LogGamma

Notations

Traditional name

Logarithm of the gamma function

Traditional notation

$\log\Gamma(z)$

Mathematica StandardForm notation

`LogGamma[z]`

Primary definition

06.11.02.0001.01

$$\log\Gamma(z) = \sum_{k=1}^{\infty}\left(\frac{z}{k} - \log\left(1 + \frac{z}{k}\right)\right) - \gamma z - \log(z)$$

The function $\log\Gamma(z)$ is equivalent to $\log(\Gamma(z))$ as a multivalued analytic function, except that it is conventionally defined with a different branch cut structure and principal sheet. The function $\log\Gamma(z)$ allows a concise formulation of many identities related to the Riemann zeta function $\zeta(z)$.

Specific values

Specialized values

06.11.03.0001.01

$$\log\Gamma(n) = \log((n-1)!) \;/;\; n \in \mathbb{N}^+$$

06.11.03.0002.01

$$\log\Gamma\left(\frac{n}{2}\right) = \log\left(\frac{2^{1-n}\sqrt{\pi}\,(n-1)!}{\frac{n-1}{2}!}\right) \;/;\; n \in \mathbb{N}^+$$

06.11.03.0003.01

$$\log\Gamma(-n) = \infty \;/;\; n \in \mathbb{N}$$

06.11.03.0004.01

$$\log\Gamma\left(n + \frac{1}{4}\right) = \log\Gamma\left(\frac{1}{4}\right) - 2n\log(2) + \sum_{k=1}^{n}\log(4k-3) \;/;\; n \in \mathbb{N}$$

06.11.03.0005.01

$$\log\Gamma\left(\frac{1}{4}-n\right) = \log\Gamma\left(\frac{1}{4}\right) + 2\log(2)\,n - \pi\,i\,n - \sum_{k=1}^{n}\log(4\,k-1)\;/;\; n\in\mathbb{N}$$

06.11.03.0006.01

$$\log\Gamma\left(n+\frac{1}{3}\right) = \log\Gamma\left(\frac{1}{3}\right) - n\log(3) + \sum_{k=1}^{n}\log(3\,k-2)\;/;\; n\in\mathbb{N}$$

06.11.03.0007.01

$$\log\Gamma\left(\frac{1}{3}-n\right) = \log\Gamma\left(\frac{1}{3}\right) + \log(3)\,n - \pi\,i\,n - \sum_{k=1}^{n}\log(3\,k-1)\;/;\; n\in\mathbb{N}$$

06.11.03.0008.01

$$\log\Gamma\left(n+\frac{1}{2}\right) = \frac{\log(\pi)}{2} - n\log(2) + \sum_{k=1}^{n}\log(2\,k-1)\;/;\; n\in\mathbb{N}$$

06.11.03.0009.01

$$\log\Gamma\left(\frac{1}{2}-n\right) = \frac{\log(\pi)}{2} + \log(2)\,n - \pi\,i\,n - \sum_{k=1}^{n}\log(2\,k-1)\;/;\; n\in\mathbb{N}$$

06.11.03.0010.01

$$\log\Gamma\left(n+\frac{2}{3}\right) = \log\Gamma\left(\frac{2}{3}\right) - n\log(3) + \sum_{k=1}^{n}\log(3\,k-1)\;/;\; n\in\mathbb{N}$$

06.11.03.0011.01

$$\log\Gamma\left(\frac{2}{3}-n\right) = \log\Gamma\left(\frac{2}{3}\right) + \log(3)\,n - \pi\,i\,n - \sum_{k=1}^{n}\log(3\,k-2)\;/;\; n\in\mathbb{N}$$

06.11.03.0012.01

$$\log\Gamma\left(n+\frac{3}{4}\right) = \log\Gamma\left(\frac{3}{4}\right) - n\log(4) + \sum_{k=1}^{n}\log(4\,k-1)\;/;\; n\in\mathbb{N}$$

06.11.03.0013.01

$$\log\Gamma\left(\frac{3}{4}-n\right) = \log\Gamma\left(\frac{3}{4}\right) + \log(4)\,n - \pi\,i\,n - \sum_{k=1}^{n}\log(4\,k-3)\;/;\; n\in\mathbb{N}$$

06.11.03.0014.01

$$\log\Gamma\left(n+\frac{p}{q}\right) = \log\Gamma\left(\frac{p}{q}\right) - n\log(q) + \sum_{k=1}^{n}\log(p+k\,q-q)\;/;\; n\in\mathbb{N}\land p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land p<q$$

06.11.03.0015.01

$$\log\Gamma\left(\frac{p}{q}-n\right) = \log\Gamma\left(\frac{p}{q}\right) + \log(q)\,n - \pi\,i\,n - \sum_{k=1}^{n}\log(q\,k-p)\;/;\; n\in\mathbb{N}\land p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land p<q$$

06.11.03.0034.01

$$\log\Gamma\left(1-\frac{p}{q}\right) + \log\Gamma\left(\frac{p}{q}\right) = \log(\pi) - \log\left(\sin\left(\frac{p\,\pi}{q}\right)\right)\;/;\; p\in\mathbb{N}^+\land q\in\mathbb{N}^+\land p<q$$

