# LegendrePGeneral



## Notations

### Traditional name

Legendre function

### Traditional notation

$P_\nu(z)$

### *Mathematica* StandardForm notation

```
LegendreP[ν, z]
```

## Primary definition

07.07.02.0001.01

$$P_\nu(z) = {}_2F_1\left(-\nu, \nu+1; 1; \frac{1-z}{2}\right)$$

## Specific values

### Specialized values

#### For fixed $\nu$

07.07.03.0001.01

$$P_\nu(0) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}+1\right)}$$

07.07.03.0002.01

$$P_\nu(1) = 1$$

07.07.03.0003.01

$$P_\nu(-1) = -\infty \sin(\pi \nu)$$

#### For fixed $z$

07.07.03.0004.01

$$P_{-\frac{1}{2}}(z) = \frac{2}{\pi} K\left(\frac{1-z}{2}\right)$$

07.07.03.0005.01

$$P_{-\frac{1}{2}}(z) = \frac{2\sqrt{2}}{\pi\sqrt{z+1}} K\left(\frac{z-1}{z+1}\right) /; z \notin (-\infty, -1)$$

07.07.03.0006.01

$$P_{\frac{1}{2}}(z) = \frac{2}{\pi}\left(2 E\left(\frac{1-z}{2}\right) - K\left(\frac{1-z}{2}\right)\right)$$

07.07.03.0007.01

$$P_0(z) = 1$$

07.07.03.0008.01

$$P_1(z) = z$$

07.07.03.0009.01

$$P_2(z) = \frac{1}{2}\left(3 z^2 - 1\right)$$

07.07.03.0010.01

$$P_3(z) = \frac{1}{2}\left(5 z^3 - 3 z\right)$$

07.07.03.0011.01

$$P_4(z) = \frac{1}{8}\left(35 z^4 - 30 z^2 + 3\right)$$

07.07.03.0012.01

$$P_5(z) = \frac{1}{8} z\left(63 z^4 - 70 z^2 + 15\right)$$

07.07.03.0013.01

$$P_6(z) = \frac{1}{16}\left(231 z^6 - 315 z^4 + 105 z^2 - 5\right)$$

07.07.03.0014.01

$$P_7(z) = \frac{1}{16} z\left(429 z^6 - 693 z^4 + 315 z^2 - 35\right)$$

07.07.03.0015.01

$$P_8(z) = \frac{1}{128}\left(6435 z^8 - 12\,012 z^6 + 6930 z^4 - 1260 z^2 + 35\right)$$

07.07.03.0016.01

$$P_9(z) = \frac{1}{128} z\left(12\,155 z^8 - 25\,740 z^6 + 18\,018 z^4 - 4620 z^2 + 315\right)$$

07.07.03.0017.01

$$P_{10}(z) = \frac{1}{256}\left(46\,189 z^{10} - 109\,395 z^8 + 90\,090 z^6 - 30\,030 z^4 + 3465 z^2 - 63\right)$$

07.07.03.0018.01

$$P_n(z) = \frac{1}{2^n} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} z^{n-2k} /; n \in \mathbb{N}$$

07.07.03.0019.01

$$P_{-n}(z) = \frac{1}{2^{n-1}} \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^k \binom{n-1}{k} \binom{2n-2k-2}{n-1} z^{n-2k-1} /; n \in \mathbb{N}^+$$

### Values at infinities

07.07.03.0020.01

$$P_n(\infty) = \infty /; n \in \mathbb{N}^+$$

07.07.03.0021.01

$$P_n(-\infty) = (-1)^n \infty /; n \in \mathbb{N}^+$$

## General characteristics

### Domain and analyticity

$P_\nu(z)$ is an analytical function of $\nu$ and $z$ which is defined over $\mathbb{C}^2$. For integer $\nu$, $P_\nu(z)$ degenerates to a polynomial in $z$.

07.07.04.0001.01

$$(\nu * z) \longrightarrow P_\nu(z) :: (\mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

### Symmetries and periodicities

#### Parity

07.07.04.0002.01

$$P_n(-z) = (-1)^n P_n(z) /; n \in \mathbb{N}$$

07.07.04.0003.01

$$P_{-\nu}(z) = P_{\nu-1}(z)$$

#### Mirror symmetry

07.07.04.0004.01

$$P_{\bar{\nu}}(\bar{z}) = \overline{P_\nu(z)} /; z \notin (-\infty, -1)$$

#### Periodicity

No periodicity

### Poles and essential singularities

#### With respect to $z$

For fixed $\nu$ /; $\nu \notin \mathbb{Z}$, the function $P_\nu(z)$ does not have poles and essential singularities.

07.07.04.0005.01

$$\mathcal{S}ing_z(P_\nu(z)) = \{\} /; \nu \notin \mathbb{Z}$$

For integer $\nu$, the function $P_\nu(z)$ is polynomial and has pole of order $\nu$ at $z = \tilde{\infty}$ (for $\nu \in \mathbb{N}^+$) or order $-\nu - 1$ at $z = \tilde{\infty}$ (for $-\nu \in \mathbb{N}^+$).

07.07.04.0006.01

$\mathcal{S}ing_z(P_\nu(z)) = \{\{\tilde{\infty}, \nu\}\} /; \nu \in \mathbb{N}^+$

07.07.04.0007.01

$\mathcal{S}ing_z(P_\nu(z)) = \{\{\tilde{\infty}, -\nu - 1\}\} /; -\nu \in \mathbb{N}^+$

### With respect to $\nu$

For fixed $z$, the function $P_\nu(z)$ has only one singular point at $\nu = \tilde{\infty}$. It is an essential singular point. .

07.07.04.0008.01

$\mathcal{S}ing_\nu(P_\nu(z)) = \{\{\tilde{\infty}, \infty\}\}$

## Branch points

### With respect to $z$

For fixed noninteger $\nu$, the function $P_\nu(z)$ has two branch points: $z = -1$, $z = \tilde{\infty}$.

For fixed integer $\nu$, the function $P_\nu(z)$ does not have branch points.

07.07.04.0009.01

$\mathcal{BP}_z(P_\nu(z)) = \{-1, \tilde{\infty}\} /; \nu \notin \mathbb{Z}$

07.07.04.0010.01

$\mathcal{BP}_z(P_\nu(z)) = \{\} /; \nu \in \mathbb{Z}$

07.07.04.0011.01

$\mathcal{R}_z(P_\nu(z), -1) = \log /; \nu \notin \mathbb{Z}$

07.07.04.0012.01

$\mathcal{R}_z(P_\nu(z), \tilde{\infty}) = \log /; \nu \notin \mathbb{Z}$

### With respect to $\nu$

For fixed $z$, the function $P_\nu(z)$ does not have branch points.

07.07.04.0013.01

$\mathcal{BP}_\nu(P_\nu(z)) = \{\}$

## Branch cuts

### With respect to $z$

For fixed noninteger $\nu$, the function $P_\nu(z)$ is a single-valued function on the $z$-plane cut along the interval $(-\infty, -1)$ where it is continuous from above.

For fixed integers $\nu$, the function $P_\nu(z)$ is a polynomial and does not have branch cuts.

07.07.04.0014.01

$\mathcal{BC}_z(P_\nu(z)) = \{\{(-\infty, -1), -i\}\} /; \nu \notin \mathbb{Z}$

07.07.04.0015.01

$\mathcal{BC}_z(P_\nu(z)) = \{\} /; \nu \in \mathbb{Z}$

07.07.04.0016.01

$$\lim_{\epsilon\to+0} P_\nu(x+i\,\epsilon) = P_\nu(x) \;/;\; x<-1$$

07.07.04.0017.01

$$\lim_{\epsilon\to+0} P_\nu(x-i\,\epsilon) = P_\nu(x) - 2\,i\sin(\pi\,\nu)\,P_\nu(-x) \;/;\; x<-1$$

### With respect to $\nu$

For fixed $z$, the function $P_\nu(z)$ does not have branch cuts.

07.07.04.0018.01

$$\mathcal{BC}_\nu(P_\nu(z)) = \{\}$$

## Series representations

### Generalized power series

#### Expansions at generic point $z = z_0$

### For the function itself

07.07.06.0029.01

$$P_\nu(z) \propto \frac{1}{2}\tan(\pi\,\nu) \left(\frac{2^{-\nu}\,\Gamma(-\nu)}{\Gamma(\nu+1)}\,{}_2\tilde{F}_1\!\left(-\nu,-\nu;-2\,\nu;\frac{2}{z_0+1}\right)\left(2\,i\left\lfloor\frac{\arg(z-z_0)}{2\,\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\,\pi}\right\rfloor(-z_0-1)^\nu+\csc(\pi\,\nu)\,(z_0+1)^\nu\right)+-\frac{2^{\nu+1}\,\Gamma(\nu+1)}{\Gamma(-\nu)}\left(2\,i\left\lfloor\frac{\arg(z-z_0)}{2\,\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\,\pi}\right\rfloor(-z_0-1)^{-\nu-1}+\csc(\pi\,\nu)\,(z_0+1)^{-\nu-1}\right){}_2\tilde{F}_1\!\left(\nu+1,\nu+1;2\,\nu+2;\frac{2}{z_0+1}\right)-\frac{1}{z_0+1}\left(\frac{2^{-\nu}\,\Gamma(1-\nu)}{\Gamma(\nu+1)}\left(2\,i\left\lfloor\frac{\arg(z-z_0)}{2\,\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\,\pi}\right\rfloor(-z_0-1)^\nu+\csc(\pi\,\nu)\,(z_0+1)^\nu\right){}_2\tilde{F}_1\!\left(1-\nu,-\nu;-2\,\nu;\frac{2}{z_0+1}\right)-\frac{2^{\nu+1}\,\Gamma(\nu+2)}{\Gamma(-\nu)}\left(2\,i\left\lfloor\frac{\arg(z-z_0)}{2\,\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\,\pi}\right\rfloor(-z_0-1)^{-\nu-1}+\csc(\pi\,\nu)\,(z_0+1)^{-\nu-1}\right){}_2\tilde{F}_1\!\left(\nu+1,\nu+2;2\,\nu+2;\frac{2}{z_0+1}\right)\right)(z-z_0)+\ldots\right) \;/;\; (z\to z_0)$$

