MathieuC

Notations

Traditional name

Even Mathieu function

Traditional notation

$\mathrm{Ce}(a, q, z)$

Mathematica StandardForm notation

`MathieuC[a, q, z]`

Primary definition

11.01.02.0001.01

$\mathrm{Ce}(a, q, z)$

$\mathrm{Ce}(a, q, z)$ is the even Mathieu function with characteristic value a and parameter q. It is defined as the even in z solution $w(z) == \mathrm{Ce}(a, q, z)$ of the Mathieu differential equation $w''(z) + (a - 2\,q\cos(2\,z))\,w(z) == 0$, which satisfies for $q == 0$ relation $\mathrm{Ce}(a, 0, z) == \cos\left(\sqrt{a}\; z\right)$. It is analytical function in the variables a, q and z. It is a periodical only in z function for special values of parameter a (so called characteristic values $a == a_r(q)$ with integer or rational numbers r, which makes even solutions of the form $e^{i r z} f(z)$ where $f(z)$ is an even function of z with period 2π).

Specific values

Specialized values

For fixed *a*, *z*

11.01.03.0001.01

$\mathrm{Ce}(a, 0, z) == \cos\left(\sqrt{a}\; z\right)$

General characteristics

Domain and analyticity

$\mathrm{Ce}(a, q, z)$ is an analytical function of a, q, z which is defined in $\mathbb{C}^3$.

11.01.04.0001.01

$(a * q * z) \longrightarrow \mathrm{Ce}(a, q, z) :: (\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$

Symmetries and periodicities

Parity

$\mathrm{Ce}(a, q, z)$ is an even function with respect to z.

11.01.04.0002.01

$$\mathrm{Ce}(a, q, -z) == \mathrm{Ce}(a, q, z)$$

Mirror symmetry

11.01.04.0003.01

$$\mathrm{Ce}(\overline{a}, \overline{q}, \overline{z}) == \overline{\mathrm{Ce}(a, q, z)}$$

Periodicity

No periodicity

Series representations

Generalized power series

Expansions at generic point $z == z_0$

For the function itself

11.01.06.0011.01

$$\mathrm{Ce}(a, q, z) \propto \mathrm{Ce}(a, q, z_0) + \mathrm{Ce}^{(0,0,1)}(a, q, z_0)(z - z_0) + \frac{1}{2}(2 q \cos(2 z_0) - a)\,\mathrm{Ce}(a, q, z_0)(z - z_0)^2 + \frac{1}{6}\left((2 q \cos(2 z_0) - a)\,\mathrm{Ce}^{(0,0,1)}(a, q, z_0) - 4 q \sin(2 z_0)\,\mathrm{Ce}(a, q, z_0)\right)(z - z_0)^3 + \frac{1}{24}\left(\left(a^2 + 4 q \cos(2 z_0)(-a + q \cos(2 z_0) - 2)\right)\mathrm{Ce}(a, q, z_0) - 8 q \sin(2 z_0)\,\mathrm{Ce}^{(0,0,1)}(a, q, z_0)\right)(z - z_0)^4 + \frac{1}{120}\left(\left(a^2 + 4 q \cos(2 z_0)(-a + q \cos(2 z_0) - 6)\right)\mathrm{Ce}^{(0,0,1)}(a, q, z_0) + 16 q (a - 2 q \cos(2 z_0) + 1)\sin(2 z_0)\,\mathrm{Ce}(a, q, z_0)\right)(z - z_0)^5 + \ldots /; (z \to z_0)$$

11.01.06.0012.01

$$\mathrm{Ce}(a, q, z) \propto \mathrm{Ce}(a, q, z_0)(1 + O(z - z_0))$$

Expansions at $z == 0$

11.01.06.0001.01

$$\mathrm{Ce}(a_{2n}(q), q, z) == \sum_{k=0}^{\infty} A_{2k}^{2n} \cos(2 k z) /; a_{2n}(q) A_0^{2n} - q A_2^{2n} == 0 \bigwedge (a_{2n}(q) - 4) A_2^{2n} - q\left(A_4^{2n} - 2 A_0^{2n}\right) == 0 \bigwedge \left(a_{2n}(q) - 4 k^2\right) A_{2k}^{2n} - q\left(A_{2k-2}^{2n} + A_{2k+2}^{2n}\right) == 0 \bigwedge 2\left(A_0^{2n}\right)^2 + \sum_{k=1}^{\infty} A_{2k}^{2n} == 1 \bigwedge n \in \mathbb{Z}$$

11.01.06.0002.01

$$\operatorname{Ce}(a_{2n+1}(q), q, z) = \sum_{k=0}^{\infty} A_{2k+1}^{2n+1} \cos((2k+1)z) \;/;$$

$$(a_{2n+1}(q) - q - 1) A_1^{2n+1} - q A_3^{2n+1} = 0 \bigwedge \left(a_{2n+1}(q) - (2k+1)^2\right) A_{2k+1}^{2n+1} - q\left(A_{2k-1}^{2n+1} + A_{2k+3}^{2n+1}\right) = 0 \bigwedge \sum_{k=0}^{\infty} A_{2k+1}^{2n+1} = 1 \bigwedge n \in \mathbb{Z}$$

Expansions at $q = 0$

11.01.06.0003.01

$$\operatorname{Ce}(a_r(q), q, z) \propto$$

$$\cos(rz) + \frac{1}{4}\left(\frac{\cos((r-2)z)}{r-1} - \frac{\cos((r+2)z)}{r+1}\right) q + \frac{1}{32}\left(\frac{\cos((r-4)z)}{(r-2)(r-1)} - \frac{2\left(r^2+1\right)\cos(rz)}{(r-1)^2(r+1)^2} + \frac{\cos((r+4)z)}{(r+1)(r+2)}\right) q^2 + \frac{1}{384}$$

$$\left(\frac{\cos((r-6)z)}{(r-3)(r-2)(r-1)} - \frac{3\left(r^3-r^2-r-11\right)\cos((r-2)z)}{(r-2)(r-1)^3(r+1)^2} + \frac{3\left(r^3+r^2-r+11\right)\cos((r+2)z)}{(r-1)^2(r+1)^3(r+2)} - \frac{\cos((r+6)z)}{(r+1)(r+2)(r+3)}\right) q^3 +$$

