NorlundB2

Notations

Traditional name

Norlund polynomial

Traditional notation

$B_n^{(z)}$

Mathematica StandardForm notation

`NorlundB[n, z]`

Primary definition

05.16.02.0001.01

$$B_n^{(z)} == n!\left([t^n]\left(\frac{t}{e^t - 1}\right)^z\right) /; n \in \mathbb{N}$$

Specific values

Specialized values

For fixed n

05.16.03.0001.01

$$B_0^{(0)} == 1$$

05.16.03.0002.01

$$B_n^{(0)} == 0 /; n > 0$$

05.16.03.0003.01

$$B_n^{(-1)} == \frac{1}{n+1}$$

For fixed z

05.16.03.0004.01

$$B_0^{(z)} == 1$$

05.16.03.0005.01

$$B_1^{(z)} == -\frac{z}{2}$$

05.16.03.0006.01

$$B_2^{(z)} = \frac{1}{12} z (3 z - 1)$$

05.16.03.0007.01

$$B_3^{(z)} = -\frac{1}{8} (z - 1) z^2$$

05.16.03.0008.01

$$B_4^{(z)} = \frac{1}{240} z \left(15 z^3 - 30 z^2 + 5 z + 2\right)$$

05.16.03.0009.01

$$B_5^{(z)} = -\frac{1}{96} z^2 \left(3 z^3 - 10 z^2 + 5 z + 2\right)$$

05.16.03.0010.01

$$B_6^{(z)} = \frac{z \left(63 z^5 - 315 z^4 + 315 z^3 + 91 z^2 - 42 z - 16\right)}{4032}$$

05.16.03.0011.01

$$B_7^{(z)} = \frac{z^2 \left(-9 z^5 + 63 z^4 - 105 z^3 - 7 z^2 + 42 z + 16\right)}{1152}$$

05.16.03.0012.01

$$B_8^{(z)} = \frac{z \left(135 z^7 - 1260 z^6 + 3150 z^5 - 840 z^4 - 2345 z^3 - 540 z^2 + 404 z + 144\right)}{34\,560}$$

05.16.03.0013.01

$$B_9^{(z)} = -\frac{z^2 \left(15 z^7 - 180 z^6 + 630 z^5 - 448 z^4 - 665 z^3 + 100 z^2 + 404 z + 144\right)}{7680}$$

05.16.03.0014.01

$$B_{10}^{(z)} = \frac{1}{101\,376} z \left(-768 - 2288 z + 2068 z^2 + 11\,792 z^3 + 8195 z^4 - 8085 z^5 - 8778 z^6 + 6930 z^7 - 1485 z^8 + 99 z^9\right)$$

General characteristics

Domain and analyticity

The function $B_n^{(z)}$ is defined over $\mathbb{N} \otimes \mathbb{C}$. For fixed n, the function $B_n^{(z)}$ is a polynomial in z of degree n.

05.16.04.0001.01

$$(n * z) \longrightarrow B_n^{(z)} :: (\mathbb{N} \otimes \mathbb{C}) \longrightarrow \mathbb{C}$$

Symmetries and periodicities

Parity

No parity

Mirror symmetry

05.16.04.0002.01

$$B_n^{(\bar{z})} = \overline{B_n^{(z)}}$$

Periodicity

No periodicity

Poles and essential singularities

The function $B_n^{(z)}$ is polynomial and has pole of order n at $z = \tilde{\infty}$.

05.16.04.0003.01

$$\mathcal{Sing}_z\left(B_n^{(z)}\right) = \{\{\tilde{\infty}, n\}\}$$

Branch points

The function $B_n^{(z)}$ does not have branch points.

05.16.04.0004.01

$$\mathcal{BP}_z\left(B_n^{(z)}\right) = \{\}$$

Branch cuts

The function $B_n^{(z)}$ does not have branch cuts.