07.07.06.0030.01

$$P_\nu(z) = \frac{\tan(\pi\,\nu)}{2}\sum_{k=0}^{\infty}\frac{(-z_0-1)^{-k}}{k!}\left(\frac{2^{-\nu}\,\Gamma(k-\nu)}{\Gamma(\nu+1)}\left(2\,i\left\lfloor\frac{\arg(z-z_0)}{2\,\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\,\pi}\right\rfloor(-z_0-1)^\nu+\csc(\pi\,\nu)\,(z_0+1)^\nu\right){}_2\tilde{F}_1\!\left(k-\nu,-\nu;-2\,\nu;\frac{2}{z_0+1}\right)-\frac{2^{\nu+1}\,\Gamma(k+\nu+1)}{\Gamma(-\nu)}\left(2\,i\left\lfloor\frac{\arg(z-z_0)}{2\,\pi}\right\rfloor\left\lfloor\frac{\arg(z_0+1)+\pi}{2\,\pi}\right\rfloor(-z_0-1)^{-\nu-1}+\csc(\pi\,\nu)\,(z_0+1)^{-\nu-1}\right){}_2\tilde{F}_1\!\left(\nu+1,k+\nu+1;2\,\nu+2;\frac{2}{z_0+1}\right)\right)(z-z_0)^k$$

07.07.06.0031.01

$$P_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k P_\nu^k(z_0)}{\left(1 - z_0^2\right)^{k/2} k!} (z - z_0)^k$$

07.07.06.0032.01

$$P_\nu(z) = \sum_{k=0}^{\infty} \frac{2^k \left(\frac{1}{2}\right)_k C_{\nu-k}^{k+\frac{1}{2}}(z_0)}{k!} (z - z_0)^k$$

07.07.06.0033.01

$$P_\nu(z) = \sum_{k=0}^{\infty} \frac{2^{-k}\,\Gamma(\nu + k + 1)}{(k!)^2\,\Gamma(\nu - k + 1)}\, {}_2F_1\!\left(k - \nu,\, k + \nu + 1;\, k + 1;\, \frac{1 - z_0}{2}\right) (z - z_0)^k$$

07.07.06.0034.01

$$P_\nu(z) \propto \frac{1}{2} \tan(\pi\,\nu) \left( \frac{2^{-\nu}\,\Gamma(-\nu)}{\Gamma(\nu + 1)}\, {}_2\tilde{F}_1\!\left(-\nu,\, -\nu;\, -2\,\nu;\, \frac{2}{z_0 + 1}\right) \left(2\,i \left\lfloor \frac{\arg(z - z_0)}{2\,\pi} \right\rfloor \left\lfloor \frac{\arg(z_0 + 1) + \pi}{2\,\pi} \right\rfloor (-z_0 - 1)^\nu + \csc(\pi\,\nu)\,(z_0 + 1)^\nu \right) + \right.$$
$$- \frac{2^{\nu+1}\,\Gamma(\nu + 1)}{\Gamma(-\nu)} \left(2\,i \left\lfloor \frac{\arg(z - z_0)}{2\,\pi} \right\rfloor \left\lfloor \frac{\arg(z_0 + 1) + \pi}{2\,\pi} \right\rfloor (-z_0 - 1)^{-\nu-1} + \csc(\pi\,\nu)\,(z_0 + 1)^{-\nu-1} \right)$$
$$\left. {}_2\tilde{F}_1\!\left(\nu + 1,\, \nu + 1;\, 2\,\nu + 2;\, \frac{2}{z_0 + 1}\right) \right) + O(z - z_0)$$

### Expansions on branch cuts

## For the function itself

07.07.06.0035.01

$$P_\nu(z) \propto \frac{1}{2} \tan(\pi\,\nu) \left( \frac{2^{-\nu}\,\Gamma(-\nu)}{\Gamma(\nu + 1)} \left(2\,i \left\lfloor \frac{\arg(z - x)}{2\,\pi} \right\rfloor (-x - 1)^\nu + (x + 1)^\nu \csc(\pi\,\nu)\right) {}_2\tilde{F}_1\!\left(-\nu,\, -\nu;\, -2\,\nu;\, \frac{2}{x + 1}\right) - \right.$$
$$\frac{2^{\nu+1}\,\Gamma(\nu + 1)}{\Gamma(-\nu)} \left(2\,i \left\lfloor \frac{\arg(z - x)}{2\,\pi} \right\rfloor (-x - 1)^{-\nu-1} + (x + 1)^{-\nu-1} \csc(\pi\,\nu)\right) {}_2\tilde{F}_1\!\left(\nu + 1,\, \nu + 1;\, 2\,\nu + 2;\, \frac{2}{x + 1}\right) -$$
$$\frac{1}{x + 1} \left( \frac{2^{-\nu}\,\Gamma(1 - \nu)}{\Gamma(\nu + 1)} \left(2\,i \left\lfloor \frac{\arg(z - x)}{2\,\pi} \right\rfloor (-x - 1)^\nu + (x + 1)^\nu \csc(\pi\,\nu)\right) {}_2\tilde{F}_1\!\left(1 - \nu,\, -\nu;\, -2\,\nu;\, \frac{2}{x + 1}\right) - \right.$$
$$\frac{2^{\nu+1}\,\Gamma(\nu + 2)}{\Gamma(-\nu)} \left(2\,i \left\lfloor \frac{\arg(z - x)}{2\,\pi} \right\rfloor (-x - 1)^{-\nu-1} + (x + 1)^{-\nu-1} \csc(\pi\,\nu)\right)$$
$$\left.\left. {}_2\tilde{F}_1\!\left(\nu + 1,\, \nu + 2;\, 2\,\nu + 2;\, \frac{2}{x + 1}\right)\right)(z - x) + \ldots \right) /;\, (z \to x) \land x \in \mathbb{R} \land x < -1$$

07.07.06.0036.01

$$P_\nu(z) = \frac{\tan(\pi\,\nu)}{2} \sum_{k=0}^{\infty} \frac{(-x - 1)^{-k}}{k!} \left( \frac{2^{-\nu}\,\Gamma(k - \nu)}{\Gamma(\nu + 1)} \left(2\,i \left\lfloor \frac{\arg(z - x)}{2\,\pi} \right\rfloor (-x - 1)^\nu + (x + 1)^\nu \csc(\pi\,\nu)\right) {}_2\tilde{F}_1\!\left(k - \nu,\, -\nu;\, -2\,\nu;\, \frac{2}{x + 1}\right) - \right.$$
$$\frac{2^{\nu+1}\,\Gamma(k + \nu + 1)}{\Gamma(-\nu)} \left(2\,i \left\lfloor \frac{\arg(z - x)}{2\,\pi} \right\rfloor (-x - 1)^{-\nu-1} + (x + 1)^{-\nu-1} \csc(\pi\,\nu)\right)$$
$$\left. {}_2\tilde{F}_1\!\left(\nu + 1,\, k + \nu + 1;\, 2\,\nu + 2;\, \frac{2}{x + 1}\right)\right) (z - x)^k \,/;\, x \in \mathbb{R} \land x < -1$$

07.07.06.0037.01

$$P_\nu(z) \propto \frac{1}{2}\tan(\pi\nu)\left(\frac{2^{-\nu}\,\Gamma(-\nu)}{\Gamma(\nu+1)}\left(2\,i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor(-x-1)^{\nu}+(x+1)^{\nu}\csc(\pi\nu)\right){}_2\tilde{F}_1\left(-\nu,-\nu;-2\nu;\frac{2}{x+1}\right)-\frac{2^{\nu+1}\,\Gamma(\nu+1)}{\Gamma(-\nu)}\left(2\,i\left\lfloor\frac{\arg(z-x)}{2\pi}\right\rfloor(-x-1)^{-\nu-1}+(x+1)^{-\nu-1}\csc(\pi\nu)\right){}_2\tilde{F}_1\left(\nu+1,\nu+1;2\nu+2;\frac{2}{x+1}\right)\right)+O(z-x)\;/;\;x\in\mathbb{R}\land x<-1$$

## Expansions at $z == 0$

07.07.06.0001.01

$$P_\nu(z) \propto -\frac{\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)\sin(\pi\nu)}{2\,\pi^{3/2}}\left(1-\frac{2\,\Gamma\left(\frac{1-\nu}{2}\right)\Gamma\left(\frac{\nu}{2}+1\right)z}{\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}-\frac{\nu(\nu+1)}{2}z^2+\ldots\right)/;\;|z|<1$$

07.07.06.0002.01

$$P_\nu(z) == \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\nu)_{j+k}\,(\nu+1)_{j+k}\,(-z)^j}{(j+k)!\,j!\,k!\,2^{j+k}}\;/;\;|z|<1$$

07.07.06.0003.01

$$P_\nu(z) == F_{1\times0\times0}^{2\times0\times0}\left(\begin{matrix}-\nu,\,1+\nu;;;\\ 1;;;\end{matrix}\;\frac{1}{2},-\frac{z}{2}\right)/;\;|z|<1$$