$$\frac{1}{6144}\left(\frac{\cos((r-8)z)}{(r-4)(r-3)(r-2)(r-1)} - \frac{4\left(r^3-r^2-7r-29\right)\cos((r-4)z)}{(r-3)(r-2)(r-1)^3(r+1)^2} + \frac{6\left(r^8-15r^6-185r^4+675r^2+316\right)\cos(rz)}{(r-2)^2(r-1)^4(r+1)^4(r+2)^2} -\right.$$

$$\left.\frac{4\left(r^3+r^2-7r+29\right)\cos((r+4)z)}{(r-1)^2(r+1)^3(r+2)(r+3)} + \frac{\cos((r+8)z)}{(r+1)(r+2)(r+3)(r+4)}\right) q^4 +$$

$$\frac{1}{122\,880}\left(\frac{\cos((r-10)z)}{(r-5)(r-4)(r-3)(r-2)(r-1)} - \frac{5\left(r^3-r^2-17r-55\right)\cos((r-6)z)}{(r-4)(r-3)(r-2)(r-1)^3(r+1)^2} +\right.$$

$$\left(10\left(r^9-r^8-31r^7-5r^6-273r^5+2457r^4+1931r^3-6335r^2-3572r-7564\right)\cos((r-2)z)\right) \Big/$$

$$\left((r-3)(r-2)^2(r-1)^5(r+1)^4(r+2)^2\right) -$$

$$\left(10\left(r^9+r^8-31r^7+5r^6-273r^5-2457r^4+1931r^3+6335r^2-3572r+7564\right)\cos((r+2)z)\right) \Big/ \left((r-2)^2(r-1)^4\right.$$

$$\left.\left.(r+1)^5(r+2)^2(r+3)\right) + \frac{5\left(r^3+r^2-17r+55\right)\cos((r+6)z)}{(r-1)^2(r+1)^3(r+2)(r+3)(r+4)} - \frac{\cos((r+10)z)}{(r+1)(r+2)(r+3)(r+4)(r+5)}\right) q^5 +$$

$$\frac{1}{2\,949\,120}\left(\frac{\cos((r-12)z)}{(r-6)(r-5)(r-4)(r-3)(r-2)(r-1)} - \frac{6\left(r^3-r^2-31r-89\right)\cos((r-8)z)}{(r-5)(r-4)(r-3)(r-2)(r-1)^3(r+1)^2} +\right.$$

$$\left(15\left(r^{10}-3r^9-53r^8+69r^7+145r^6+8211r^5-16\,879r^4-32\,025r^3+32\,954r^2+16\,404r+71\,528\right)\right.$$

$$\left.\cos((r-4)z)\right) \Big/ \left((r-4)(r-3)(r-2)^3(r-1)^5(r+1)^4(r+2)^2\right) -$$

$$\left(20\left(r^{14}-72r^{12}+597r^{10}+75\,244r^8-718\,317r^6+153\,312r^4+4\,883\,287r^2+1\,329\,084\right)\cos(rz)\right) \Big/$$

$$\left((r-3)^2(r-2)^2(r-1)^6(r+1)^6(r+2)^2(r+3)^2\right) +$$

$$\left(15\left(r^{10}+3r^9-53r^8-69r^7+145r^6-8211r^5-16\,879r^4+32\,025r^3+32\,954r^2-16\,404r+71\,528\right)\right.$$

$$\left.\cos((r+4)z)\right) \Big/ \left((r-2)^2(r-1)^4(r+1)^5(r+2)^3(r+3)(r+4)\right) - \frac{6\left(r^3+r^2-31r+89\right)\cos((r+8)z)}{(r-1)^2(r+1)^3(r+2)(r+3)(r+4)(r+5)} +$$

$$\left.\frac{\cos((r+12)z)}{(r+1)(r+2)(r+3)(r+4)(r+5)(r+6)}\right) q^6 + O\left(q^7\right) \;/;\; \neg\,(r \in \mathbb{Z} \land -6 \le r \le 6)$$

11.01.06.0004.01

$$\operatorname{Ce}(a_0(q), q, z) \propto \frac{1}{\sqrt{2}}\left(1 - \frac{1}{2}\cos(2z)\,q - \left(\frac{1}{16} - \frac{1}{32}\cos(4z)\right)q^2 + \left(\frac{11}{128}\cos(2z) - \frac{\cos(6z)}{1152}\right)q^3 + \left(\frac{79}{4096} - \frac{29\cos(4z)}{4608} + \frac{\cos(8z)}{73\,728}\right)q^4 - \left(\frac{1891\cos(2z)}{73\,728} - \frac{55\cos(6z)}{294\,912} + \frac{\cos(10z)}{7\,372\,800}\right)q^5 - \left(\frac{36\,919}{5\,308\,416} - \frac{8941\cos(4z)}{4\,718\,592} + \frac{89\cos(8z)}{29\,491\,200} - \frac{\cos(12z)}{1\,061\,683\,200}\right)q^6 + \left(\frac{1\,521\,691\cos(2z)}{169\,869\,312} - \frac{26\,641\cos(6z)}{471\,859\,200} + \frac{131\cos(10z)}{4\,246\,732\,800} - \frac{\cos(14z)}{208\,089\,907\,200}\right)q^7 + \left(\frac{58\,304\,143}{21\,743\,271\,936} - \frac{11\,239\,489\cos(4z)}{16\,986\,931\,200} + \frac{62\,299\cos(8z)}{67\,947\,724\,800} - \frac{181\cos(12z)}{832\,359\,628\,800} + \frac{\cos(16z)}{53\,271\,016\,243\,200}\right)q^8 - \left(\frac{3\,664\,886\,951\cos(2z)}{1\,087\,163\,596\,800} - \frac{48\,249\,209\cos(6z)}{2\,446\,118\,092\,800} + \frac{125\,119\cos(10z)}{13\,317\,754\,060\,800} - \frac{239\cos(14z)}{213\,084\,064\,972\,800} + \frac{\cos(18z)}{17\,259\,809\,262\,796\,800}\right)q^9 - \left(\frac{233\,552\,934\,751}{217\,432\,719\,360\,000} - \frac{7\,785\,932\,591\cos(4z)}{31\,310\,311\,587\,840} + \frac{153\,642\,299\cos(8z)}{479\,439\,146\,188\,800} - \frac{226\,201\cos(12z)}{3\,409\,345\,039\,564\,800} + \frac{61\cos(16z)}{13\,807\,847\,410\,237\,440} - \frac{\cos(20z)}{6\,903\,923\,705\,118\,720\,000}\right)q^{10} + O[q]^{11}\right)$$

