 - 402 \lambda(z)^5 + 1380 \lambda(z)^4 - 2340 \lambda(z)^3 + 2175 \lambda(z)^2 - 1060 \lambda(z) + 212) E(\lambda(z)) - K(\lambda(z))^4 (\lambda(z) - 2) (64 \lambda(z)^{10} - 64 \lambda(z)^9 - 20 \lambda(z)^8 + 149 \lambda(z)^7 - 131 \lambda(z)^6 + 926 \lambda(z)^5 - 3440 \lambda(z)^4 + 6086 \lambda(z)^3 - 5774 \lambda(z)^2 + 2835 \lambda(z) - 567) \right)$$

Operations

Limit operation

09.50.25.0001.01

$$\lim_{\epsilon \rightarrow +0} J(i \epsilon) = \infty$$

Representations through equivalent functions

With inverse function

09.50.27.0001.01

$$z = \frac{i(r-s)}{r+s} /; \{r, s\} = \left\{ \Gamma\left(\frac{5}{12}\right)^2 {}_2F_1\left(\frac{1}{12}, \frac{1}{12}; \frac{1}{2}; 1-\lambda\right), 2(\sqrt{3}-2)\Gamma\left(\frac{11}{12}\right)^2 \sqrt{\lambda-1} {}_2F_1\left(\frac{7}{12}, \frac{7}{12}; \frac{3}{2}; 1-\lambda\right) \right\} \wedge$$

$$|z| \geq 1 \wedge -\frac{1}{2} \leq \operatorname{Re}(z) \leq 0 \wedge \lambda = J(z)$$

With related functions

Involving Weierstrass functions

09.50.27.0002.01

$$J\left(\frac{\omega_3}{\omega_1}\right) = \frac{g_2^3}{g_2^3 - 27g_3^2} /; \{g_2, g_3\} = \{g_2(\omega_1, \omega_3), g_3(\omega_1, \omega_3)\} \wedge \operatorname{Im}\left(\frac{\omega_3}{\omega_1}\right) > 0$$

09.50.27.0003.01

$$J(z) = \frac{g_2^3}{g_2^3 - 27g_3^2} /; \{g_2, g_3\} = \{g_2(1, z), g_3(1, z)\} \wedge \operatorname{Im}(z) > 0$$

Involving theta functions

09.50.27.0004.01

$$J(z) = \frac{\left(\vartheta_2(0, e^{\pi iz})^8 + \vartheta_3(0, e^{\pi iz})^8 + \vartheta_4(0, e^{\pi iz})^8\right)^3}{54 \left(\vartheta_2(0, e^{\pi iz}) \vartheta_3(0, e^{\pi iz}) \vartheta_4(0, e^{\pi iz})\right)^8} /; \operatorname{Im}(z) > 0$$

Involving other related functions

09.50.27.0005.01

$$J(z) = \frac{(\eta(z)^{24} + 256 \eta(2z)^{24})^3}{1728 \eta(z)^{48} \eta(2z)^{24}}$$

09.50.27.0006.01

$$J(z) = \frac{1}{1728} \left(\frac{256 \eta(2z)^{16}}{\eta(z)^{16}} + \frac{\eta(z)^8}{\eta(2z)^8} \right)^3 /; \operatorname{Im}(z) > 0$$

09.50.27.0008.01

$$J(2z) = \frac{1}{1728} \left(\frac{\eta(z)^{16}}{\eta(2z)^{16}} + \frac{16 \eta(2z)^8}{\eta(z)^8} \right)^3$$

09.50.27.0007.01

$$J(z) = \frac{4(\lambda(z)^2 - \lambda(z) + 1)^3}{27 \lambda(z)^2 (1 - \lambda(z))^2}$$

09.50.27.0009.01

$$J(z) = \frac{(\eta(z)^{12} + 27 \eta(3z)^{12})(\eta(z)^{12} + 243 \eta(3z)^{12})^3}{1728 \eta(z)^{36} \eta(3z)^{12}}$$

09.50.27.0010.01

$$J(3z) = \frac{(\eta(z)^{12} + 3\eta(3z)^{12})^3 (\eta(z)^{12} + 27\eta(3z)^{12})}{1728 \eta(z)^{12} \eta(3z)^{36}}$$

09.50.27.0011.01

$$J(z) = \frac{(\eta(z)^{12} + 250\eta(5z)^6 \eta(z)^6 + 3125\eta(5z)^{12})^3}{1728 \eta(z)^{30} \eta(5z)^6}$$

09.50.27.0012.01

$$J(5z) = \frac{(\eta(z)^{12} + 10\eta(5z)^6 \eta(z)^6 + 5\eta(5z)^{12})^3}{1728 \eta(z)^6 \eta(5z)^{30}}$$

Zeros

09.50.30.0001.01

$$J(z) = 0 \text{ ; } z = e^{\frac{\pi i}{3}} \vee z = e^{\frac{2\pi i}{3}}$$

Theorems

Property of modular functions

Every modular function $m(\tau)$ (meaning a function that is invariant under all argument substitutions of the form $z \rightarrow (az + b)/(cz + d)$ where $a, b, c,$ and d are integers and $ad - bc = 1$) is a rational function of $J(z)$.

Transcendentality of Klein invariant J for algebraic argument

The value $J(\alpha)$ is transcendental for any algebraic α where $\text{Im}(\alpha) > 0$ and α not being a quadratic irrational.

Moonshine conjecture

The dimensions of the representations of the monster group (the largest simple sporadic group) of order $2^{46} \times 3^{20} \times 5^9 \times 11^2 \times 13^3 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71$ are simple combinations the Fourier coefficients of $1728J(z)$.

Solution of Chazy equation

The function $w(z) = \frac{(4-7J(\tau))J'(\tau)^2+6(J(\tau)-1)J(\tau)J''(\tau)}{2(J(\tau)-1)J(\tau)J'(\tau)}$ is a solution of Chazy equation $w'''(z) = 2w''(z)w(z) - 3w(z)^2$.

Replicability of $1728J(\tau) - 744$

The function $1728J(\tau) - 744$ is completely replicable:

$$J(\sigma) - J(\tau) = \frac{e^{-2\pi i \sigma}}{1728} \prod_{n=-1}^{\infty} \prod_{m=1}^{\infty} (1 - e^{2\pi i(m\sigma+n\tau)})^{c(nm)} \text{ ; } 1728J(\tau) - 744 = \sum_{n=-1}^{\infty} c(n) e^{2\pi i \tau}.$$

History

- R. Dedekind (1877)
- F. Klein (1878)
- R. Fricke (1890–1892)

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