07.07.06.0004.01

$$P_\nu(z) \propto \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu}{2}+1\right)}(1+O(z))\;/;\;(z\to0)$$

07.07.06.0005.01

$$P_n(z) == \frac{1}{2^n}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\binom{n}{k}\binom{2n-2k}{n}z^{n-2k}\;/;\;n\in\mathbb{N}$$

07.07.06.0006.01

$$P_n(z) \propto \frac{(-1)^{\left\lfloor\frac{n}{2}\right\rfloor}}{2^n}\binom{n}{\left\lfloor\frac{n}{2}\right\rfloor}(n+1)^{n-2\left\lfloor\frac{n}{2}\right\rfloor}z^{n-2\left\lfloor\frac{n}{2}\right\rfloor}\left(1+O\left(z^2\right)\right)/;\;(z\to0)\land n\in\mathbb{N}$$

## Expansions at $z == 1$

07.07.06.0007.01

$$P_\nu(z) \propto 1-\frac{-\nu(1+\nu)}{2}(z-1)+\frac{(-\nu)(1-\nu)(1+\nu)(2+\nu)}{16}(z-1)^2-\ldots\;/;\;\left|\frac{1-z}{2}\right|<1$$

07.07.06.0008.01

$$P_\nu(z) == \sum_{k=0}^{\infty}\frac{(-\nu)_k\,(\nu+1)_k}{k!^2}\left(\frac{1-z}{2}\right)^k/;\;\left|\frac{1-z}{2}\right|<1$$

07.07.06.0009.01

$$P_\nu(z) \propto 1+O(z-1)\;/;\;(z\to1)$$

07.07.06.0010.01

$$P_n(z) == \sum_{k=0}^{n}\frac{(-n)_k\,(n+1)_k}{k!^2}\left(\frac{1-z}{2}\right)^k/;\;n\in\mathbb{N}$$

## Expansions at $z == -1$

07.07.06.0011.01

$$P_\nu(z) \propto \frac{\sin(\pi\nu)}{\pi}\log\left(\frac{z+1}{2}\right)\left(1-\frac{\nu(\nu+1)}{2}(z+1)-\frac{(1-\nu)\nu(\nu+1)(\nu+2)}{16}(z+1)^2+\ldots\right)- \frac{\sin(\pi\nu)}{\pi}\left(-\psi(-\nu)-\psi(\nu+1)-2\gamma-\frac{\nu(\nu+1)}{2}(2(1-\gamma)-\psi(1-\nu)-\psi(\nu+2))(z+1)- \frac{(1-\nu)\nu(\nu+1)(\nu+2)}{16}\left(2\left(\frac{3}{2}-\gamma\right)-\psi(2-\nu)-\psi(\nu+3)\right)(z+1)^2+\ldots\right) /; \left|\frac{z+1}{2}\right|<1 \bigwedge \nu\notin\mathbb{Z}$$

07.07.06.0012.01

$$P_\nu(z) == \frac{\sin(\pi\nu)}{\pi}\log\left(\frac{z+1}{2}\right)\sum_{k=0}^{\infty}\frac{(-\nu)_k(\nu+1)_k}{k!^2}\left(\frac{z+1}{2}\right)^k - \frac{\sin(\pi\nu)}{\pi}\sum_{k=0}^{\infty}\frac{(-\nu)_k(\nu+1)_k(2\psi(k+1)-\psi(k+\nu+1)-\psi(k-\nu))}{k!^2}\left(\frac{z+1}{2}\right)^k /; \left|\frac{z+1}{2}\right|<1 \bigwedge \nu\notin\mathbb{Z}$$

07.07.06.0013.01

$$P_\nu(z) \propto \frac{\sin(\pi\nu)}{\pi}\log\left(\frac{z+1}{2}\right)(1+O(z+1))+\frac{\sin(\pi\nu)}{\pi}(-\pi\cot(\pi\nu)+2\psi(-\nu)+2\gamma)(1+O(z+1)) /; (z\to-1)\bigwedge \nu\notin\mathbb{Z}$$

07.07.06.0014.01

$$P_n(z) == (-1)^n\sum_{k=0}^{n}\frac{(-n)_k(n+1)_k}{k!^2}\left(\frac{z+1}{2}\right)^k /; n\in\mathbb{N}$$

07.07.06.0015.01

$$P_n(z) \propto (-1)^n(1+O(z+1)) /; n\in\mathbb{N}$$

## Expansions at $z == \infty$

07.07.06.0016.01

$$P_\nu(z) \propto \frac{2^{-\nu-1}\Gamma\left(-\nu-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(-\nu)}(z-1)^{-\nu-1}\left(1-\frac{\nu+1}{z-1}+\frac{(1+\nu)(2+\nu)^2}{(3+2\nu)(z-1)^2}-\ldots\right)+ \frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu+1)}(z-1)^\nu\left(1+\frac{\nu}{z-1}-\frac{(1-\nu)^2\nu}{(1-2\nu)(z-1)^2}+\ldots\right) /; \left|\frac{1-z}{2}\right|>1\bigwedge 2\nu\notin\mathbb{Z}$$

07.07.06.0017.01

$$P_\nu(z) == \frac{2^{-\nu-1}\Gamma\left(-\nu-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(-\nu)}(z-1)^{-\nu-1}\sum_{k=0}^{\infty}\frac{(\nu+1)_k^2}{k!\,(2\nu+2)_k}\left(\frac{2}{1-z}\right)^k+\frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu+1)}(z-1)^\nu\sum_{k=0}^{\infty}\frac{(-\nu)_k^2}{k!\,(-2\nu)_k}\left(\frac{2}{1-z}\right)^k /; \left|\frac{1-z}{2}\right|>1\bigwedge 2\nu\notin\mathbb{Z}$$

07.07.06.0018.01

$$P_\nu(z) == \frac{2^{-\nu-1}\Gamma\left(-\nu-\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(-\nu)}(z-1)^{-\nu-1}\,{}_2F_1\left(\nu+1,\nu+1;2\nu+2;\frac{2}{1-z}\right)+\frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu+1)}(z-1)^\nu\,{}_2F_1\left(-\nu,-\nu;-2\nu;\frac{2}{1-z}\right) /; z\notin(-1,1)\bigwedge 2\nu\notin\mathbb{Z}$$

07.07.06.0019.01

$$P_\nu(z) \propto \frac{2^{-\nu-1}\Gamma\left(-\nu-\frac{1}{2}\right)z^{-\nu-1}}{\sqrt{\pi}\,\Gamma(-\nu)}\left(1+O\left(\frac{1}{z}\right)\right)+\frac{2^\nu\Gamma\left(\nu+\frac{1}{2}\right)z^\nu}{\sqrt{\pi}\,\Gamma(\nu+1)}\left(1+O\left(\frac{1}{z}\right)\right) /; (|z|\to\infty)\bigwedge 2\nu\notin\mathbb{Z}$$

07.07.06.0020.01

$$P_\nu(z) = -\frac{2^{-\nu}\sin(\pi\nu)(z-1)^{-\nu-1}}{\sqrt{\pi}\,\Gamma(-\nu)\,\Gamma\left(\nu+\frac{3}{2}\right)}\log\left(\frac{z-1}{2}\right)\sum_{k=0}^{\infty}\frac{(\nu+1)_k^{\,2}}{k!\,(2\nu+2)_k}\left(\frac{2}{1-z}\right)^k +$$
$$\frac{2^{-\nu}\,\Gamma(\nu+1)}{\pi^{3/2}\,\Gamma\left(\nu+\frac{3}{2}\right)}(z-1)^{-\nu-1}\sum_{k=0}^{\infty}\frac{(\nu+1)_k^{\,2}\,(\psi(k+1)-\psi(k+\nu+1)+\psi(k+2\nu+2)-\psi(-k-\nu))}{k!\,(2\nu+2)_k}\left(\frac{2}{1-z}\right)^k +$$
$$\frac{2^{-\nu}(z-1)^{\nu}}{\Gamma(\nu+1)}\sum_{k=0}^{2\nu}\frac{(2\nu-k)!\,(-\nu)_k}{k!\,\Gamma(\nu-k+1)}\left(\frac{2}{1-z}\right)^k /;\ \left|\frac{1-z}{2}\right|>1\bigwedge\nu+\frac{1}{2}\in\mathbb{N}$$

07.07.06.0021.01

$$P_{-\frac{1}{2}}(z)\propto\frac{\sqrt{2}}{\pi\sqrt{z}}\left(\log\left(\frac{z-1}{2}\right)-2\left(\psi\left(\frac{1}{2}\right)+\gamma\right)\right)\left(1+O\left(\frac{1}{z}\right)\right) /;\ (|z|\to\infty)$$

07.07.06.0022.01

$$P_\nu(z)\propto\frac{2^{-\nu}(2\nu)!\,z^{\nu}}{\Gamma(\nu+1)^2}\left(1+O\left(\frac{1}{z}\right)\right)+\frac{2^{-\nu}z^{-\nu-1}}{\sqrt{\pi}\,\Gamma(-\nu)\,\Gamma\left(\nu+\frac{3}{2}\right)}$$
$$(\csc(\pi\nu)(\psi(-\nu)+\psi(\nu+1)-\psi(2\nu+2)+\gamma)+(\log(2)-\log(z-1))\sin(\pi\nu))\left(1+O\left(\frac{1}{z}\right)\right) /;\ (|z|\to\infty)\bigwedge\nu-\frac{1}{2}\in\mathbb{N}$$