11.01.06.0005.01

$$\operatorname{Ce}(a_1(q), q, z) \propto \cos(z) - \frac{1}{8}\cos(3z)\, q + \left(-\frac{\cos(z)}{128} - \frac{1}{64}\cos(3z) + \frac{1}{192}\cos(5z)\right) q^2 + \left(-\frac{\cos(z)}{512} + \frac{\cos(3z)}{3072} + \frac{\cos(5z)}{1152} - \frac{\cos(7z)}{9216}\right) q^3 + \left(-\frac{37\cos(z)}{294\,912} + \frac{49\cos(3z)}{73\,728} - \frac{\cos(7z)}{49\,152} + \frac{\cos(9z)}{737\,280}\right) q^4 + \left(\frac{121\cos(z)}{1\,769\,472} + \frac{317\cos(3z)}{2\,359\,296} - \frac{41\cos(5z)}{1\,179\,648} - \frac{\cos(7z)}{5\,898\,240} + \frac{\cos(9z)}{3\,686\,400} - \frac{\cos(11z)}{88\,473\,600}\right) q^5 + \left(\frac{8105\cos(z)}{339\,738\,624} - \frac{103\cos(3z)}{56\,623\,104} - \frac{731\cos(5z)}{94\,371\,840} + \frac{379\cos(7z)}{471\,859\,200} + \frac{\cos(9z)}{283\,115\,520} - \frac{\cos(11z)}{424\,673\,280} + \frac{\cos(13z)}{14\,863\,564\,800}\right) q^6 + \left(\frac{481\cos(z)}{226\,492\,416} - \frac{102\,547\cos(3z)}{13\,589\,544\,960} - \frac{659\cos(5z)}{11\,324\,620\,800} + \frac{12\,677\cos(7z)}{67\,947\,724\,800} - \frac{181\cos(9z)}{16\,986\,931\,200} - \frac{\cos(11z)}{26\,424\,115\,200} + \frac{\cos(13z)}{69\,363\,302\,400} - \frac{\cos(15z)}{3\,329\,438\,515\,200}\right) q^7 + \left(-\frac{1\,237\,783\cos(z)}{1\,449\,551\,462\,400} - \frac{940\,781\cos(3z)}{543\,581\,798\,400} + \frac{322\,897\cos(5z)}{815\,372\,697\,600} + \frac{9413\cos(7z)}{3\,261\,490\,790\,400} - \frac{2143\cos(9z)}{845\,571\,686\,400} + \frac{1229\cos(11z)}{13\,317\,754\,060\,800} + \frac{\cos(13z)}{3\,805\,072\,588\,800} - \frac{\cos(15z)}{15\,220\,290\,355\,200} + \frac{\cos(17z)}{958\,878\,292\,377\,600}\right) q^8 + \left(-\frac{11\,221\,967\cos(z)}{32\,614\,907\,904\,000} + \frac{506\,831\cos(3z)}{104\,367\,705\,292\,800} + \frac{310\,133\cos(5z)}{3\,131\,031\,158\,784} - \frac{5\,039\,101\cos(7z)}{547\,930\,452\,787\,200} - \frac{16\,013\cos(9z)}{319\,626\,097\,459\,200} + \frac{3167\cos(11z)}{142\,056\,043\,315\,200} - \frac{481\cos(13z)}{852\,336\,259\,891\,200} - \frac{\cos(15z)}{767\,102\,633\,902\,080} + \frac{\cos(17z)}{4\,314\,952\,315\,699\,200} - \frac{\cos(19z)}{345\,196\,185\,255\,936\,000}\right) q^9 + \left(-\frac{4\,539\,285\,691\cos(z)}{125\,241\,246\,351\,360\,000} + \frac{1\,282\,939\,901\cos(3z)}{12\,524\,124\,635\,136\,000} + \frac{10\,304\,813\cos(5z)}{5\,844\,591\,496\,396\,800} - \frac{48\,740\,801\cos(7z)}{20\,456\,070\,237\,388\,800} + \frac{7\,516\,703\cos(9z)}{61\,368\,210\,712\,166\,400} + \frac{30\,773\cos(11z)}{61\,368\,210\,712\,166\,400} - \frac{25\,379\cos(13z)}{184\,104\,632\,136\,499\,200} + \frac{2839\cos(15z)}{1\,104\,627\,792\,818\,995\,200} + \frac{\cos(17z)}{204\,560\,702\,373\,888\,000} - \frac{\cos(19z)}{1\,534\,205\,267\,804\,160\,000} + \frac{\cos(21z)}{151\,886\,321\,512\,611\,840\,000}\right) q^{10} + O(q^{11})$$