Values at fixed points

06.11.03.0016.01

$$\log\Gamma(-3) = \infty$$

06.11.03.0017.01

$$\log\Gamma\left(-\frac{5}{2}\right) = \log\left(\frac{8\sqrt{\pi}}{15}\right) - 3\,i\,\pi$$

06.11.03.0018.01

$$\log\Gamma(-2) = \infty$$

06.11.03.0019.01

$$\log\Gamma\left(-\frac{3}{2}\right) = \log\left(\frac{4\sqrt{\pi}}{3}\right) - 2\,i\,\pi$$

06.11.03.0020.01

$$\log\Gamma(-1) = \infty$$

06.11.03.0021.01

$$\log\Gamma\left(-\frac{1}{2}\right) = \log\left(2\sqrt{\pi}\right) - i\,\pi$$

06.11.03.0022.01

$$\log\Gamma(0) = \infty$$

06.11.03.0023.01

$$\log\Gamma\left(\frac{1}{2}\right) = \frac{\log(\pi)}{2}$$

06.11.03.0024.01

$$\log\Gamma(1) = 0$$

06.11.03.0025.01

$$\log\Gamma\left(\frac{3}{2}\right) = \log\left(\frac{\sqrt{\pi}}{2}\right)$$

06.11.03.0026.01

$$\log\Gamma(2) = 0$$

06.11.03.0027.01

$$\log\Gamma\left(\frac{5}{2}\right) = \log\left(\frac{3\sqrt{\pi}}{4}\right)$$

06.11.03.0028.01

$$\log\Gamma(3) = \log(2)$$

06.11.03.0035.01

$$\log\Gamma\left(\frac{1}{4}\right) + \log\Gamma\left(\frac{3}{4}\right) = \frac{\log(2)}{2} + \log(\pi)$$

06.11.03.0036.01

$$\log\Gamma\left(\frac{1}{3}\right) + \log\Gamma\left(\frac{2}{3}\right) = \log(2\,\pi) - \frac{\log(3)}{2}$$

Values at infinities

06.11.03.0029.01

$$\log\Gamma(\infty) = \infty$$

06.11.03.0030.01

$\log\Gamma(-\infty) = \tilde{\infty}$

06.11.03.0031.01

$\log\Gamma(i\,\infty) = \tilde{\infty}$

06.11.03.0032.01

$\log\Gamma(-i\,\infty) = \tilde{\infty}$

06.11.03.0033.01

$\log\Gamma(\tilde{\infty}) = \tilde{\infty}$

General characteristics

Domain and analyticity

$\log\Gamma(z)$ is an analytical function of z which is defined over the whole complex z-plane. It has one infinitely long branch cut.

06.11.04.0001.01

$z \longrightarrow \log\Gamma(z) :: \mathbb{C} \longrightarrow \mathbb{C}$

Symmetries and periodicities

Mirror symmetry

06.11.04.0002.01

$\log\Gamma(\bar{z}) = \overline{\log\Gamma(z)} \;/;\; z \notin (-\infty, 0)$

Periodicity

No periodicity

Poles and essential singularities

The function $\log\Gamma(z)$ does not have poles and essential singularities.

06.11.04.0003.01

$\mathcal{Sing}_z(\log\Gamma(z)) = \{\}$

Branch points

The function $\log\Gamma(z)$ has infinitely many branch points: $z = -n \;/;\; n \in \mathbb{N}$ and $z = \tilde{\infty}$. All these are logarithmic type branch points.

06.11.04.0004.01

$\mathcal{BP}_z(\log\Gamma(z)) = \{0, \tilde{\infty}\}$

06.11.04.0005.01

$\mathcal{R}_z(\log\Gamma(z), 0) = \log$

06.11.04.0006.01

$\mathcal{R}_z(\log\Gamma(z), \tilde{\infty}) = \log$

Branch cuts

The function $\log\Gamma(z)$ is a single-valued function on the z-plane cut along the interval $(-\infty, 0)$, where $\log\Gamma(z)$ is continuous from above. This interval includes an infinite set of branch cut lines of combined logarithmic type along $(-\infty, -n) /; n \in \mathbb{N}$.