07.07.06.0023.01

$$P_n(z) = \frac{2^n\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,n!}(z-1)^n\sum_{k=0}^{n}\frac{(-n)_k^{\,2}}{k!\,(-2n)_k}\left(\frac{2}{1-z}\right)^k /;\ n\in\mathbb{N}$$

07.07.06.0024.01

$$P_n(z)\propto\frac{2^n\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,n!}z^n\left(1+O\left(\frac{1}{z}\right)\right) /;\ (|z|\to\infty)\land n\in\mathbb{N}$$

### Other series representations

07.07.06.0025.01

$$P_n(z) = \sum_{k=0}^{n}\frac{(-1)^k\,(k+n)!}{(n-k)!\,k!^2\,2^{k+1}}\left((1-z)^k+(-1)^n(z+1)^k\right) /;\ n\in\mathbb{N}$$

07.07.06.0026.01

$$P_n(z) = \left(\frac{z-1}{2}\right)^n\sum_{k=0}^{n}\binom{n}{k}^2\left(\frac{z+1}{z-1}\right)^k /;\ n\in\mathbb{N}$$

07.07.06.0027.01

$$P_n(\cos(\theta)) = (-1)^n\sum_{k=0}^{n}\binom{-\frac{1}{2}}{k}\binom{-\frac{1}{2}}{n-k}\cos((n-2k)\theta) /;\ n\in\mathbb{N}$$

07.07.06.0028.01

$$P_n\left(x\,y-\sqrt{1-x^2}\,\sqrt{1-y^2}\,\cos(\alpha)\right) = P_n(x)\,P_n(y)+2\sum_{k=1}^{n}\frac{(-1)^k\cos(k\,\alpha)\,(n-k)!}{(k+n)!}P_n^k(x)\,P_n^k(y) /;\ n\in\mathbb{N}$$

## Integral representations

### On the real axis

#### Of the direct function

07.07.07.0001.01

$$P_n(z) == \frac{1}{\pi}\int_0^{\pi}\left(z-\sqrt{z^2-1}\,\cos(t)\right)^n \,dt \,/;\, n\in\mathbb{N}$$

07.07.07.0002.01

$$P_n(z) == \frac{1}{\pi}\int_0^{\pi}\left(z+i\sqrt{1-z^2}\,\cos(t)\right)^n \,dt \,/;\, n\in\mathbb{N}$$

07.07.07.0003.01

$$P_n(z) == \frac{2^n}{\pi}\int_{-\infty}^{\infty}\frac{(i\,t+z)^n}{\left(t^2+1\right)^{n+1}} \,dt \,/;\, n\in\mathbb{N}$$

07.07.07.0004.01

$$P_\nu(z) == \frac{1}{\pi}\int_0^{\pi}\left(z+\sqrt{z^2-1}\,\cos(t)\right)^{-\nu-1} \,dt \,/;\, \operatorname{Re}(z)>0$$

### Integral representations of negative integer order

Rodrigues-type formula.

07.07.07.0005.01

$$P_n(z) == \frac{(-1)^n}{2^n\,n!}\,\frac{\partial^n\left(1-z^2\right)^n}{\partial z^n} \,/;\, n\in\mathbb{N}$$

07.07.07.0006.01

$$P_n^m(z) == \frac{(-1)^{m+n}\left(1-z^2\right)^{m/2}}{2^n\,n!}\,\frac{\partial^{m+n}\left(1-z^2\right)^n}{\partial z^{m+n}} \,/;\, n\in\mathbb{N}\land m\in\mathbb{N}$$

## Generating functions

07.07.11.0001.01

$$P_n(z) == \left([t^n]\,\frac{1}{\sqrt{t^2-2\,z\,t+1}}\right) /;\, n\in\mathbb{N}\land -1<z<1$$

## Differential equations

### Ordinary linear differential equations and wronskians

#### For the direct function itself

07.07.13.0001.01

$$\left(1-z^2\right)w''(z)-2\,z\,w'(z)+(\nu+1)\,\nu\,w(z)==0 \,/;\, w(z)== c_1\,P_\nu(z)+c_2\,Q_\nu(z)$$

07.07.13.0002.02

$$W_z\left(P_\nu(z),\,Q_\nu(z)\right) == \frac{1}{1-z^2}$$

07.07.13.0003.01

$$g'(z)\,w''(z) - \left(\frac{2\,g(z)\,g'(z)^2}{1-g(z)^2} + g''(z)\right) w'(z) + \frac{\nu\,(\nu+1)\,g'(z)^3}{1-g(z)^2}\,w(z) = 0 \,/;\, w(z) = c_1\,P_\nu(g(z)) + c_2\,Q_\nu(g(z))$$

07.07.13.0004.01

$$W_z\big(P_\nu(g(z)),\,Q_\nu(g(z))\big) = \frac{g'(z)}{1-g(z)^2}$$

07.07.13.0005.01

$$g'(z)\,h(z)^2\,w''(z) - \left(\left(\frac{2\,g(z)\,g'(z)^2}{1-g(z)^2} + g''(z)\right)h(z)^2 + 2\,g'(z)\,h'(z)\,h(z)\right)w'(z) + \left(\frac{\nu\,(\nu+1)\,h(z)^2\,g'(z)^3}{1-g(z)^2} + 2\,h'(z)^2\,g'(z) + h(z)\left(h'(z)\left(\frac{2\,g(z)\,g'(z)^2}{1-g(z)^2} + g''(z)\right) - g'(z)\,h''(z)\right)\right)w(z) = 0 \,/;\, w(z) = c_1\,h(z)\,P_\nu(g(z)) + c_2\,h(z)\,Q_\nu(g(z))$$

07.07.13.0006.01

$$W_z\big(h(z)\,P_\nu(g(z)),\,h(z)\,Q_\nu(g(z))\big) = \frac{h(z)^2\,g'(z)}{1-g(z)^2}$$

07.07.13.0007.01

$$z^2\,w''(z) - z\left(2\,s + \frac{r\left(a^2\,z^{2r}+1\right)}{1-a^2\,z^{2r}} - 1\right)w'(z) + \left(-\frac{a^2\,\nu\,(\nu+1)\,r^2\left(a^2\,z^{2r}-1\right)z^{2r}}{\left(1-a^2\,z^{2r}\right)^2} + s^2 + \frac{r\,s\left(a^2\,z^{2r}+1\right)}{1-a^2\,z^{2r}}\right)w(z) = 0 \,/;\, w(z) = c_1\,z^s\,P_\nu(a\,z^r) + c_2\,z^s\,Q_\nu(a\,z^r)$$

07.07.13.0008.01

$$W_z\big(z^s\,P_\nu(a\,z^r),\,z^s\,Q_\nu(a\,z^r)\big) = \frac{a\,r\,z^{r+2s-1}}{1-a^2\,z^{2r}}$$

07.07.13.0009.01

$$w''(z) - \frac{a^2\,(\log(r) - 2\log(s))\,r^{2z} + \log(r) + 2\log(s)}{1-a^2\,r^{2z}}\,w'(z) + \left(\frac{a^2\,\nu\,(\nu+1)\log^2(r)\,r^{2z}}{1-a^2\,r^{2z}} + \log^2(s) + \frac{\left(a^2\,r^{2z}+1\right)\log(r)\log(s)}{1-a^2\,r^{2z}}\right)w(z) = 0 \,/;\, w(z) = c_1\,s^z\,P_\nu(a\,r^z) + c_2\,s^z\,Q_\nu(a\,r^z)$$

07.07.13.0010.01

$$W_z\big(s^z\,P_\nu(a\,r^z),\,s^z\,Q_\nu(a\,r^z)\big) = \frac{a\,r^z\,s^{2z}\log(r)}{1-a^2\,r^{2z}}$$

# Transformations

## Transformations and argument simplifications

### Argument involving basic arithmetic operations

07.07.16.0001.01

$$P_{-\nu-1}(z) = P_\nu(z)$$

07.07.16.0002.01

$$P_n(-z) = (-1)^n\,P_n(z) \,/;\, n \in \mathbb{N}$$

## Products, sums, and powers of the direct function

### Products of the direct function

07.07.16.0003.01

$$P_n(z)\,P_m(z) = \sum_{k=|m-n|}^{m+n} b(n,m,k)\,P_k(z) \;/;$$

$$b(n,m,k) = \left(\delta_{0,\left(\frac{1}{2}(k+m+n)\right) \bmod 1}\,(2k+1)\,(k+m-n-1)!!\,(k-m+n-1)!!\,(m+n-k-1)!!\,(k+m+n)!!\right)\Big/ \left((k+m-n)!!\,(k-m+n)!!\,(m+n-k)!!\,(k+m+n+1)!!\right) \bigwedge n \in \mathbb{N} \bigwedge m \in \mathbb{N}$$

07.07.16.0004.01

$$P_\nu(x)\,P_\nu(y) = \frac{\sin(\nu\pi)}{\pi} \sum_{k=0}^{\infty} (-1)^k \left(\frac{1}{\nu-k} - \frac{1}{k+\nu+1}\right) P_k(x)\,P_k(y) \;/;$$

$$x \in \mathbb{R} \wedge -1 < x \leq 1 \wedge y \in \mathbb{R} \wedge -1 < y \leq 1 \wedge x+y > 0 \wedge \nu \notin \mathbb{Z}$$

# Identities

## Recurrence identities

### Consecutive neighbors

07.07.17.0001.01

$$P_\nu(z) = \frac{(2\nu+3)\,z}{\nu+1} P_{\nu+1}(z) - \frac{\nu+2}{\nu+1} P_{\nu+2}(z)$$