05.16.04.0005.01

$$\mathcal{BC}_z\left(B_n^{(z)}\right) = \{\}$$

Series representations

Generalized power series

Expansions at generic point $z = z_0$

05.16.06.0001.01

$$B_n^{(z)} \propto B_n^{(z_0)} + \left(\frac{\partial B_n^{(z)}}{\partial z} \Big|_{z=z_0}\right)(z-z_0) + \frac{1}{2}\left(\frac{\partial^2 B_n^{(z)}}{\partial z^2} \Big|_{z=z_0}\right)(z-z_0)^2 + \ldots /; (z \to z_0)$$

05.16.06.0002.01

$$B_n^{(z)} = B_n^{(z_0)} + \sum_{k=1}^{n} \frac{1}{k!}\left(\frac{\partial^k B_n^{(z)}}{\partial z^k} \Big|_{z=z_0}\right)(z-z_0)^k$$

05.16.06.0003.01

$$B_n^{(z)} = \delta_n + \sum_{k=0}^{n} \frac{(-1)^{k+n-1}}{k!} \sum_{i=0}^{n-k} (-1)^i (i+1)_k \sum_{r=1}^{i+k} S_{n+1}^{(r)} \sum_{j=1}^{n} (-1)^j j^{r-i-k-1} \binom{n}{j} p_{j,n} z_0^i (z-z_0)^k /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.16.06.0004.01

$$B_n^{(z)} \propto B_n^{(z_0)}(1 + O(z - z_0))$$

Expansions at $z == 0$

05.16.06.0005.01

$$B_n^{(z)} \propto \delta_n - n! \sum_{j=1}^{n} \frac{(-1)^{j+1}}{j} \binom{n}{j} p_{j,n}\, z - n! \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} \left(\frac{1}{j} - H_n\right) p_{j,n}\, z^2 + n! \sum_{j=1}^{n} \frac{(-1)^j \binom{n}{j}}{j} \left(-\frac{H_n}{j} + \frac{1}{2}\left(H_n^2 - H_n^{(2)}\right) + \frac{1}{j^2}\right) p_{j,n}\, z^3 + \ldots \;/;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.16.06.0006.01

$$B_n^{(z)} = \delta_n - (-1)^n \sum_{i=0}^{n} \sum_{k=1}^{i} S_{n+1}^{(k)} \sum_{j=1}^{n} (-1)^{i+j}\, j^{k-i-1} \binom{n}{j} p_{j,n}\, z^i \;/;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.16.06.0007.01

$$B_n^{(z)} \propto \delta_n + z \left(\frac{\delta_{n-1}}{2} + \left(n - 2\left\lfloor\frac{n}{2}\right\rfloor + 1\right) n! \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} p_{j,n} - \left(n - 2\left\lfloor\frac{n}{2}\right\rfloor\right) n! \sum_{j=1}^{n} \frac{(-1)^j}{j} \binom{n}{j} \left(\frac{1}{j} - H_n\right) p_{j,n}\, z\right) (1 + O(z)) \;/;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

Expansions at $z == \infty$

05.16.06.0008.01

$$B_n^{(z)} \propto \delta_n + (-1)^n\, 2^{-n}\, z^n \left(1 + 2^n \sum_{k=1}^{n-1} S_{n+1}^{(k)} \sum_{j=1}^{n} (-1)^{j+n}\, j^{k-n} \binom{n}{j} p_{j,n} \frac{1}{z} - 2^n \sum_{k=1}^{n-2} S_{n+1}^{(k)} \sum_{j=1}^{n} (-1)^{j+n}\, j^{k-n+1} \binom{n}{j} p_{j,n} \frac{1}{z^2} + \ldots\right) \;/;$$

$$(|z| \to \infty) \bigwedge p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N} \bigwedge n > 0$$