06.11.04.0007.01

$$\mathcal{BC}_z(\log\Gamma(z)) == \{\{(-\infty, 0), -i\}\}$$

06.11.04.0008.01

$$\lim_{\epsilon\to+0} \log\Gamma(x + i\,\epsilon) == \log\Gamma(x) /; x < 0$$

06.11.04.0009.01

$$\lim_{\epsilon\to+0} \log\Gamma(x - i\,\epsilon) == \log\Gamma(x) - 2\,i\,\pi\,\lfloor x\rfloor /; x < 0$$

Series representations

Generalized power series

Expansions on branch cuts

06.11.06.0019.01

$$\log\Gamma(z) \propto \log\Gamma(x) + 2\,i\,\pi\left\lfloor\frac{\arg(z-x)}{2\,\pi}\right\rfloor\lfloor x\rfloor + \left(\psi(x) + \frac{1}{2}\,\psi^{(1)}(x)\,(z-x) + \frac{1}{6}\,\psi^{(2)}(x)\,(z-x)^2 + \ldots\right)(z-x) /; (z\to x)\land x\in\mathbb{R}\land x<0$$

06.11.06.0020.01

$$\log\Gamma(z) \propto \log\Gamma(x) + 2\,i\,\pi\left\lfloor\frac{\arg(z-x)}{2\,\pi}\right\rfloor\lfloor x\rfloor + \psi(x)\,(z-x) + \frac{1}{2}\,\psi^{(1)}(x)\,(z-x)^2 + \frac{1}{6}\,\psi^{(2)}(x)\,(z-x)^3 + O\left((z-x)^4\right) /; x\in\mathbb{R}\land x<0$$

06.11.06.0021.01

$$\log\Gamma(z) = \log\Gamma(x) + 2\,i\,\pi\left\lfloor\frac{\arg(z-x)}{2\,\pi}\right\rfloor\lfloor x\rfloor + \sum_{k=1}^{\infty}\frac{\psi^{(k-1)}(x)}{k!}\,(z-x)^k /; x\in\mathbb{R}\land x<0$$

06.11.06.0022.01

$$\log\Gamma(z) \propto \left(\log\Gamma(x) + 2\,i\,\pi\left\lfloor\frac{\arg(z-x)}{2\,\pi}\right\rfloor\lfloor x\rfloor\right)(1 + O(z-x)) /; x\in\mathbb{R}\land x<0$$

Expansions at $z == 0$

For the function itself

06.11.06.0002.02

$$\log\Gamma(z) \propto -\log(z) - \gamma\,z + \frac{\pi^2\,z^2}{12} - \frac{\zeta(3)\,z^3}{3} + \frac{\pi^4\,z^4}{360} - \ldots /; (z\to 0)$$

06.11.06.0023.01

$$\log\Gamma(z) \propto -\log(z) - \gamma\,z + \frac{\pi^2\,z^2}{12} - \frac{\zeta(3)\,z^3}{3} + \frac{\pi^4\,z^4}{360} - O\left(z^5\right)$$

06.11.06.0003.01

$$\log\Gamma(z) = -\log(z) - \gamma z + \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(j+2) z^{j+2}}{j+2} /; |z| < 1$$

06.11.06.0001.01

$$\log\Gamma(z) = -\log(z) - \gamma z + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j (k+1)^{-j-2} z^{j+2}}{j+2} /; |z| < 1$$

06.11.06.0004.02

$$\log\Gamma(z) \propto -\log(z) - \gamma z (1 + O(z))$$

Expansions at $z = z_0$ /; $\neg (z_0 \in \mathbb{R} \land z_0 \leq 0)$

For the function itself

06.11.06.0007.02

$$\log\Gamma(z) \propto \log\Gamma(z_0) + \psi(z_0)(z - z_0) + \frac{\zeta(2, z_0)}{2} (z - z_0)^2 - \frac{\zeta(3, z_0)}{3} (z - z_0)^3 + \ldots /; (z \to z_0) \land \neg (z_0 \in \mathbb{R} \land z_0 \leq 0)$$

06.11.06.0024.01

$$\log\Gamma(z) \propto \log\Gamma(z_0) + \psi(z_0)(z - z_0) + \frac{\zeta(2, z_0)}{2} (z - z_0)^2 - \frac{\zeta(3, z_0)}{3} (z - z_0)^3 + O\left((z - z_0)^4\right) /; \neg (z_0 \in \mathbb{R} \land z_0 \leq 0)$$

06.11.06.0008.02

$$\log\Gamma(z) = \log\Gamma(z_0) + \psi(z_0)(z - z_0) + \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(j+2, z_0)}{j+2} (z - z_0)^{j+2} /; \neg (z_0 \in \mathbb{R} \land z_0 \leq 0)$$

06.11.06.0025.01

$$\log\Gamma(z) = \sum_{k=0}^{\infty} \frac{\psi^{(k-1)}(z_0)}{k!} (z - z_0)^k /; \neg (z_0 \in \mathbb{R} \land z_0 \leq 0)$$

06.11.06.0005.02

$$\log\Gamma(z) = \log\Gamma(z_0) + \psi(z_0)(z - z_0) + \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^j (z - z_0)^{j+2}}{(j+2)(k + z_0)^{j+2}} /; \neg (z_0 \in \mathbb{R} \land z_0 \leq 0)$$

06.11.06.0006.02

$$\log\Gamma(z) = \log\Gamma(z_0) + \psi(z_0)(z - z_0) + \sum_{k=0}^{\infty} \left(\frac{z - z_0}{k + z_0} - \log\left(\frac{k + z}{k + z_0} \right) \right) /; \neg (z_0 \in \mathbb{R} \land z_0 \leq 0)$$