07.07.17.0002.01

$$P_\nu(z) = \frac{(2\nu-1)\,z}{\nu} P_{\nu-1}(z) - \frac{\nu-1}{\nu} P_{\nu-2}(z)$$

### Distant neighbors

07.07.17.0006.01

$$P_\nu(z) = C_n(\nu,z)\,P_{\nu+n}(z) - \frac{n+\nu+1}{n+\nu} C_{n-1}(\nu,z)\,P_{\nu+n+1}(z) \;/;$$

$$C_0(\nu,z) = 1 \bigwedge C_1(\nu,z) = \frac{(2\nu+3)\,z}{\nu+1} \bigwedge C_n(\nu,z) = \frac{z\,(2n+2\nu+1)}{n+\nu} C_{n-1}(\nu,z) - \frac{n+\nu}{n+\nu-1} C_{n-2}(\nu,z) \bigwedge n \in \mathbb{N}^+$$

07.07.17.0007.01

$$P_\nu(z) = C_n(\nu,z)\,P_{\nu-n}(z) + \frac{n-\nu}{\nu-n+1} C_{n-1}(\nu,z)\,P_{\nu-n-1}(z) \;/;$$

$$C_0(\nu,z) = 1 \bigwedge C_1(\nu,z) = \frac{(2\nu-1)\,z}{\nu} \bigwedge C_n(\nu,z) = \frac{z\,(2n-2\nu-1)}{n-\nu-1} C_{n-1}(\nu,z) - \frac{n-\nu-1}{n-\nu-2} C_{n-2}(\nu,z) \bigwedge n \in \mathbb{N}^+$$

## Functional identities

### Relations between contiguous functions

## Recurrence relations

07.07.17.0003.01

$$\nu P_{\nu-1}(z) + (\nu+1) P_{\nu+1}(z) = (2\nu+1) z P_\nu(z)$$

07.07.17.0004.01

$$P_\nu(z) = \frac{1}{(2\nu+1) z} (\nu P_{\nu-1}(z) + (\nu+1) P_{\nu+1}(z))$$

### Normalized recurrence relation

07.07.17.0005.01

$$z\, p(\nu, z) = \frac{\nu^2}{4\nu^2-1} p(\nu-1, z) + p(\nu+1, z) /; p(\nu, z) = \frac{2^{-\nu} \sqrt{\pi}\ \nu!}{\Gamma\left(\nu+\frac{1}{2}\right)} P_\nu(z)$$

## Complex characteristics

### Real part

07.07.19.0001.01

$$\operatorname{Re}(P_n(x+i y)) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^j 2^{2j}\ y^{2j}}{(2j)!} \left(\frac{1}{2}\right)_{2j} C_{n-2j}^{\left(2j+\frac{1}{2}\right)}(x) /; x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge n \in \mathbb{N}$$

### Imaginary part

07.07.19.0002.01

$$\operatorname{Im}(P_n(x+i y)) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^j 2^{2j+1}\ y^{2j+1}}{(2j+1)!} \left(\frac{1}{2}\right)_{2j+1} C_{-2j+n-1}^{\left(2j+\frac{3}{2}\right)}(x) /; x \in \mathbb{R} \bigwedge y \in \mathbb{R} \bigwedge n \in \mathbb{N}$$

## Differentiation

### Low-order differentiation

#### With respect to $\nu$

07.07.20.0001.01

$$\frac{\partial P_\nu(z)}{\partial \nu} = \pi \cot(\pi \nu) P_\nu(z) - \sum_{k=0}^{\infty} \frac{(-\nu)_k (\nu+1)_k}{k!^2} (\psi(k-\nu) - \psi(k+\nu+1)) \left(\frac{1-z}{2}\right)^k /; \left|\frac{1-z}{2}\right| < 1 \bigwedge \nu \notin \mathbb{Z}$$

07.07.20.0002.01

$$\frac{\partial P_\nu(z)}{\partial \nu} = \sum_{k=0}^{\infty} \frac{1}{k!^2} \left(\frac{1-z}{2}\right)^k \sum_{j=1}^{k} S_k^{(j)} \nu^j \sum_{r=1}^{k} (-1)^r S_k^{(r)} \left(\frac{j}{\nu} + \frac{r}{\nu+1}\right) (\nu+1)^r /; \left|\frac{1-z}{2}\right| < 1$$

07.07.20.0003.01

$$\frac{\partial P_\nu(z)}{\partial \nu} = -\frac{2\nu+1}{2} (1-z) F_{2\times 0\times 2}^{2\times 1\times 3}\left(\begin{matrix} 1-\nu, \nu+2; 1; 1, -\nu, \nu+1; \\ 2, 2;; \nu+2, 1-\nu; \end{matrix} \frac{1-z}{2}, \frac{1-z}{2}\right)$$

07.07.20.0004.01

$$\frac{\partial^2 P_\nu(z)}{\partial \nu^2} = \sum_{k=0}^{\infty} \frac{(-\nu)_k (\nu+1)_k}{k!^2} \left(\frac{1-z}{2}\right)^k \left(\psi(k-\nu)^2 - 2(\pi\cot(\pi\nu) + \psi(k+\nu+1))\psi(k-\nu) + \psi(k+\nu+1)^2 + 2\pi\cot(\pi\nu)\psi(k+\nu+1) + \psi^{(1)}(k-\nu) + \psi^{(1)}(k+\nu+1)\right) - \pi^2 P_\nu(z) /; \left|\frac{1-z}{2}\right| < 1$$

07.07.20.0005.01

$$\frac{\partial^2 P_\nu(z)}{\partial \nu^2} = \sum_{k=0}^{\infty} \frac{1}{k!^2} \left(\frac{1-z}{2}\right)^k \sum_{i=1}^{k} S_k^{(i)} \nu^{i-2} \sum_{r=1}^{k} (-1)^r (\nu+1)^{r-2} \left((r-1) r \nu^2 + i^2 (\nu+1)^2 + ((2r-1)\nu - 1) i (\nu+1)\right) S_k^{(r)} /; \left|\frac{1-z}{2}\right| < 1$$

### With respect to $z$

07.07.20.0006.01

$$\frac{\partial P_\nu(z)}{\partial z} = \frac{\nu}{z^2-1} (z P_\nu(z) - P_{\nu-1}(z))$$

07.07.20.0007.01

$$\frac{\partial^2 P_\nu(z)}{\partial z^2} = \frac{\nu}{(z^2-1)^2} \left(2 z P_{\nu-1}(z) + \left((\nu-1) z^2 - \nu - 1\right) P_\nu(z)\right)$$

## Symbolic differentiation

### With respect to $\nu$

07.07.20.0008.02

$$\frac{\partial^m P_\nu(z)}{\partial \nu^m} = \sum_{k=0}^{\infty} \frac{1}{k!^2} \left(\frac{1-z}{2}\right)^k \sum_{j=0}^{m} \binom{m}{j} \sum_{i=1}^{k} S_k^{(i)} (i-j+1)_j \nu^{i-j} \sum_{r=1}^{k} (-1)^r S_k^{(r)} (j-m+r+1)_{m-j} (\nu+1)^{j-m+r} /; \left|\frac{1-z}{2}\right| < 1 \bigwedge m \in \mathbb{N}$$

### With respect to $z$

07.07.20.0009.02

$$\frac{\partial^m P_\nu(z)}{\partial z^m} = 2^m \left(\frac{1}{2}\right)_m C_{\nu-m}^{m+\frac{1}{2}}(z) /; m \in \mathbb{N}$$

07.07.20.0010.02

$$\frac{\partial^m P_\nu(z)}{\partial z^m} = (z-1)^{-m} {}_2\tilde{F}_1\left(-\nu, \nu+1; 1-m; \frac{1-z}{2}\right) /; m \in \mathbb{N}$$

07.07.20.0011.02

$$\frac{\partial^m P_\nu(z)}{\partial z^m} = \frac{2^{-m} \Gamma(m+\nu+1)}{m! \Gamma(\nu-m+1)} {}_2F_1\left(m-\nu, m+\nu+1; m+1; \frac{1-z}{2}\right) /; m \in \mathbb{N}$$

07.07.20.0013.01

$$\frac{\partial^m P_\nu(z)}{\partial z^m} = (-1)^m \left(1-z^2\right)^{-\frac{m}{2}} P_\nu^m(z) /; m \in \mathbb{N}$$

07.07.20.0014.01

$$\frac{\partial^m P_\nu(z)}{\partial z^m} = \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} \left(1-z^2\right)^{-\frac{m}{2}} P_\nu^{-m}(z) /; m \in \mathbb{N}$$

07.07.20.0015.01

$$\frac{\partial^m P_n(z)}{\partial z^m} == (2\,m-1)\,!!\sum_{i_1=0}^{n-m}\ldots\sum_{i_{2\,m+1}=0}^{n-m}\delta_{\sum_{j=1}^{2\,m+1} i_j,\,n-m}\prod_{j=1}^{2\,m+1} P_{i_j}(z)\;/;\;m\in\mathbb{N}\land n\in\mathbb{N}$$

### Fractional integro-differentiation

#### With respect to z

07.07.20.0012.01

$$\frac{\partial^\alpha P_\nu(z)}{\partial z^\alpha} == z^{-\alpha}\,F_{1\times 0\times 1}^{2\times 0\times 1}\left(\begin{matrix}-\nu,\ \nu+1;;\ 1; \\ 1;;\ 1-\alpha;\end{matrix}\ \frac{1}{2},\ -\frac{z}{2}\right)$$