05.16.06.0009.01

$$B_n^{(z)} = \delta_n - z^n \sum_{i=0}^{n} \sum_{k=1}^{n-i} S_{n+1}^{(k)} \sum_{j=1}^{n} (-1)^{i+j}\, j^{i+k-n-1} \binom{n}{j} p_{j,n}\, z^{-i} \;/;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.16.06.0010.01

$$B_n^{(z)} \propto (-1)^n\, 2^{-n}\, z^n \left(1 + O\left(\frac{1}{z}\right)\right)$$

Other series representations

05.16.06.0011.01

$$B_n^{(z)} = (z)_{n+1} \sum_{j=0}^{n} \frac{(-1)^j}{j+z} \binom{n}{j} p_{j,n} \text{ /; } p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\, m + m - k)\, a_m\, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.16.06.0012.01

$$B_n^{(z)} = \sum_{k=1}^{n} \frac{(-1)^k (z)_k}{(n+1)_k} b_{k+n,k} \text{ /; }$$

$$\left(b_{n,k} = 0 \text{ /; } k > \frac{n}{2} \bigwedge k < 0\right) \bigwedge b_{0,0} = 1 \bigwedge b_{n,1} = 1 \bigwedge b_{n,k} = 1 \bigwedge b_{n,k} = (n-1)\, b_{n-2,k-1} + k\, b_{n-1,k} \bigwedge k \in \mathbb{N}$$

Generating functions

05.16.11.0001.01

$$B_n^{(z)} = n! \left([t^n] \left(\frac{t}{e^t - 1}\right)^z\right) \text{ /; } n \in \mathbb{N}$$

Identities

Functional identities

Relations between contiguous functions

Recurrence relations

05.16.17.0001.01

$$B_n^{(m)} = \frac{m-n-1}{m-1} B_n^{(m-1)} - n\, B_{n-1}^{(m-1)} \text{ /; } n \in \mathbb{N} \land m \in \mathbb{Z} \land m > 1$$

05.16.17.0002.01

$$B_n^{(m)} = \frac{n\, m}{m-n} B_{n-1}^{(m)} + \frac{m}{m-n} B_n^{(m+1)} \text{ /; } n \in \mathbb{N} \land m \in \mathbb{N} \land m \neq n$$

05.16.17.0003.01

$$B_n^{(n-z)} = \frac{(-1)^n}{\binom{z}{n}} \sum_{k=0}^{n} \binom{n-z}{k+n} \binom{n+z}{n-k} \binom{k+n-1}{n} B_n^{(k+n)} \text{ /; } n \in \mathbb{N}$$

Differentiation

Low-order differentiation

Forward shift operator:

05.16.20.0001.01

$$\frac{\partial B_n^{(z)}}{\partial z} = -\frac{n}{2} B_{n-1}^{(z)} - \sum_{k=2}^{n} \frac{\binom{n}{k} B_k}{k} B_{n-k}^{(z)}$$

Pavlyk O. (2006)

05.16.20.0002.01

$$\frac{\partial B_n^{(z)}}{\partial z} = (-1)^{n-1} \sum_{i=1}^{n} i \sum_{k=1}^{i} S_{n+1}^{(k)} \sum_{j=1}^{n} (-1)^{i+j} \, j^{k-i-1} \binom{n}{j} p_{j,n} \, z^{i-1} \,/;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{v=1}^{k} (j\,v + v - k)\, a_v \, p_{j,k-v} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

05.16.20.0003.01

$$\frac{n\,(n-1)}{4} B_{n-2}^{(z)} + \sum_{k=2}^{n-1} \frac{n}{k} \binom{n-1}{k} B_k \, B_{n-k-1}^{(z)} + \sum_{m=0}^{n-2} \sum_{k=2}^{m} \frac{n!\, B_{n-m} \; B_k}{k\,(n-m)\,k!\,(m-k)!\,(n-m)!} B_{m-k}^{(z)}$$