06.11.06.0009.02

$$\log\Gamma(z) \propto \log\Gamma(z_0) + \psi(z_0)(z - z_0)(1 + O(z - z_0)) /; \neg (z_0 \in \mathbb{R} \land z_0 \leq 0)$$

Expansions at $z = -n$

For the function itself

06.11.06.0026.01

$$\log\Gamma(z) \propto -\log(z + n) - 2 i n \pi \left\lfloor \frac{\arg(z + n)}{2\pi} \right\rfloor - \log\Gamma(n + 1) - n i \pi + \psi(n + 1)(z + n) + \frac{1}{2} \left(\frac{\pi^2}{3} - \psi^{(1)}(n + 1) \right) (z + n)^2 + \ldots /;$$
$$(z \to -n) \land -n \in \mathbb{N}$$

06.11.06.0027.01

$$\log\Gamma(z) \propto -\log(z+n) - 2\,i\,n\,\pi\left\lfloor\frac{\arg(z+n)}{2\,\pi}\right\rfloor - \log\Gamma(n+1) - n\,i\,\pi + \psi(n+1)\,(z+n) + \frac{1}{2}\left(\frac{\pi^2}{3} - \psi^{(1)}(n+1)\right)(z+n)^2 + O\!\left((z+n)^3\right) /;\ -n \in \mathbb{N}$$

06.11.06.0028.01

$$\log\Gamma(z) \propto -\log(z+n) - 2\,i\,\pi\left\lfloor\frac{\arg(z+n)}{2\,\pi}\right\rfloor n - i\,\pi\,n + \sum_{k=0}^{\infty}\frac{\left(\left(1-(-1)^{k-1}\right)\psi^{(k-1)}(1) + (-1)^{k-1}\,\psi^{(k-1)}(n+1)\right)}{k!}\,(z+n)^k /;\ (z \to -n) \land -n \in \mathbb{N}$$

06.11.06.0010.01

$$\log\Gamma(z) \propto -\log(z+n) + \log\Gamma\,(z+n+1) - \sum_{k=0}^{n-1}\log(z+k) /;\ (z \to -n) \land n \in \mathbb{N}$$

06.11.06.0029.01

$$\log\Gamma(z) = -\gamma\,z - \log(z) - \sum_{k=1}^{\infty}\left(\log\left(1+\frac{z}{k}\right) - \frac{z}{k}\right)$$

06.11.06.0011.01

$$\log\Gamma(z) \propto -\log(z+n) - \sum_{k=0}^{n-1}\log(z+k)\,(1+O(z+n)) /;\ (z \to -n) \land n \in \mathbb{N}$$

06.11.06.0030.01

$$\log\Gamma(z) \propto -\log(z+n) - \left(2\,i\,\pi\left\lfloor\frac{\arg(n+z)}{2\,\pi}\right\rfloor n + i\,\pi\,n + \log\Gamma(n+1)\right)(1+O(z+n)) /;\ -n \in \mathbb{N}$$

Exponential Fourier series

06.11.06.0012.01

$$\log\Gamma(x) = \frac{1}{2}\,(\log(\pi) - \log(\sin(\pi\,x))) + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(\log(2\,k\,\pi)+\gamma)\sin(2\,k\,\pi\,x)}{k} /;\ 0 < x < 1$$

06.11.06.0013.01

$$\log\Gamma(x) = \left(\frac{1}{2} - x\right)(\log(2)+\gamma) + (1-x)\log(\pi) - \frac{\log(\sin(\pi\,x))}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\log(k)\sin(2\,k\,\pi\,x)}{k} /;\ 0 < x < 1$$

06.11.06.0014.01

$$\log\Gamma(x) = \frac{\log(2\,\pi)}{2} + \sum_{k=1}^{\infty}\left(\frac{\cos(2\,k\,\pi\,x)}{2\,k} + \frac{(\log(2\,k\,\pi)+\gamma)\sin(2\,k\,\pi\,x)}{k\,\pi}\right) /;\ 0 < x < 1$$

Asymptotic series expansions

06.11.06.0015.01

$$\log\Gamma(z) \propto \left(z - \frac{1}{2}\right)\log(z) - z + \frac{\log(2\,\pi)}{2} + \sum_{k=1}^{\infty}\frac{B_{2\,k}}{2\,k\,(2\,k-1)\,z^{2\,k-1}} /;\ \neg\,(z \in \mathbb{Z} \land z < 0) \land (|z| \to \infty)$$

06.11.06.0016.01

$$\log\Gamma(z) \propto \left(z - \frac{1}{2}\right)\log(z) - z + \frac{\log(2\,\pi)}{2} + \frac{1}{12\,z}\left(1 + O\left(\frac{1}{z^2}\right)\right) /;\ \neg\,(z \in \mathbb{Z} \land z < 0) \land (|z| \to \infty)$$

Other series representations

06.11.06.0017.01

$$\log\Gamma(z) = -\log(z) - \gamma z + \frac{1}{2}\log\left(\frac{\pi z}{\sin(\pi z)}\right) - \sum_{k=1}^{\infty} \frac{\zeta(2k+1) z^{2k+1}}{2k+1} \;/;\; |z| < 1$$