## Integration

### Indefinite integration

#### Involving only one direct function

07.07.21.0001.01

$$\int P_\nu(z)\,dz == \frac{P_{\nu+1}(z)-P_{\nu-1}(z)}{2\,\nu+1}$$

#### Involving one direct function and elementary functions

### Involving power function

07.07.21.0002.01

$$\int z^{\alpha-1}\,P_\nu(z)\,dz == \frac{z^\alpha}{\alpha}\,F_{1\times 0\times 1}^{2\times 0\times 1}\left(\begin{matrix}-\nu,\ 1+\nu;;\ \alpha; \\ 1;;\ \alpha+1;\end{matrix}\ \frac{1}{2},\ -\frac{z}{2}\right)$$

### Involving algebraic functions

07.07.21.0003.01

$$\int \left(1-z^2\right)^{\frac{1}{2}(-\nu-3)}\,P_\nu(z)\,dz == \frac{\left(1-z^2\right)^{\frac{1}{2}(-\nu-1)}}{\nu+1}\,P_{\nu+1}(z)$$

07.07.21.0004.01

$$\int \left(1-z^2\right)^{\frac{\nu}{2}-1}\,P_\nu(z)\,dz == -\frac{\left(1-z^2\right)^{\nu/2}}{\nu}\,P_{\nu-1}(z)$$

### Involving logarithm

07.07.21.0005.01

$$\int \log\left(\frac{1+z}{1-z}\right)P_\nu(z)\,dz == \frac{2\,P_\nu(z)}{\nu^2+\nu}+\frac{1}{2\,\nu+1}\log\left(\frac{1+z}{1-z}\right)\left(P_{\nu+1}(z)-P_{\nu-1}(z)\right)$$

### Definite integration

#### Involving the direct function

07.07.21.0006.01

$$\int_1^\infty \frac{P_{i\tau-\frac{1}{2}}(t)}{t+z}\, dt = \pi \operatorname{sech}(\pi\tau)\, P_{i\tau-\frac{1}{2}}(z) /; \tau \in \mathbb{R} \land \tau < 1$$

Orthogonality:

07.07.21.0007.01

$$\int_{-1}^1 P_m(t)\, P_n(t)\, dt = \frac{2\,\delta_{n,m}}{2\,n+1} /; m \in \mathbb{N} \land n \in \mathbb{N}$$

07.07.21.0008.01

$$\int_0^\pi P_m(\cos(t))\, P_n(\cos(t))\, P_k(\cos(t)) \sin(t)\, dt = 2 \begin{pmatrix} m & n & k \\ 0 & 0 & 0 \end{pmatrix}^2 /; m \in \mathbb{N} \bigwedge n \in \mathbb{N} \bigwedge k \in \mathbb{N}$$

## Summation

### Finite summation

07.07.23.0001.01

$$\sum_{k=1}^n \cos\left(k \cos^{-1}(z)\right) P_{n-k}(z) = n\, P_n(z) /; n \in \mathbb{N}$$

### Infinite summation

07.07.23.0002.01

$$\sum_{n=0}^\infty P_n(z)\, w^n = \frac{1}{\sqrt{w^2 - 2\,z\,w + 1}} /; -1 < z < 1 \land |w| < 1$$

07.07.23.0003.01

$$\sum_{n=0}^\infty \frac{1}{n!^2} P_n(z)\, w^n = {}_0F_1\left(; 1; \frac{1}{2}(z-1)\,w\right) {}_0F_1\left(; 1; \frac{1}{2}(z+1)\,w\right) /; -1 < z < 1 \land |w| < 1$$

07.07.23.0004.01

$$\sum_{n=0}^\infty \frac{1}{n!} P_n(z)\, w^n = e^{w\,z}\, {}_0F_1\left(; 1; \frac{1}{4}\left(z^2-1\right) w^2\right) /; -1 < z < 1 \land |w| < 1$$

07.07.23.0005.01

$$\sum_{n=0}^\infty \frac{(\gamma)_n\,(1-\gamma)_n}{n!^2} P_n(z)\, w^n = {}_2F_1\left(\gamma, 1-\gamma; 1; \frac{1}{2}\left(1 - \sqrt{w^2 - 2\,z\,w + 1} - w\right)\right) {}_2F_1\left(\gamma, 1-\gamma; 1; \frac{1}{2}\left(1 - \sqrt{w^2 - 2\,z\,w + 1} + w\right)\right)$$

07.07.23.0006.01

$$\sum_{n=0}^\infty \frac{(\gamma)_n}{n!} P_n(z)\, w^n = (1 - w\,z)^{-\gamma}\, {}_2F_1\left(\frac{\gamma}{2}, \frac{\gamma+1}{2}; 1; \frac{\left(z^2-1\right) w^2}{(1-w\,z)^2}\right) /; -1 < z < 1 \land |w| < 1$$

07.07.23.0007.01

$$\sum_{k=0}^\infty (-1)^k \left(\frac{1}{\nu-k} - \frac{1}{k+\nu+1}\right) P_k(x) = \frac{\pi\, P_\nu(x)}{\sin(\nu\,\pi)} /; x \in \mathbb{R} \bigwedge -1 < x \leq 1 \bigwedge \nu \notin \mathbb{Z}$$

07.07.23.0008.01

$$\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\nu-k}-\frac{1}{k+\nu+1}\right)P_k(x)\,P_k(y)=\frac{\pi}{\sin(\nu\,\pi)}\,P_\nu(x)\,P_\nu(y)\;/;$$
$$x\in\mathbb{R}\land -1<x\leq 1\land y\in\mathbb{R}\land -1<y\leq 1\land x+y>0\land \nu\notin\mathbb{Z}$$

07.07.23.0009.01

$$\sum_{n=0}^{\infty}(2\,n+1)\,P_n(x)\,P_n(y)=2\,\delta(x-y)\;/;\;-1<x<1\land -1<y<1$$

## Operations

### Orthogonality, completeness, and Fourier expansions

The set of functions $P_n(x)$, $n = 0, 1, \ldots$, forms a complete, orthogonal (with weight $\frac{2n+1}{2}$) system on the interval $(-1, 1)$.

07.07.25.0001.01

$$\sum_{n=0}^{\infty}\left(\sqrt{\frac{2\,n+1}{2}}\;P_n(x)\right)\left(\sqrt{\frac{2\,n+1}{2}}\;P_n(y)\right)=\delta(x-y)\;/;\;-1<x<1\land -1<y<1$$

07.07.25.0002.01

$$\int_{-1}^{1}\left(\sqrt{\frac{2\,m+1}{2}}\;P_m(t)\right)\left(\sqrt{\frac{2\,n+1}{2}}\;P_n(t)\right)dt=\delta_{m,n}$$

Any sufficiently smooth function $f(x)$ can be expanded in the system $\{P_n(x)\}_{n=0,1,\ldots}$ as a generalized Fourier series, with its sum converging to $f(x)$ almost everywhere.

07.07.25.0003.01

$$f(x)=\sum_{n=0}^{\infty}c_n\,\psi_n(x)\;/;\;c_n=\int_{-1}^{1}\psi_n(t)\,f(t)\,dt\bigwedge\psi_n(x)=\sqrt{\frac{2\,n+1}{2}}\;P_n(x)\bigwedge -1<x<1$$

## Representations through more general functions

### Through hypergeometric functions

#### Involving ${}_2\tilde{F}_1$

07.07.26.0036.01

$$P_\nu(z)={}_2\tilde{F}_1\left(-\nu,\,\nu+1;\,1;\,\frac{1-z}{2}\right)$$

07.07.26.0037.01

$$P_\nu(z)=2^{-\nu}\,(z+1)^\nu\,{}_2\tilde{F}_1\left(-\nu,\,-\nu;\,1;\,\frac{z-1}{z+1}\right)$$

07.07.26.0038.01

$$P_\nu(z) = \frac{2^{-\nu-1}\,\Gamma\left(-\nu-\frac{1}{2}\right)\Gamma(2\,\nu+2)}{\sqrt{\pi}\;\Gamma(-\nu)}\,(z-1)^{-\nu-1}\,{}_2\tilde{F}_1\left(\nu+1,\,\nu+1;\,2\,\nu+2;\,\frac{2}{1-z}\right)+$$
$$\frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(-2\,\nu)}{\sqrt{\pi}\;\Gamma(\nu+1)}\,(z-1)^\nu\,{}_2\tilde{F}_1\left(-\nu,\,-\nu;\,-2\,\nu;\,\frac{2}{1-z}\right)\;/;\;z\notin(-1,\,1)\bigwedge 2\,\nu\notin\mathbb{Z}$$

07.07.26.0039.01

$$P_\nu(z) = \pi\left(\frac{1}{\Gamma\left(\frac{1-\nu}{2}\right)\Gamma\left(\frac{\nu+2}{2}\right)}\,{}_2\tilde{F}_1\left(\frac{\nu+1}{2},\,-\frac{\nu}{2};\,\frac{1}{2};\,z^2\right)-\frac{z}{\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}\,{}_2\tilde{F}_1\left(\frac{1-\nu}{2},\,\frac{\nu+2}{2};\,\frac{3}{2};\,z^2\right)\right)$$