11.01.06.0006.01

$$\mathrm{Ce}(a_2(q),\, q,\, z) \propto \cos(z) - \frac{1}{8}\cos(3\,z)\,q + \left(-\frac{\cos(z)}{128} - \frac{1}{64}\cos(3\,z) + \frac{1}{192}\cos(5\,z)\right)q^2 +$$
$$\left(-\frac{\cos(z)}{512} + \frac{\cos(3\,z)}{3072} + \frac{\cos(5\,z)}{1152} - \frac{\cos(7\,z)}{9216}\right)q^3 + \left(-\frac{37\cos(z)}{294\,912} + \frac{49\cos(3\,z)}{73\,728} - \frac{\cos(7\,z)}{49\,152} + \frac{\cos(9\,z)}{737\,280}\right)q^4 +$$
$$\left(\frac{121\cos(z)}{1\,769\,472} + \frac{317\cos(3\,z)}{2\,359\,296} - \frac{41\cos(5\,z)}{1\,179\,648} - \frac{\cos(7\,z)}{5\,898\,240} + \frac{\cos(9\,z)}{3\,686\,400} - \frac{\cos(11\,z)}{88\,473\,600}\right)q^5 +$$
$$\left(\frac{8105\cos(z)}{339\,738\,624} - \frac{103\cos(3\,z)}{56\,623\,104} - \frac{731\cos(5\,z)}{94\,371\,840} + \frac{379\cos(7\,z)}{471\,859\,200} + \frac{\cos(9\,z)}{283\,115\,520} - \frac{\cos(11\,z)}{424\,673\,280} + \frac{\cos(13\,z)}{14\,863\,564\,800}\right)q^6 +$$
$$\left(\frac{481\cos(z)}{226\,492\,416} - \frac{102\,547\cos(3\,z)}{13\,589\,544\,960} - \frac{659\cos(5\,z)}{11\,324\,620\,800} + \frac{12\,677\cos(7\,z)}{67\,947\,724\,800} - \right.$$
$$\left.\frac{181\cos(9\,z)}{16\,986\,931\,200} - \frac{\cos(11\,z)}{26\,424\,115\,200} + \frac{\cos(13\,z)}{69\,363\,302\,400} - \frac{\cos(15\,z)}{3\,329\,438\,515\,200}\right)q^7 +$$
$$\left(-\frac{1\,237\,783\cos(z)}{1\,449\,551\,462\,400} - \frac{940\,781\cos(3\,z)}{543\,581\,798\,400} + \frac{322\,897\cos(5\,z)}{815\,372\,697\,600} + \frac{9413\cos(7\,z)}{3\,261\,490\,790\,400} - \frac{2143\cos(9\,z)}{845\,571\,686\,400} + \right.$$
$$\left.\frac{1229\cos(11\,z)}{13\,317\,754\,060\,800} + \frac{\cos(13\,z)}{3\,805\,072\,588\,800} - \frac{\cos(15\,z)}{15\,220\,290\,355\,200} + \frac{\cos(17\,z)}{958\,878\,292\,377\,600}\right)q^8 +$$
$$\left(-\frac{11\,221\,967\cos(z)}{32\,614\,907\,904\,000} + \frac{506\,831\cos(3\,z)}{104\,367\,705\,292\,800} + \frac{310\,133\cos(5\,z)}{3\,131\,031\,158\,784} - \frac{5\,039\,101\cos(7\,z)}{547\,930\,452\,787\,200} - \right.$$
$$\frac{16\,013\cos(9\,z)}{319\,626\,097\,459\,200} + \frac{3167\cos(11\,z)}{142\,056\,043\,315\,200} - \frac{481\cos(13\,z)}{852\,336\,259\,891\,200} -$$
$$\left.\frac{\cos(15\,z)}{767\,102\,633\,902\,080} + \frac{\cos(17\,z)}{4\,314\,952\,315\,699\,200} - \frac{\cos(19\,z)}{345\,196\,185\,255\,936\,000}\right)q^9 +$$
$$\left(-\frac{4\,539\,285\,691\cos(z)}{125\,241\,246\,351\,360\,000} + \frac{1\,282\,939\,901\cos(3\,z)}{12\,524\,124\,635\,136\,000} + \frac{10\,304\,813\cos(5\,z)}{5\,844\,591\,496\,396\,800} - \frac{48\,740\,801\cos(7\,z)}{20\,456\,070\,237\,388\,800} + \right.$$
$$\frac{7\,516\,703\cos(9\,z)}{61\,368\,210\,712\,166\,400} + \frac{30\,773\cos(11\,z)}{61\,368\,210\,712\,166\,400} - \frac{25\,379\cos(13\,z)}{184\,104\,632\,136\,499\,200} + \frac{2839\cos(15\,z)}{1\,104\,627\,792\,818\,995\,200} +$$
$$\left.\frac{\cos(17\,z)}{204\,560\,702\,373\,888\,000} - \frac{\cos(19\,z)}{1\,534\,205\,267\,804\,160\,000} + \frac{\cos(21\,z)}{151\,886\,321\,512\,611\,840\,000}\right)q^{10} + O(q^{11})$$

11.01.06.0007.01

$$
\begin{aligned}
&\operatorname{Ce}(a_3(q),\, q,\, z) \propto \cos(3z) + \left(\frac{\cos(z)}{8} - \frac{1}{16}\cos(5z)\right) q + \\
&\left(\frac{\cos(z)}{64} - \frac{5}{512}\cos(3z) + \frac{1}{640}\cos(7z)\right) q^2 + \left(-\frac{\cos(z)}{4096} - \frac{1}{512}\cos(3z) + \frac{11\cos(5z)}{40960} - \frac{\cos(9z)}{46080}\right) q^3 + \\
&\left(-\frac{21\cos(z)}{32768} - \frac{1621\cos(3z)}{13107200} + \frac{\cos(5z)}{16384} - \frac{11\cos(7z)}{2949120} + \frac{\cos(11z)}{5160960}\right) q^4 + \\
&\left(-\frac{14061\cos(z)}{104857600} + \frac{9\cos(3z)}{131072} + \frac{12329\cos(5z)}{1887436800} - \frac{3\cos(7z)}{3276800} + \frac{\cos(9z)}{33030144} - \frac{\cos(13z)}{825753600}\right) q^5 + \\
&\left(\frac{699\cos(z)}{838860800} + \frac{13050583\cos(3z)}{543581798400} - \frac{533\cos(5z)}{209715200} - \frac{76679\cos(7z)}{528482304000} + \frac{17\cos(9z)}{2123366400} - \frac{\cos(11z)}{6606028800} + \right. \\
&\qquad \left. \frac{\cos(15z)}{178362777600}\right) q^6 + \left(\frac{31826419\cos(z)}{4348654387200} + \frac{70123\cos(3z)}{33554432000} - \frac{326021051\cos(5z)}{304405807104000} + \frac{1319\cos(7z)}{27179089920} + \right. \\
&\qquad \left. \frac{7831\cos(9z)}{4227858432000} - \frac{37\cos(11z)}{832359628800} + \frac{\cos(13z)}{2283043553280} - \frac{\cos(17z)}{49941577728000}\right) q^7 + \\
&\left(\frac{300245939\cos(z)}{173946175488000} - \frac{1193766593741\cos(3z)}{1363738015825920000} - \frac{10194121\cos(5z)}{86973087744000} + \frac{55617547\cos(7z)}{2435246456832000} - \frac{30253\cos(9z)}{53271016243200} - \right. \\
&\qquad \left. \frac{28361\cos(11z)}{1826434842624000} + \frac{17\cos(13z)}{106542032486400} - \frac{\cos(15z)}{3196260974592000} + \frac{\cos(19z)}{17579435360256000}\right) q^8 + \\
&\left(\frac{64306539779\cos(z)}{10909904126607360000} - \frac{1087026917\cos(3z)}{3131031158784000} + \frac{767116375621\cos(5z)}{21819808253214720000} + \frac{468897223\cos(7z)}{170467251978240000} - \right. \\
&\qquad \frac{21665887\cos(9z)}{75144747810816000} + \frac{7663\cos(11z)}{1704672519782400} + \frac{189053\cos(13z)}{2045607023738880000} - \\
&\qquad \left. \frac{23\cos(15z)}{69039237051187200} - \frac{\cos(17z)}{281270965764096000} - \frac{\cos(21z)}{7594316075630592000}\right) q^9 + \\
&\left(-\frac{78221421983189\cos(z)}{785513097115729920000} - \frac{3555989290829\cos(3z)}{99747694871838720000} + \frac{1595308088447\cos(5z)}{98189137139466240000} - \right. \\
&\qquad \frac{16802983705367\cos(7z)}{23565392913471897600000} - \frac{50000417\cos(9z)}{1363738015825920000} + \frac{44442089\cos(11z)}{18410463213649920000} - \\
&\qquad \frac{113461\cos(13z)}{4418511171275980800} - \frac{74069\cos(15z)}{180013418089021440000} + \frac{\cos(17z)}{13807847410237440000} + \\
&\qquad \left. \frac{\cos(19z)}{48603622884035788800} + \frac{\cos(23z)}{3949044359327907840000}\right) q^{10} + O\left(q^{11}\right)
\end{aligned}
$$