06.11.06.0018.01

$$\log\Gamma(z) = \frac{\log(2\pi)}{2} + \left(z - \frac{1}{2}\right)\log(z) - z + \frac{1}{2}\sum_{k=2}^{\infty} \frac{(k-1)\zeta(k, z+1)}{k(k+1)} \;/;\; \mathrm{Re}(z) > 0$$

Integral representations

On the real axis

Of the direct function

06.11.07.0001.01

$$\log\Gamma(z) = -\int_0^{\infty} \frac{e^{-t}}{t}\left(\frac{e^{tz}-1}{1-e^{-t}} - z\right)dt + \log(\pi) - \log(\sin(\pi z)) \;/;\; \mathrm{Re}(z) < 1$$

06.11.07.0002.01

$$\log\Gamma(z) = \int_0^{\infty} \frac{1}{t}\left((z-1)e^{-t} + \frac{e^{-tz} - e^{-t}}{1-e^{-t}}\right)dt \;/;\; \mathrm{Re}(z) > 0$$

06.11.07.0003.01

$$\log\Gamma(z) = \int_0^{\infty} \frac{e^{-tz}}{t}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)dt + \frac{\log(2\pi)}{2} + \left(z - \frac{1}{2}\right)\log(z) - z \;/;\; \mathrm{Re}(z) > 0$$

06.11.07.0004.01

$$\log\Gamma(z) = \int_0^{\infty} \frac{e^{-t}}{t}\left(z - 1 - \frac{1 - e^{-t(z-1)}}{1 - e^{-t}}\right)dt \;/;\; \mathrm{Re}(z) > 0$$

06.11.07.0005.01

$$\log\Gamma(z) = \int_0^{\infty} \frac{e^{-tz} - e^{-t}}{t}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right)dt + \left(z - \frac{1}{2}\right)\log(z) + 1 - z \;/;\; \mathrm{Re}(z) > 0$$

06.11.07.0006.01

$$\log\Gamma(z) = \frac{1}{2}\int_0^{\infty} \frac{1}{t}\left(\mathrm{csch}\left(\frac{t}{2}\right)\sinh\left(t\left(\frac{1}{2} - z\right)\right) - e^{-t}(1-2z)\right)dt + \frac{\log(\pi)}{2} - \frac{1}{2}\log(\sin(\pi z)) \;/;\; 0 < \mathrm{Re}(z) < 1$$

06.11.07.0007.01

$$\log\Gamma(z) = \int_0^{\infty} \frac{1}{t}\left(e^{-t}(z-1) + \frac{1}{\log(t+1)}\left((t+1)^{-z} - \frac{1}{t+1}\right)\right)dt \;/;\; \mathrm{Re}(z) > 0$$

06.11.07.0008.01

$$\log\Gamma(z) = 2\int_0^{\infty} \frac{\tan^{-1}\left(\frac{t}{z}\right)}{e^{2\pi t} - 1}dt + \frac{\log(2\pi)}{2} + \left(z - \frac{1}{2}\right)\log(z) - z \;/;\; \mathrm{Re}(z) > 0$$

Limit representations

06.11.09.0001.01

$$\log\Gamma(z) == \lim_{n\to\infty}\left(z\log(n) - \sum_{k=1}^{n}\log\left(1+\frac{z}{k}\right)\right) - \log(z)$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

06.11.13.0001.01

$$\frac{\partial w(z)}{\partial z} == \psi(z) \,/;\, w(z) == \log\Gamma(z)$$

Transformations

Transformations and argument simplifications

Argument involving basic arithmetic operations

06.11.16.0001.01

$$\log\Gamma(1-z) == \log(\pi) - \log(\sin(\pi z)) - \log\Gamma(z) \,/;\, -\frac{1}{2} < \mathrm{Re}(z) \leq \frac{3}{2}$$

06.11.16.0002.01

$$\log\Gamma(1-z) == \log(\pi) - \log(\sin(\pi z)) - \log\Gamma(z) + 2\,i\,\pi\left\lfloor\frac{2\,\mathrm{Re}(z)+1}{4}\right\rfloor\left(\mathrm{sgn}(\mathrm{Im}(z)) + \left(\mathrm{sgn}(\mathrm{Im}(z))^2 - 1\right)\mathrm{sgn}(\mathrm{Re}(z))\right) \,/;\, \frac{2\,\mathrm{Re}(z)+1}{4} \notin \mathbb{Z}$$

06.11.16.0009.01

$$\log\Gamma(1-z) = -\pi\,i\,|\lfloor\mathrm{Re}(z)\rfloor|\,(1 - |\mathrm{sgn}(\mathrm{Im}(z))|) + \log(\pi) - \log(\sin(\pi\,(z - \lfloor\mathrm{Re}(z)\rfloor))) - \log\Gamma(z) + i\,\pi\,\lfloor\mathrm{Re}(z)\rfloor\,\mathrm{sgn}(\mathrm{Im}(z))$$