05.16.20.0004.01

$$\frac{\partial^2 B_n^{(z)}}{\partial z^2} = (-1)^{n-1} \sum_{i=0}^{n-2} (-1)^i \,(i+1)\,(i+2) \sum_{k=1}^{i+2} S_{n+1}^{(k)} \sum_{j=1}^{n} (-1)^j \, j^{k-i-3} \binom{n}{j} p_{j,n} \, z^{i-1} \,/;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{v=1}^{k} (j\,v + v - k)\, a_v \, p_{j,k-v} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

Symbolic differentiation

05.16.20.0005.01

$$\frac{\partial^m B_n^{(z)}}{\partial z^m} = \delta_m \, \delta_n - (-1)^{n+m} \sum_{i=0}^{n-m} (-1)^i \,(i+1)_m \sum_{k=1}^{i+m} S_{n+1}^{(k)} \sum_{j=1}^{n} (-1)^j \, j^{k-m-i-1} \binom{n}{j} p_{j,n} \, z^i \,/;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\,m + m - k)\, a_m \, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N} \bigwedge m \in \mathbb{N}$$

Fractional integro-differentiation

05.16.20.0006.01

$$\frac{\partial^\alpha B_n^{(z)}}{\partial z^\alpha} = \frac{\delta_n \, z^{-\alpha}}{\Gamma(1-\alpha)} - (-1)^n \sum_{i=0}^{n} \frac{i!}{\Gamma(i-\alpha+1)} \sum_{k=1}^{i} S_{n+1}^{(k)} \sum_{j=1}^{n} (-1)^{i+j} \, j^{k-i-1} \binom{n}{j} p_{j,n} \, z^{i-\alpha} \,/;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\,m + m - k)\, a_m \, p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

Integration

Indefinite integration

Involving only one direct function

05.16.21.0001.01

$$\int B_n^{(z)}\,dz = \delta_n\,z - (-1)^n \sum_{i=0}^{n} \sum_{k=1}^{i} S_{n+1}^{(k)} \sum_{j=1}^{n} \frac{(-1)^{i+j}\,j^{k-i-1}}{i+1} \binom{n}{j} p_{j,n}\,z^{i+1} /;$$

$$p_{j,0} = 1 \bigwedge p_{j,k} = \frac{1}{k} \sum_{m=1}^{k} (j\,m + m - k)\,a_m\,p_{j,k-m} \bigwedge a_k = \frac{1}{(k+1)!} \bigwedge k \in \mathbb{N}$$

Summation

Finite summation

05.16.23.0001.01

$$\sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} B_n^{(k\,z)} = B_n^{(-z)}$$

05.16.23.0002.01

$$\sum_{k=0}^{n} \binom{n-z}{k+n} \binom{n+z}{n-k} \binom{k+n-1}{n} B_n^{(k+n)} = (-1)^n \binom{z}{n} B_n^{(n-z)}$$

Infinite summation

05.16.23.0003.01

$$\sum_{k=0}^{\infty} \frac{B_k^{(k)}\,z^k}{k!} = \frac{z}{(z+1)\log(z+1)}$$

05.16.23.0004.01

$$\sum_{k=1}^{\infty} \frac{B_k^{(k)}\,z^k}{k\,k!} = -\log\left(\frac{z}{\log(z+1)}\right)$$

Representations through more general functions

Through other functions

05.16.26.0001.01

$$B_n^{(z)} = B_n^{(z)}(0)$$

Representations through equivalent functions

With related functions

05.16.27.0001.01

$$B_n^{(m)} = \frac{S_m^{(m-n)}}{\binom{m-1}{n}} /; m \in \mathbb{N}^+ \bigwedge n \in \mathbb{N} \bigwedge n < m$$

05.16.27.0002.01

$$B_n^{(-m)} = \frac{\mathcal{S}_{m+n}^{(m)}}{\binom{m+n}{n}} \;/;\; m \in \mathbb{N} \land n \in \mathbb{N} \land n \leq m$$

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