11.01.06.0008.01

$$\begin{aligned}
&\mathrm{Ce}(a_4(q),\, q,\, z) \propto \cos(3\,z) + \left(\frac{\cos(z)}{8} - \frac{1}{16}\cos(5\,z)\right) q + \\
&\left(\frac{\cos(z)}{64} - \frac{5}{512}\cos(3\,z) + \frac{1}{640}\cos(7\,z)\right) q^2 + \left(-\frac{\cos(z)}{4096} - \frac{1}{512}\cos(3\,z) + \frac{11\cos(5\,z)}{40\,960} - \frac{\cos(9\,z)}{46\,080}\right) q^3 + \\
&\left(-\frac{21\cos(z)}{32\,768} - \frac{1621\cos(3\,z)}{13\,107\,200} + \frac{\cos(5\,z)}{16\,384} - \frac{11\cos(7\,z)}{2\,949\,120} + \frac{\cos(11\,z)}{5\,160\,960}\right) q^4 + \\
&\left(-\frac{14\,061\cos(z)}{104\,857\,600} + \frac{9\cos(3\,z)}{131\,072} + \frac{12\,329\cos(5\,z)}{1\,887\,436\,800} - \frac{3\cos(7\,z)}{3\,276\,800} + \frac{\cos(9\,z)}{33\,030\,144} - \frac{\cos(13\,z)}{825\,753\,600}\right) q^5 + \\
&\left(\frac{699\cos(z)}{838\,860\,800} + \frac{13\,050\,583\cos(3\,z)}{543\,581\,798\,400} - \frac{533\cos(5\,z)}{209\,715\,200} - \frac{76\,679\cos(7\,z)}{528\,482\,304\,000} + \frac{17\cos(9\,z)}{2\,123\,366\,400} - \frac{\cos(11\,z)}{6\,606\,028\,800} + \right. \\
&\qquad \left. \frac{\cos(15\,z)}{178\,362\,777\,600}\right) q^6 + \left(\frac{31\,826\,419\cos(z)}{4\,348\,654\,387\,200} + \frac{70\,123\cos(3\,z)}{33\,554\,432\,000} - \frac{326\,021\,051\cos(5\,z)}{304\,405\,807\,104\,000} + \frac{1319\cos(7\,z)}{27\,179\,089\,920} + \right. \\
&\qquad \left. \frac{7831\cos(9\,z)}{4\,227\,858\,432\,000} - \frac{37\cos(11\,z)}{832\,359\,628\,800} + \frac{\cos(13\,z)}{2\,283\,043\,553\,280} - \frac{\cos(17\,z)}{49\,941\,577\,728\,000}\right) q^7 + \\
&\left(\frac{300\,245\,939\cos(z)}{173\,946\,175\,488\,000} - \frac{1\,193\,766\,593\,741\cos(3\,z)}{1\,363\,738\,015\,825\,920\,000} - \frac{10\,194\,121\cos(5\,z)}{86\,973\,087\,744\,000} + \frac{55\,617\,547\cos(7\,z)}{2\,435\,246\,456\,832\,000} - \frac{30\,253\cos(9\,z)}{53\,271\,016\,243\,200} - \right. \\
&\qquad \left. \frac{28\,361\cos(11\,z)}{1\,826\,434\,842\,624\,000} + \frac{17\cos(13\,z)}{106\,542\,032\,486\,400} - \frac{\cos(15\,z)}{3\,196\,260\,974\,592\,000} + \frac{\cos(19\,z)}{17\,579\,435\,360\,256\,000}\right) q^8 + \\
&\left(\frac{64\,306\,539\,779\cos(z)}{10\,909\,904\,126\,607\,360\,000} - \frac{1\,087\,026\,917\cos(3\,z)}{3\,131\,031\,158\,784\,000} + \frac{767\,116\,375\,621\cos(5\,z)}{21\,819\,808\,253\,214\,720\,000} + \frac{468\,897\,223\cos(7\,z)}{170\,467\,251\,978\,240\,000} - \right. \\
&\qquad \frac{21\,665\,887\cos(9\,z)}{75\,144\,747\,810\,816\,000} + \frac{7663\cos(11\,z)}{1\,704\,672\,519\,782\,400} + \frac{189\,053\cos(13\,z)}{2\,045\,607\,023\,738\,880\,000} - \\
&\qquad \left. \frac{23\cos(15\,z)}{69\,039\,237\,051\,187\,200} - \frac{\cos(17\,z)}{281\,270\,965\,764\,096\,000} - \frac{\cos(21\,z)}{7\,594\,316\,075\,630\,592\,000}\right) q^9 + \\
&\left(-\frac{78\,221\,421\,983\,189\cos(z)}{785\,513\,097\,115\,729\,920\,000} - \frac{3\,555\,989\,290\,829\cos(3\,z)}{99\,747\,694\,871\,838\,720\,000} + \frac{1\,595\,308\,088\,447\cos(5\,z)}{98\,189\,137\,139\,466\,240\,000} - \right. \\
&\qquad \frac{16\,802\,983\,705\,367\cos(7\,z)}{23\,565\,392\,913\,471\,897\,600\,000} - \frac{50\,000\,417\cos(9\,z)}{1\,363\,738\,015\,825\,920\,000} + \frac{44\,442\,089\cos(11\,z)}{18\,410\,463\,213\,649\,920\,000} - \\
&\qquad \frac{113\,461\cos(13\,z)}{4\,418\,511\,171\,275\,980\,800} - \frac{74\,069\cos(15\,z)}{180\,013\,418\,089\,021\,440\,000} + \frac{\cos(17\,z)}{13\,807\,847\,410\,237\,440\,000} + \\
&\qquad \left. \frac{\cos(19\,z)}{48\,603\,622\,884\,035\,788\,800} + \frac{\cos(23\,z)}{3\,949\,044\,359\,327\,907\,840\,000}\right) q^{10} + O\left(q^{11}\right)
\end{aligned}$$