06.11.16.0010.01

$$\log\Gamma(-z) = -\log\Gamma(z) + i\,\pi\,z + 2\,i\,\pi\left\lfloor\frac{3}{4} - \frac{\arg(z)}{2\pi}\right\rfloor - \log\left(1 - e^{2\,i\,\pi\,z}\right) + \log(-2\,i\,\pi) - \log(-z) \,/;\, 0 < \arg(z) \leq \pi \bigvee -1 < \mathrm{Re}(z) \leq 1$$

06.11.16.0011.01

$$\log\Gamma(-z) = -\log\Gamma(z) + i\,\pi\,z + 2\,i\,\pi\left\lfloor\frac{3}{4} - \frac{\arg(z)}{2\pi}\right\rfloor - 2\,\pi\,i\,\theta(-\mathrm{Im}(z))\,(\theta(\mathrm{Re}(z))\,\lfloor\mathrm{Re}(z)\rfloor - \theta(-\mathrm{Re}(z))\,\lfloor-\mathrm{Re}(z)\rfloor) - 2\,\pi\,i\,\lfloor-\mathrm{Re}(z)\rfloor\,\delta_{\arg(z)-\pi} - \log\left(1 - e^{2\,i\,\pi\,z}\right) + \log(-2\,i\,\pi) - \log(-z)$$

06.11.16.0012.01

$$\log\Gamma(-z) = -\pi\,i\,|\lfloor\mathrm{Re}(z)\rfloor|\,(1 - |\mathrm{sgn}(\mathrm{Im}(z))|) + \log(\pi) - \log(-z) - \log(\sin(\pi\,(z - \lfloor\mathrm{Re}(z)\rfloor))) - \log\Gamma(z) + i\,\pi\,\lfloor\mathrm{Re}(z)\rfloor\,\mathrm{sgn}(\mathrm{Im}(z))$$

06.11.16.0003.01

$$\log\Gamma(z+1) == \log\Gamma(z) + \log(z)$$

06.11.16.0004.01

$$\log\Gamma(z-1) == \log\Gamma(z) - \log(z-1)$$

06.11.16.0005.01

$$\log\Gamma(z+n) == \log\Gamma(z) + \sum_{k=0}^{n-1}\log(k+z) \,/;\, n \in \mathbb{N}$$

06.11.16.0006.01

$$\log\Gamma(z-n) = \log\Gamma(z) - \sum_{k=1}^{n} \log(z-k) \;/;\; n \in \mathbb{N}$$

Multiple arguments

Argument involving numeric multiples of variable

06.11.16.0007.01

$$\log\Gamma(2z) = \log\Gamma\left(z+\frac{1}{2}\right) + \log\Gamma(z) + (2z-1)\log(2) - \frac{\log(\pi)}{2}$$

06.11.16.0013.01

$$\log\Gamma(3z) = \log\Gamma(z) + \log\Gamma\left(z+\frac{1}{3}\right) + \log\Gamma\left(z+\frac{2}{3}\right) + 3z\log(3) - \frac{\log(3)}{2} - \log(2\pi)$$

Argument involving symbolic multiples of variable

06.11.16.0008.01

$$\log\Gamma(mz) = \sum_{k=0}^{m-1} \log\Gamma\left(z+\frac{k}{m}\right) + mz\log(m) - \frac{1}{2}(\log(m) + (m-1)\log(2\pi)) \;/;\; m \in \mathbb{N}^+$$

Identities

Recurrence identities

Consecutive neighbors

06.11.17.0002.01

$$\log\Gamma(z) = \log\Gamma(z+1) - \log(z)$$

06.11.17.0001.01

$$\log\Gamma(z) = \log\Gamma(z-1) + \log(z-1)$$

Distant neighbors

06.11.17.0003.01

$$\log\Gamma(z) = \log\Gamma(z+n) - \sum_{k=0}^{n-1} \log(z+k) \;/;\; n \in \mathbb{N}$$

06.11.17.0004.01

$$\log\Gamma(z) = \log\Gamma(z-n) + \sum_{k=1}^{n} \log(z-k) \;/;\; n \in \mathbb{N}$$

Functional identities

Relations of special kind

06.11.17.0005.01

$$\log\Gamma(-z) = \log(\pi) - \log(-z) - \log(\sin(\pi z)) - \log\Gamma(z) \;/;\; -\frac{1}{2} \leq \operatorname{Re}(z) < \frac{\pi}{2}$$

06.11.17.0006.01

$$\log\Gamma(-z) = \log(\pi) - \log(-z) - \log(\sin(\pi z)) - \log\Gamma(z) + 2 i \pi \left\lfloor \frac{2 \operatorname{Re}(z) + 1}{4} \right\rfloor \left(\operatorname{sgn}(\operatorname{Im}(z)) + \left(\operatorname{sgn}(\operatorname{Im}(z))^2 - 1\right) \operatorname{sgn}(\operatorname{Re}(z))\right) /; \frac{2 \operatorname{Re}(z) + 1}{4} \notin \mathbb{Z}$$