07.07.26.0040.01

$$P_\nu(z) = -\frac{2^{-\nu-3}\,(1-\cos(2\,\pi\,\nu))\sec(\pi\,\nu)}{\sqrt{\pi}}\left(2^{2\,\nu+1}\,\Gamma(-\nu)\left(-z^2\right)^{\nu/2}\left(\csc\left(\frac{\pi\,\nu}{2}\right)-\frac{\sqrt{-z^2}}{z}\sec\left(\frac{\pi\,\nu}{2}\right)\right){}_2\tilde{F}_1\left(-\frac{\nu}{2},\,\frac{1-\nu}{2};\,\frac{1}{2}-\nu;\,\frac{1}{z^2}\right)-\right.$$
$$\left.\frac{1}{z}\,\Gamma(\nu+1)\left(-z^2\right)^{-\frac{\nu}{2}}\left(\csc\left(\frac{\pi\,\nu}{2}\right)+\frac{\sqrt{-z^2}}{z}\sec\left(\frac{\pi\,\nu}{2}\right)\right){}_2\tilde{F}_1\left(\frac{\nu+1}{2},\,\frac{\nu+2}{2};\,\nu+\frac{3}{2};\,\frac{1}{z^2}\right)\right)/;\;z\notin(-1,\,0)$$

### Involving ${}_2F_1$

07.07.26.0001.01

$$P_\nu(z) = {}_2F_1\left(-\nu,\,\nu+1;\,1;\,\frac{1-z}{2}\right)$$

07.07.26.0041.01

$$P_\nu(z) = 2^{-\nu}\,(z+1)^\nu\,{}_2F_1\left(-\nu,\,-\nu;\,1;\,\frac{z-1}{z+1}\right)/;\;z\notin(-\infty,\,-1)$$

07.07.26.0002.01

$$P_\nu(z) = \frac{2^{-\nu-1}\,\Gamma\left(-\nu-\frac{1}{2}\right)}{\sqrt{\pi}\;\Gamma(-\nu)}\,(z-1)^{-\nu-1}\,{}_2F_1\left(\nu+1,\,\nu+1;\,2\,\nu+2;\,\frac{2}{1-z}\right)+\frac{2^\nu\,\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\;\Gamma(\nu+1)}\,(z-1)^\nu\,{}_2F_1\left(-\nu,\,-\nu;\,-2\,\nu;\,\frac{2}{1-z}\right)\;/;$$
$$z\notin(-1,\,1)\bigwedge 2\,\nu\notin\mathbb{Z}$$

07.07.26.0042.01

$$P_\nu(z) = \sqrt{\pi}\left(\frac{1}{\Gamma\left(\frac{1-\nu}{2}\right)\Gamma\left(\frac{\nu+2}{2}\right)}\,{}_2F_1\left(\frac{\nu+1}{2},\,-\frac{\nu}{2};\,\frac{1}{2};\,z^2\right)-\frac{2\,z}{\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}\,{}_2F_1\left(\frac{1-\nu}{2},\,\frac{\nu+2}{2};\,\frac{3}{2};\,z^2\right)\right)$$

07.07.26.0043.01

$$P_\nu(z) = \frac{2^{-\nu}\sin(\pi\,\nu)}{(2\,\nu+1)\,\pi^{3/2}}\left(\frac{1}{z}\left(\left(-z^2\right)^{-\frac{\nu}{2}}\,\Gamma(\nu+1)\,\Gamma\left(\frac{1}{2}-\nu\right)\right)\left(\cos\left(\frac{\pi\,\nu}{2}\right)+\frac{\sqrt{-z^2}}{z}\sin\left(\frac{\pi\,\nu}{2}\right)\right){}_2F_1\left(\frac{\nu+1}{2},\,\frac{\nu+2}{2};\,\nu+\frac{3}{2};\,\frac{1}{z^2}\right)\right)-$$
$$2^{2\,\nu+1}\left(-z^2\right)^{\nu/2}\,\Gamma(-\nu)\,\Gamma\left(\nu+\frac{3}{2}\right)\left(\cos\left(\frac{\pi\,\nu}{2}\right)-\frac{\sqrt{-z^2}}{z}\sin\left(\frac{\pi\,\nu}{2}\right)\right){}_2F_1\left(-\frac{\nu}{2},\,\frac{1-\nu}{2};\,\frac{1}{2}-\nu;\,\frac{1}{z^2}\right)\Bigg)/;\;z\notin(-1,\,0)$$

## Through hypergeometric functions of two variables

07.07.26.0003.01

$$P_\nu(z) = F_{1\times0\times0}^{2\times0\times0}\left(\begin{matrix}-\nu,\,1+\nu;;; \\ 1;;;\end{matrix}\;\frac{1}{2},\,-\frac{z}{2}\right)\;/;\;|z|<1$$

## Through Meijer *G*

### Classical cases for the direct function itself

07.07.26.0004.01

$$P_\nu(z) = -\frac{\sin(\pi\,\nu)}{\pi}\,G_{2,2}^{1,2}\left(\frac{z-1}{2}\;\middle|\;\begin{matrix}\nu+1,\,-\nu\\0,\,0\end{matrix}\right)\;/;\;\nu\notin\mathbb{Z}$$

07.07.26.0005.01

$$P_n(z) = -\frac{1}{\pi}\lim_{\nu\to n}\sin(\pi\,\nu)\,G_{2,2}^{1,2}\left(\frac{z-1}{2}\;\middle|\;\begin{matrix}\nu+1,\,-\nu\\0,\,0\end{matrix}\right)\;/;\;n\in\mathbb{Z}$$

07.07.26.0006.01

$$P_\nu(2\,z+1) = -\frac{\sin(\pi\,\nu)}{\pi}\,G_{2,2}^{1,2}\left(z\;\middle|\;\begin{matrix}\nu+1,\,-\nu\\0,\,0\end{matrix}\right)\;/;\;\nu\notin\mathbb{Z}$$

### Classical cases involving algebraic functions

07.07.26.0007.01

$$(z+1)^{-\nu-1}\,P_\nu\left(\frac{1-z}{1+z}\right) = \frac{1}{\Gamma(\nu+1)^2}\,G_{2,2}^{1,2}\left(z\;\middle|\;\begin{matrix}-\nu,\,-\nu\\0,\,0\end{matrix}\right)\;/;\;z\notin(-\infty,\,-1)$$

07.07.26.0008.01

$$(z+1)^{-\nu-1}\,P_\nu\left(\frac{z-1}{z+1}\right) = \frac{1}{\Gamma(\nu+1)^2}\,G_{2,2}^{2,1}\left(z\;\middle|\;\begin{matrix}-\nu,\,-\nu\\0,\,0\end{matrix}\right)\;/;\;z\notin(-1,\,0)$$

07.07.26.0009.01

$$(z+1)^{-\frac{\nu+1}{2}}\,P_\nu\left(\frac{1}{\sqrt{z+1}}\right) = \frac{2^\nu}{\Gamma(\nu+1)\sqrt{\pi}}\,G_{2,2}^{1,2}\left(z\;\middle|\;\begin{matrix}-\frac{\nu}{2},\,\frac{1-\nu}{2}\\0,\,0\end{matrix}\right)$$

07.07.26.0010.01

$$(z+1)^{-\frac{\nu+1}{2}}\,P_\nu\left(\sqrt{\frac{z}{z+1}}\right) = \frac{2^\nu}{\Gamma(\nu+1)\sqrt{\pi}}\,G_{2,2}^{2,1}\left(z\;\middle|\;\begin{matrix}\frac{1-\nu}{2},\,\frac{1-\nu}{2}\\0,\,\frac{1}{2}\end{matrix}\right)\;/;\;z\notin(-1,\,0)$$

07.07.26.0011.01

$$(z+1)^{-\frac{\nu+1}{2}}\,P_\nu\left(\frac{z+2}{2\sqrt{z+1}}\right) = \frac{1}{\Gamma(\nu+1)\sqrt{\pi}}\,G_{2,2}^{1,2}\left(z\;\middle|\;\begin{matrix}\frac{1}{2},\,-\nu\\0,\,0\end{matrix}\right)$$

07.07.26.0012.01

$$(z+1)^{-\frac{\nu+1}{2}}\,P_\nu\left(\frac{2\,z+1}{2\sqrt{z}\,\sqrt{z+1}}\right) = \frac{1}{\Gamma(\nu+1)\sqrt{\pi}}\,G_{2,2}^{2,1}\left(z\;\middle|\;\begin{matrix}\frac{1-\nu}{2},\,\frac{1-\nu}{2}\\-\frac{\nu}{2},\,\frac{\nu+1}{2}\end{matrix}\right)\;/;\;z\notin(-1,\,0)$$

### Classical cases involving unit step $\theta$

07.07.26.0013.01

$$\theta(1-|z|)\,P_\nu(2\,z-1) = G_{2,2}^{2,0}\left(z\;\middle|\;\begin{matrix}\nu+1,\,-\nu\\0,\,0\end{matrix}\right)\;/;\;z\notin(-1,\,0)$$

07.07.26.0014.01

$$\theta(|z|-1)\, P_\nu(2z-1) = G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \nu+1, -\nu \\ 0, 0 \end{matrix}\right)$$

07.07.26.0015.01

$$\theta(1-|z|)\, P_\nu\left(\frac{2}{z}-1\right) = G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} 1, 1 \\ \nu+1, -\nu \end{matrix}\right)$$

07.07.26.0016.01

$$\theta(|z|-1)\, P_\nu\left(\frac{2}{z}-1\right) = G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} 1, 1 \\ \nu+1, -\nu \end{matrix}\right) /;\, z \notin (-\infty, -1)$$

07.07.26.0017.01

$$\theta(1-|z|)\, P_\nu\left(\frac{z+1}{2\sqrt{z}}\right) = G_{2,2}^{2,0}\left(z \,\middle|\, \begin{matrix} \frac{1-\nu}{2}, \frac{\nu}{2}+1 \\ \frac{\nu+1}{2}, -\frac{\nu}{2} \end{matrix}\right) /;\, z \notin (-1, 0)$$