11.01.06.0009.01

$$\operatorname{Ce}(a_5(q),\, q,\, z) \propto \cos(3\,z) + \left(\frac{\cos(z)}{8} - \frac{1}{16}\cos(5\,z)\right) q + \left(\frac{\cos(z)}{64} - \frac{5}{512}\cos(3\,z) + \frac{1}{640}\cos(7\,z)\right) q^2 + \left(-\frac{\cos(z)}{4096} - \frac{1}{512}\cos(3\,z) + \frac{11\cos(5\,z)}{40\,960} - \frac{\cos(9\,z)}{46\,080}\right) q^3 +$$
$$\left(-\frac{21\cos(z)}{32\,768} - \frac{1621\cos(3\,z)}{13\,107\,200} + \frac{\cos(5\,z)}{16\,384} - \frac{11\cos(7\,z)}{2\,949\,120} + \frac{\cos(11\,z)}{5\,160\,960}\right) q^4 +$$
$$\left(-\frac{14\,061\cos(z)}{104\,857\,600} + \frac{9\cos(3\,z)}{131\,072} + \frac{12\,329\cos(5\,z)}{1\,887\,436\,800} - \frac{3\cos(7\,z)}{3\,276\,800} + \frac{\cos(9\,z)}{33\,030\,144} - \frac{\cos(13\,z)}{825\,753\,600}\right) q^5 +$$
$$\left(\frac{699\cos(z)}{838\,860\,800} + \frac{13\,050\,583\cos(3\,z)}{543\,581\,798\,400} - \frac{533\cos(5\,z)}{209\,715\,200} - \frac{76\,679\cos(7\,z)}{528\,482\,304\,000} + \frac{17\cos(9\,z)}{2\,123\,366\,400} - \frac{\cos(11\,z)}{6\,606\,028\,800} + \frac{\cos(15\,z)}{178\,362\,777\,600}\right) q^6 +$$
$$\left(\frac{31\,826\,419\cos(z)}{4\,348\,654\,387\,200} + \frac{70\,123\cos(3\,z)}{33\,554\,432\,000} - \frac{326\,021\,051\cos(5\,z)}{304\,405\,807\,104\,000} + \frac{1319\cos(7\,z)}{27\,179\,089\,920} + \frac{7831\cos(9\,z)}{4\,227\,858\,432\,000} - \frac{37\cos(11\,z)}{832\,359\,628\,800} + \frac{\cos(13\,z)}{2\,283\,043\,553\,280} - \frac{\cos(17\,z)}{49\,941\,577\,728\,000}\right) q^7 +$$
$$\left(\frac{300\,245\,939\cos(z)}{173\,946\,175\,488\,000} - \frac{1\,193\,766\,593\,741\cos(3\,z)}{1\,363\,738\,015\,825\,920\,000} - \frac{10\,194\,121\cos(5\,z)}{86\,973\,087\,744\,000} + \frac{55\,617\,547\cos(7\,z)}{2\,435\,246\,456\,832\,000} - \frac{30\,253\cos(9\,z)}{53\,271\,016\,243\,200} - \frac{28\,361\cos(11\,z)}{1\,826\,434\,842\,624\,000} + \frac{17\cos(13\,z)}{106\,542\,032\,486\,400} - \frac{\cos(15\,z)}{3\,196\,260\,974\,592\,000} + \frac{\cos(19\,z)}{17\,579\,435\,360\,256\,000}\right) q^8 +$$
$$\left(\frac{64\,306\,539\,779\cos(z)}{10\,909\,904\,126\,607\,360\,000} - \frac{1\,087\,026\,917\cos(3\,z)}{3\,131\,031\,158\,784\,000} + \frac{767\,116\,375\,621\cos(5\,z)}{21\,819\,808\,253\,214\,720\,000} + \frac{468\,897\,223\cos(7\,z)}{170\,467\,251\,978\,240\,000} - \frac{21\,665\,887\cos(9\,z)}{75\,144\,747\,810\,816\,000} + \frac{7663\cos(11\,z)}{1\,704\,672\,519\,782\,400} + \frac{189\,053\cos(13\,z)}{2\,045\,607\,023\,738\,880\,000} - \frac{23\cos(15\,z)}{69\,039\,237\,051\,187\,200} - \frac{\cos(17\,z)}{281\,270\,965\,764\,096\,000} - \frac{\cos(21\,z)}{7\,594\,316\,075\,630\,592\,000}\right) q^9 +$$
$$\left(-\frac{78\,221\,421\,983\,189\cos(z)}{785\,513\,097\,115\,729\,920\,000} - \frac{3\,555\,989\,290\,829\cos(3\,z)}{99\,747\,694\,871\,838\,720\,000} + \frac{1\,595\,308\,088\,447\cos(5\,z)}{98\,189\,137\,139\,466\,240\,000} - \frac{16\,802\,983\,705\,367\cos(7\,z)}{23\,565\,392\,913\,471\,897\,600\,000} - \frac{50\,000\,417\cos(9\,z)}{1\,363\,738\,015\,825\,920\,000} + \frac{44\,442\,089\cos(11\,z)}{18\,410\,463\,213\,649\,920\,000} - \frac{113\,461\cos(13\,z)}{4\,418\,511\,171\,275\,980\,800} - \frac{74\,069\cos(15\,z)}{180\,013\,418\,089\,021\,440\,000} + \frac{\cos(17\,z)}{13\,807\,847\,410\,237\,440\,000} + \frac{\cos(19\,z)}{48\,603\,622\,884\,035\,788\,800} + \frac{\cos(23\,z)}{3\,949\,044\,359\,327\,907\,840\,000}\right) q^{10} + O\!\left(q^{11}\right)$$