Complex characteristics

Real part

06.11.19.0001.01

$$\operatorname{Re}(\log\Gamma(x + i y)) = \sum_{k=1}^{\infty} \left(\frac{x}{k} - \frac{1}{2} \log\left(1 + \frac{x^2 + 2 k x + y^2}{k^2}\right)\right) - \gamma x - \frac{1}{2} \log\left(x^2 + y^2\right)$$

Imaginary part

06.11.19.0002.01

$$\operatorname{Im}(\log\Gamma(x + i y)) = \sum_{k=1}^{\infty} \left(\frac{y}{k} - \tan^{-1}\left(\frac{k + x}{k}, \frac{y}{k}\right)\right) - \gamma y - \tan^{-1}(x, y)$$

Differentiation

Low-order differentiation

06.11.20.0001.01

$$\frac{\partial \log\Gamma(z)}{\partial z} = \psi(z)$$

06.11.20.0002.01

$$\frac{\partial^2 \log\Gamma(z)}{\partial z^2} = \psi^{(1)}(z)$$

Symbolic differentiation

06.11.20.0003.02

$$\frac{\partial^n \log\Gamma(z)}{\partial z^n} = \psi^{(n-1)}(z) /; n \in \mathbb{N}$$

Fractional integro-differentiation

06.11.20.0004.01

$$\frac{\partial^\alpha \log\Gamma(z)}{\partial z^\alpha} = \psi^{(\alpha-1)}(z)$$

06.11.20.0005.01

$$\frac{\partial^\alpha \log\Gamma(z)}{\partial z^\alpha} = -\frac{\gamma z^{1-\alpha}}{\Gamma(2-\alpha)} + z^{2-\alpha} \sum_{k=1}^{\infty} \frac{1}{k^2} \, {}_2\tilde{F}_1\left(1, 2; 3 - \alpha; -\frac{z}{k}\right) - \mathcal{FC}_{\log}^{(\alpha)}(z) \, z^{-\alpha}$$

06.11.20.0006.01

$$\frac{\partial^{\alpha}\log\Gamma(a\,z+b)}{\partial z^{\alpha}} = -\frac{a\,\gamma\,z^{1-\alpha}}{\Gamma(2-\alpha)} + z^{-\alpha}\sum_{k=1}^{\infty}\left(\frac{a\,z}{k\,\Gamma(2-\alpha)} - \frac{a\,z}{b+k}\,{}_2\tilde{F}_1\left(1,\,1;\,2-\alpha;\,-\frac{a\,z}{b+k}\right) + \frac{1}{\Gamma(1-\alpha)}\left(\frac{b}{k} - \log\left(\frac{b+k}{k}\right)\right)\right) - \frac{(\gamma\,b+\log(b))\,z^{-\alpha}}{\Gamma(1-\alpha)} - \frac{a\,z^{1-\alpha}}{b}\,{}_2\tilde{F}_1\left(1,\,1;\,2-\alpha;\,-\frac{a\,z}{b}\right)$$

Integration

Indefinite integration

Involving only one direct function

06.11.21.0006.01

$$\int \log\Gamma(z)\,dz = \psi^{(-2)}(z)$$

06.11.21.0001.01

$$\int \log\Gamma(z)\,dz = (1-\log(z))\,z - \frac{\gamma\,z^2}{2} + \sum_{k=1}^{\infty}\left(z\left(1+\frac{z}{2\,k}\right) - (k+z)\log\left(1+\frac{z}{k}\right)\right)$$

Involving one direct function and elementary functions

Involving power function

06.11.21.0002.01

$$\int z^{\alpha-1}\log\Gamma(z)\,dz = \frac{z^{\alpha}}{\alpha}\log\Gamma(z) + \frac{z^{\alpha}}{\alpha^2} + \frac{\gamma\,z^{\alpha+1}}{\alpha\,(\alpha+1)} - \frac{z^{\alpha+2}}{\alpha\,(\alpha+2)}\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}\,{}_3F_2\left(1,\,2,\,\alpha+2;\,2,\,\alpha+3;\,-\frac{z}{k+1}\right)$$

06.11.21.0007.01

$$\int z^n\log\Gamma(z)\,dz = \sum_{k=0}^{n}(-n)_k\,z^{n-k}\,\psi^{(-k-2)}(z)\;/;\;n\in\mathbb{N}$$

06.11.21.0008.01

$$\int z^n\log\Gamma(a+b\,z)\,dz = n!\sum_{j=0}^{n}\frac{(-1)^j\,\psi^{(-j-2)}(a+b\,z)\,b^{-j-1}\,z^{n-j}}{(n-j)!}\;/;\;n\in\mathbb{N}$$

Definite integration

For the direct function itself

06.11.21.0003.01

$$\int_z^{z+1}\log\Gamma(t)\,dt = (\log(z)-1)\,z + \frac{\log(2\,\pi)}{2}\;/;\;z\notin(-\infty,\,0)$$