07.07.26.0018.01

$$\theta(|z|-1)\, P_\nu\left(\frac{z+1}{2\sqrt{z}}\right) = G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} \frac{\nu}{2}+1, \frac{1-\nu}{2} \\ \frac{\nu+1}{2}, -\frac{\nu}{2} \end{matrix}\right)$$

07.07.26.0019.01

$$\theta(|z|-1)\left(\frac{z-1}{z}\right)^{\frac{1}{2}\left(n-2\left\lfloor\frac{n}{2}\right\rfloor-1\right)} P_n\left(\sqrt{\frac{z-1}{z}}\right) = \frac{(-1)^{\left\lfloor\frac{n}{2}\right\rfloor}}{\left\lfloor\frac{n}{2}\right\rfloor!}\,\Gamma\left(n-\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2}\right) G_{2,2}^{0,2}\left(z \,\middle|\, \begin{matrix} 1, 1 \\ \left\lfloor\frac{n}{2}\right\rfloor+1, -n+\left\lfloor\frac{n}{2}\right\rfloor+\frac{1}{2} \end{matrix}\right) /;\, n \in \mathbb{N}$$

## Classical cases for products of Legendre *P*

07.07.26.0020.01

$$P_\nu\left(\sqrt{z+1}-\sqrt{z}\right) P_\nu\left(\sqrt{z}+\sqrt{z+1}\right) = \frac{\sin(\nu\pi)}{2\pi^{3/2}}\, G_{4,4}^{1,4}\left(z \,\middle|\, \begin{matrix} \frac{\nu+1}{2}, \frac{\nu}{2}+1, -\frac{\nu}{2}, \frac{1-\nu}{2} \\ 0, 0, 0, \frac{1}{2} \end{matrix}\right)$$

07.07.26.0021.01

$$P_\nu\left(\frac{\sqrt{z+1}-1}{\sqrt{z}}\right) P_\nu\left(\frac{\sqrt{z+1}+1}{\sqrt{z}}\right) = -\frac{\sin(\nu\pi)}{2\pi^{3/2}}\, G_{4,4}^{4,1}\left(z \,\middle|\, \begin{matrix} 1, \frac{1}{2}, 1, 1 \\ -\frac{\nu}{2}, \frac{1-\nu}{2}, \frac{\nu+1}{2}, \frac{\nu}{2}+1 \end{matrix}\right) /;\, z \notin (-1, 0)$$

## Generalized cases involving algebraic functions

07.07.26.0022.01

$$\left(z^2+1\right)^{-\frac{\nu+1}{2}} P_\nu\left(\frac{z}{\sqrt{z^2+1}}\right) = \frac{2^\nu}{\Gamma(\nu+1)\sqrt{\pi}}\, G_{2,2}^{2,1}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1-\nu}{2}, \frac{1-\nu}{2} \\ 0, \frac{1}{2} \end{matrix}\right) /;\, \mathrm{Re}(z) > 0$$

07.07.26.0023.01

$$\left(z^2+1\right)^{-\frac{\nu+1}{2}} P_\nu\left(\frac{2z^2+1}{2z\sqrt{z^2+1}}\right) = \frac{1}{\Gamma(\nu+1)\sqrt{\pi}}\, G_{2,2}^{2,1}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{1-\nu}{2}, \frac{1-\nu}{2} \\ -\frac{\nu}{2}, \frac{\nu+1}{2} \end{matrix}\right) /;\, \mathrm{Re}(z) > 0$$

## Generalized cases involving unit step $\theta$

07.07.26.0024.01

$$\theta(1-|z|)\, P_\nu(z) = G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{matrix} \frac{\nu}{2}+1, \frac{1-\nu}{2} \\ 0, \frac{1}{2} \end{matrix}\right)$$

07.07.26.0025.01

$$\theta(|z|-1)\,P_\nu(z) = G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} \frac{\nu}{2}+1, \frac{1-\nu}{2} \\ 0, \frac{1}{2} \end{array}\right)$$

07.07.26.0026.01

$$\theta(1-|z|)\,P_\nu\left(\frac{1}{z}\right) = G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} 1, \frac{1}{2} \\ -\frac{\nu}{2}, \frac{\nu+1}{2} \end{array}\right)$$

07.07.26.0027.01

$$\theta(|z|-1)\,P_\nu\left(\frac{1}{z}\right) = G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} \frac{1}{2}, 1 \\ -\frac{\nu}{2}, \frac{\nu+1}{2} \end{array}\right)$$

07.07.26.0028.01

$$\theta(1-|z|)\,P_\nu\left(\frac{z^2+1}{2\,z}\right) = G_{2,2}^{2,0}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} \frac{1-\nu}{2}, \frac{\nu}{2}+1 \\ \frac{\nu+1}{2}, -\frac{\nu}{2} \end{array}\right)$$

07.07.26.0029.01

$$\theta(|z|-1)\,P_\nu\left(\frac{z^2+1}{2\,z}\right) = G_{2,2}^{0,2}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} \frac{\nu}{2}+1, \frac{1-\nu}{2} \\ \frac{\nu+1}{2}, -\frac{\nu}{2} \end{array}\right)$$

### Generalized cases for products of Legendre *P*

07.07.26.0030.01

$$P_\nu\left(\sqrt{z^2+1}-z\right) P_\nu\left(z+\sqrt{z^2+1}\right) = -\frac{\sin(\nu\,\pi)}{2\,\pi^{3/2}}\, G_{4,4}^{1,4}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} \frac{\nu+1}{2}, \frac{\nu}{2}+1, -\frac{\nu}{2}, \frac{1-\nu}{2} \\ 0, 0, 0, \frac{1}{2} \end{array}\right)$$

07.07.26.0031.01

$$P_\nu\left(\frac{\sqrt{z^2+1}-1}{z}\right) P_\nu\left(\frac{\sqrt{z^2+1}+1}{z}\right) = -\frac{\sin(\nu\,\pi)}{2\,\pi^{3/2}}\, G_{4,4}^{4,1}\left(z, \frac{1}{2} \,\middle|\, \begin{array}{c} 1, \frac{1}{2}, 1, 1 \\ -\frac{\nu}{2}, \frac{1-\nu}{2}, \frac{\nu+1}{2}, \frac{\nu}{2}+1 \end{array}\right) /;\ \mathrm{Re}(z) > 0$$

## Through other functions

### Involving some hypergeometric-type functions

07.07.26.0032.01

$$P_\nu(z) = P_\nu^0(z)$$

07.07.26.0033.01

$$P_\nu(z) = \mathcal{P}_\nu^0(z)$$

07.07.26.0034.01

$$P_\nu(z) = P_\nu^{(0,0)}(z)$$

07.07.26.0035.01

$$P_\nu(z) = C_\nu^{\frac{1}{2}}(z)$$

### Involving spheroidal functions

07.07.26.0044.01

$$P_\nu(z) = PS_{\nu,0}(0, z)$$

# Representations through equivalent functions

## With related functions

07.07.27.0001.01

$$\frac{Q_{\nu-\frac{1}{2}}(z)}{P_{\nu-\frac{1}{2}}(z)} - \frac{Q_{\nu+\frac{1}{2}}(z)}{P_{\nu+\frac{1}{2}}(z)} = \frac{1}{\left(\nu+\frac{1}{2}\right) P_{\nu-\frac{1}{2}}(z)\, P_{\nu+\frac{1}{2}}(z)}$$

# Zeros

07.07.30.0001.01

$$\frac{P_n(z)}{z-z_0} = \frac{1}{\left(1-z_0^2\right)\left(\frac{\partial P_n(x)}{\partial x}\,|_{z=z_0}\right)} \sum_{k=0}^{n-1} (2\,k+1)\, P_k(z)\, P_k(z_0)\ /;\ n \in \mathbb{N} \bigwedge P_n(z_0) = 0$$

# Theorems

## Expansions in generalized Fourier series

$$f(x) = \sum_{k=0}^{\infty} c_k\, \psi_k(x)\ /;\quad c_k = \int_{-1}^{1} f(t)\, \psi_k(t)\, dt,\ \ \psi_k(x) = \sqrt{\frac{2\,n+1}{2}}\ P_k(x),\ \ k \in \mathbb{N}.$$

## Mehler-Fock Transformation

$$\hat{f}(y) = \int_{1}^{\infty} f(x)\, P_{-1/2+i\,y}(x)\, dx \Leftrightarrow f(x) = \int_{0}^{\infty} \hat{f}(y)\, y \tanh(\pi\, y)\, P_{-1/2+i\,y}(x)\, dy.$$

## Gauss' numerical integration methods

$$\int_a^b f(x)\, dx = \frac{b-a}{2} \sum_{k=1}^{n} w_k\, f(y_k) + \frac{2^{2\,n+1}\, n!^4}{(2\,n+1)\,(2\,n)!^3}\ f^{(2\,n)}(\xi)\ /;$$

$$y_k = \frac{b-a}{2}\, x_k + \frac{b+a}{2} \bigwedge P_n(x_k) = 0 \bigwedge w_k = \frac{2}{1-x_k^2}\left(P_n'(x_k)\right)^2 \bigwedge n \in \mathbb{Z}^+,\ a,\ b \in \mathbb{R} \bigwedge a < \xi < b.$$

# History

– D. Bernoulli (1748)
– A. M. Legendre (1782, 1785)
– E. Heine (1842)
– P. L. Chebyshev (1855)
– L. Schläfli (1881)
– I. Todhunter (1875) introduced the notation $P_n(z)$

## Copyright