11.01.06.0010.01

$$\mathrm{Ce}(a_6(q),\,q,\,z)\propto\cos(3z)+\left(\frac{\cos(z)}{8}-\frac{1}{16}\cos(5z)\right)q+\left(\frac{\cos(z)}{64}-\frac{5}{512}\cos(3z)+\frac{1}{640}\cos(7z)\right)q^2+\left(-\frac{\cos(z)}{4096}-\frac{1}{512}\cos(3z)+\frac{11\cos(5z)}{40960}-\frac{\cos(9z)}{46080}\right)q^3+\left(-\frac{21\cos(z)}{32768}-\frac{1621\cos(3z)}{13107200}+\frac{\cos(5z)}{16384}-\frac{11\cos(7z)}{2949120}+\frac{\cos(11z)}{5160960}\right)q^4+\left(-\frac{14061\cos(z)}{104857600}+\frac{9\cos(3z)}{131072}+\frac{12329\cos(5z)}{1887436800}-\frac{3\cos(7z)}{3276800}+\frac{\cos(9z)}{33030144}-\frac{\cos(13z)}{825753600}\right)q^5+\left(\frac{699\cos(z)}{838860800}+\frac{13050583\cos(3z)}{543581798400}-\frac{533\cos(5z)}{209715200}-\frac{76679\cos(7z)}{528482304000}+\frac{17\cos(9z)}{2123366400}-\frac{\cos(11z)}{6606028800}+\frac{\cos(15z)}{178362777600}\right)q^6+\left(\frac{31826419\cos(z)}{4348654387200}+\frac{70123\cos(3z)}{33554432000}-\frac{326021051\cos(5z)}{304405807104000}+\frac{1319\cos(7z)}{27179089920}+\frac{7831\cos(9z)}{4227858432000}-\frac{37\cos(11z)}{832359628800}+\frac{\cos(13z)}{2283043553280}-\frac{\cos(17z)}{49941577728000}\right)q^7+\left(\frac{300245939\cos(z)}{173946175488000}-\frac{1193766593741\cos(3z)}{1363738015825920000}-\frac{10194121\cos(5z)}{86973087744000}+\frac{55617547\cos(7z)}{2435246456832000}-\frac{30253\cos(9z)}{53271016243200}-\frac{28361\cos(11z)}{1826434842624000}+\frac{17\cos(13z)}{106542032486400}-\frac{\cos(15z)}{3196260974592000}+\frac{\cos(19z)}{17579435360256000}\right)q^8+\left(\frac{64306539779\cos(z)}{10909904126607360000}-\frac{1087026917\cos(3z)}{3131031158784000}+\frac{767116375621\cos(5z)}{21819808253214720000}+\frac{468897223\cos(7z)}{170467251978240000}-\frac{21665887\cos(9z)}{75144747810816000}+\frac{7663\cos(11z)}{1704672519782400}+\frac{189053\cos(13z)}{2045607023738880000}-\frac{23\cos(15z)}{69039237051187200}-\frac{\cos(17z)}{281270965764096000}-\frac{\cos(21z)}{7594316075630592000}\right)q^9+\left(-\frac{78221421983189\cos(z)}{785513097115729920000}-\frac{3555989290829\cos(3z)}{99747694871838720000}+\frac{1595308088447\cos(5z)}{98189137139466240000}-\frac{16802983705367\cos(7z)}{23565392913471897600000}-\frac{50000417\cos(9z)}{1363738015825920000}+\frac{44442089\cos(11z)}{18410463213649920000}-\frac{113461\cos(13z)}{4418511171275980800}-\frac{74069\cos(15z)}{180013418089021440000}+\frac{\cos(17z)}{13807847410237440000}+\frac{\cos(19z)}{48603622884035788800}+\frac{\cos(23z)}{3949044359327907840000}\right)q^{10}+O(q^{11})$$

Differential equations

Ordinary linear differential equations and wronskians

For the direct function itself

11.01.13.0001.01

$$w''(z)+(a-2q\cos(2z))\,w(z)=0\;/;\;w(z)=c_1\,\mathrm{Ce}(a,q,z)+c_2\,\mathrm{Se}(a,q,z)$$

11.01.13.0002.01

$$W_z(\text{Ce}(a, q, z), \text{Se}(a, q, z)) = \text{Ce}(a, q, 0)\, \text{Se}'(a, q, 0) - \text{Ce}'(a, q, 0)\, \text{Se}(a, q, 0)$$

11.01.13.0003.01

$$w''(z) - \frac{g''(z)}{g'(z)} w'(z) + (a - 2\, q \cos(2\, g(z)))\, g'(z)^2\, w(z) = 0 \;/;\; w(z) = c_1\, \text{Ce}(a, q, g(z)) + c_2\, \text{Se}(a, q, g(z))$$

11.01.13.0004.01

$$W_z(\text{Ce}(a, q, g(z)), \text{Se}(a, q, g(z))) = g'(z) \left(\text{Ce}(a, q, 0)\, \text{Se}'(a, q, 0) - \text{Ce}'(a, q, 0)\, \text{Se}(a, q, 0)\right)$$