Involving the direct function

06.11.21.0004.01

$$\int_0^1\log\Gamma(t)\sin(2\,\pi\,n\,t)\,dt = \frac{\log(2\,\pi\,n)+\gamma}{2\,\pi\,n}\;/;\;n\in\mathbb{N}^+$$

06.11.21.0005.01

$$\int_0^1 \log\Gamma(t)\cos(2\pi n t)\,dt = \frac{1}{4n} /; n \in \mathbb{N}^+$$

Operations

Limit operation

06.11.25.0001.01

$$\lim_{\epsilon \to 0} (\log\Gamma(x + i\epsilon) - \log(\Gamma(x + i\epsilon))) = -2\pi i k /;$$
$$x \in \mathbb{R} \bigwedge z_{2k-1} < x < z_{2k+1} \bigwedge \psi(z_k) = 0 \bigwedge 1.4 < z_0 < 1.5 \bigwedge -0.6 < z_1 < -0.5 \bigwedge -1.6 < z_2 < -1.5 \bigwedge -2.7 < z_3 < -2.6 \bigwedge -3.7 < z_4 < -3.6 \bigwedge -4.7 < z_5 < -4.6 \bigwedge -5.7 < z_6 < -5.6 \bigwedge -6.7 < z_7 < -6.6 \bigwedge \ldots \bigwedge k \in \mathbb{N}^+$$

06.11.25.0002.01

$$\lim_{\epsilon \to 0} (\log\Gamma(x - i\epsilon) - \log(\Gamma(x - i\epsilon))) = 2\pi i k /;$$
$$x \in \mathbb{R} \bigwedge z_{2k-1} < x < z_{2k+1} \bigwedge \psi(z_k) = 0 \bigwedge 1.4 < z_0 < 1.5 \bigwedge -0.6 < z_1 < -0.5 \bigwedge -1.6 < z_2 < -1.5 \bigwedge -2.7 < z_3 < -2.6 \bigwedge -3.7 < z_4 < -3.6 \bigwedge -4.7 < z_5 < -4.6 \bigwedge -5.7 < z_6 < -5.6 \bigwedge -6.7 < z_7 < -6.6 \bigwedge \ldots \bigwedge k \in \mathbb{N}^+$$

Representations through more general functions

Through other functions

06.11.26.0001.01

$$\log\Gamma(z) = \int_1^z \psi(t)\,dt$$

06.11.26.0002.01

$$\log\Gamma(z) = \frac{\partial^{-\nu-1}\psi^{(\nu)}(z)}{\partial z^{-\nu-1}}$$

06.11.26.0003.01

$$\log\Gamma(z) = \zeta^{(1,0)}(0, z) + \frac{1}{2}\log(2\pi) /; \mathrm{Re}(z) > 0$$

06.11.26.0004.01

$$\log\Gamma(z) = \zeta^{(1,0)}(0, z) + \frac{1}{2}\log(2\pi) + (2\,\theta(\mathrm{Im}(z)) - 1)\, i\pi\, \mathrm{Re}(\lfloor z \rfloor) + \frac{1}{2}\left(1 + (-1)^{\lfloor -\mathrm{Re}(z) \rfloor + \lfloor \mathrm{Re}(z) \rfloor}\right) i\pi\, \theta(-\mathrm{Im}(z))\, \theta(-\mathrm{Re}(z)) /; \mathrm{Re}(z) > 0$$

Representations through equivalent functions

With related functions

06.11.27.0001.01

$$\log\Gamma(z) = \log(\Gamma(z)) /; 0 < \mathrm{Re}(z) \leq 2 \bigwedge |\mathrm{Im}(z)| \leq \frac{7}{2}$$

06.11.27.0002.01

$$\log\Gamma(z) = 2 i\pi k(z) + \log(\Gamma(z)) /; k(z) = \int_0^z \theta(-\mathrm{Re}(\Gamma(t)))\, |\mathrm{Im}(\Gamma(t)\,\psi(t))|\, \delta(\mathrm{Im}(\Gamma(t)))\,dt \in \mathbb{Z}$$

06.11.27.0003.01

$$\log\Gamma(x) = \log(\Gamma(x)) + 2\,i\,\pi \left\lfloor \frac{x}{2} \right\rfloor \theta(-x) \;/;\; x \in \mathbb{R} \land \neg\,(x \in \mathbb{Z} \land x \leq 0)$$

History

– J. Stirling (1730) used series for $\log(n!)$ to derive his famous asymptotic formula

– C. Siegel

– K. F. Gauss

– C. J. Malmstén

– O. Schlömilch

– J. P. M. Binet (1843)

– E. E. Kummer (1847)

– G. Plana (1847)

– M.A. Stern (1847) proved convergence of the Stirling series for derivative of $\log(\Gamma(z))$

– Ch. Hermite (1900) proved convergence of Stirling's series for $\log(\Gamma(z+1))$ to this function if z is a complex number

– J. Keiper (1990) (introduced logΓ(z) in *Mathematica*)

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