11.01.13.0005.01

$$h(z)\, w''(z) + \left(-2\, h'(z) - \frac{h(z)\, g''(z)}{g'(z)}\right) w'(z) + \left((a - 2\, q \cos(2\, g(z)))\, h(z)\, g'(z)^2 + \frac{2\, h'(z)^2}{h(z)} - h''(z) + \frac{h'(z)\, g''(z)}{g'(z)}\right) w(z) = 0 \;/;$$
$$w(z) = c_1\, h(z)\, \text{Ce}(a, q, g(z)) + c_2\, h(z)\, \text{Se}(a, q, g(z))$$

11.01.13.0006.01

$$W_z(h(z)\, \text{Ce}(a, q, g(z)), h(z)\, \text{Se}(a, q, g(z))) = h(z)^2\, g'(z) \left(\text{Ce}(a, q, 0)\, \text{Se}'(a, q, 0) - \text{Ce}'(a, q, 0)\, \text{Se}(a, q, 0)\right)$$

11.01.13.0007.01

$$w''(z) + \frac{1 - r - 2\, s}{z} w'(z) + \left(b^2\, r^2\, (a - 2\, q \cos(2\, b\, z^r))\, z^{2\, r - 2} + \frac{s\, (r + s)}{z^2}\right) w(z) = 0 \;/;\; w(z) = c_1\, z^s\, \text{Ce}(a, q, b\, z^r) + c_2\, z^s\, \text{Se}(a, q, b\, z^r)$$

11.01.13.0008.01

$$W_z(z^s\, \text{Ce}(a, q, b\, z^r), z^s\, \text{Se}(a, q, b\, z^r)) = b\, r\, z^{r + 2\, s - 1} \left(\text{Ce}(a, q, 0)\, \text{Se}'(a, q, 0) - \text{Ce}'(a, q, 0)\, \text{Se}(a, q, 0)\right)$$

11.01.13.0009.01

$$w''(z) - (\log(r) + 2 \log(s))\, w'(z) + \left(b^2\, (a - 2\, q \cos(2\, b\, r^z)) \log^2(r)\, r^{2\, z} + \log(s)\, (\log(r) + \log(s))\right) w(z) = 0 \;/;$$
$$w(z) = c_1\, s^z\, \text{Ce}(a, q, b\, r^z) + c_2\, s^z\, \text{Se}(a, q, b\, r^z)$$

11.01.13.0010.01

$$W_z(s^z\, \text{Ce}(a, q, b\, r^z), s^z\, \text{Se}(a, q, b\, r^z)) = b\, r^z\, s^{2\, z} \log(r) \left(\text{Ce}(a, q, 0)\, \text{Se}'(a, q, 0) - \text{Ce}'(a, q, 0)\, \text{Se}(a, q, 0)\right)$$

Differentiation

Low-order differentiation

With respect to *z*

11.01.20.0001.01

$$\frac{\partial \text{Ce}(a, q, z)}{\partial z} = \text{Ce}'(a, q, z)$$

11.01.20.0002.01

$$\frac{\partial^2 \text{Ce}(a, q, z)}{\partial z^2} = (2\, q \cos(2\, z) - a)\, \text{Ce}(a, q, z)$$

Integration

Definite integration

Involving the direct function

11.01.21.0001.01

$$\int_{-\pi}^{\pi} \text{Ce}(a_n(q), q, t)\, \text{Ce}(a_m(q), q, t)\, dt = \pi\, \delta_{n,m} \;/;\; n \in \mathbb{Z} \land m \in \mathbb{Z} \land q \in \mathbb{R}$$

Operations

Limit operation

11.01.25.0001.01

$$\lim_{a\to\infty} \text{Ce}\left(a, q, \frac{z}{\sqrt{a}}\right) = \cos(z) \;/;\; q \in \mathbb{R}$$

Representations through more general functions

Through other functions

Involving spheroidal functions

11.01.26.0001.01

$$\text{Ce}(a_r(q), q, z) = \sqrt{\pi}\, \sqrt{\sin(z)}\, PS_{r-\frac{1}{2},\frac{1}{2}}\left(2\sqrt{q}, \cos(z)\right)$$

Representations through equivalent functions

With related functions

11.01.27.0001.01

$$\text{Ce}(a, q, z) = (-1)^n\, \text{Se}\left(a, -q, z + \frac{\pi}{2}\right) \;/;\; a = a_{2n+1}(q) \lor a = b_{2n+1}(q) \land n \in \mathbb{N}$$

Theorems

The odd and even solutions for the eigenmodes of an elliptical membrane

The odd and even solutions for the eigenmodes $\psi(r, \phi)$ of an elliptical membrane fixed at the boundary in elliptical coordinates $\{x = c\cosh(r)\cos(\phi),\; y = c\sinh(r)\sin(\phi)\}$ with semi-axes a and b are given by

$$\psi(r, \phi) \propto \text{Ce}(a_n(\tilde{q}), \tilde{q}, \phi)\; \text{Ce}(a_n(\tilde{q}), \tilde{q}, i\, r) \;/;\; a = c\cosh(r_0),\; b = c\sinh(r_0) \land \text{Ce}(a_n(\tilde{q}), \tilde{q}, i\, r_0) = 0 \land n \in \mathbb{N},$$

$$\psi(r, \phi) \propto \text{Se}(b_n(\tilde{q}), \tilde{q}, \phi)\; \text{Se}(b_n(\tilde{q}), \tilde{q}, i\, r) \;/;\; a = c\cosh(r_0),\; b = c\sinh(r_0) \land \text{Ce}(a_n(\tilde{q}), \tilde{q}, i\, r_0) = 0 \land n \in \mathbb{N}^+.$$

Other information

11.01.33.0001.01

$$\frac{\partial \frac{\int_{-\pi}^{\pi} e^{2 i q \cos(t)\cos(z)}\, \text{Ce}(a_r(q), q, t)\, dt}{\text{Ce}(a_r(q), q, z)}}{\partial z} = 0 \;/;\; r \in \mathbb{Q}$$

History

– E. L. Mathieu (1868, 1873)
– H. Weber (1869)
– G.W. Hill (1877)
– E. Heine (1878)
– G. Floquet (1883)
– R. C. Maclaurin (1898)
– J. Dougall (1916, 1926)

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