Pi

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Notations

Traditional name

π

Traditional notation

π

Mathematica StandardForm notation

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Pi
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Primary definition

02.03.02.0001.01

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

Specific values

02.03.03.0001.01

$$\pi = 3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342 \ldots$$

Above approximate numerical value of π shows 90 decimal digits.

General characteristics

The pi π is a constant. It is irrational and transcendental over $\mathbb{Q}$ positive real number.

Series representations

Generalized power series

Expansions for π

02.03.06.0001.01

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(-\frac{2r+1}{8k+5} - \frac{2r+1}{8k+6} + \frac{8r+4}{8k+1} + \frac{r}{8k+7} - \frac{8r}{8k+2} - \frac{4r}{8k+3} - \frac{8r+2}{8k+4} \right) /; r \in \mathbb{N}^+$$

02.03.06.0002.01

$$\pi == 2\log(2) + 4\sum_{k=0}^{\infty} \frac{1}{k+1} (-1)^{\left\lfloor \frac{k+1}{2} \right\rfloor}$$

02.03.06.0003.01

$$\pi == 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

02.03.06.0004.01

$$\pi == \frac{4}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{1}{2k+1} (-1)^{\left\lfloor \frac{k}{2} \right\rfloor}$$

02.03.06.0005.01

$$\pi == 16\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\, 5^{2k+1}} - 4\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\, 239^{2k+1}}$$

02.03.06.0006.01

$$\pi == 3\sqrt{3} \sum_{k=0}^{\infty} \frac{(-1)^k}{3k+1} - \log(2)\sqrt{3}$$

02.03.06.0007.01

$$\pi == 4\sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{4k+1} - 2\log\left(1+\sqrt{2}\right)$$

02.03.06.0008.01

$$\pi == \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \left(\frac{2}{4k+2} + \frac{1}{4k+3} + \frac{2}{4k+1} \right)$$

02.03.06.0009.01

$$\pi == \sum_{k=1}^{\infty} \frac{1}{k^3} \left(\frac{285}{2(2k+1)} - \frac{667}{32(4k+1)} - \frac{5103}{16(4k+3)} + \frac{35625}{32(4k+5)} - \frac{238}{k+1} \right)$$

02.03.06.0010.01

$$\pi == \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{8^k} \left(\frac{1}{6k+3} + \frac{1}{6k+5} + \frac{4}{6k+1} \right)$$

02.03.06.0011.01

$$\pi == \frac{8}{1+\sqrt{2}} \sum_{k=0}^{\infty} \left(\frac{1}{8k+1} - \frac{1}{8k+7} \right)$$

02.03.06.0012.01

$$\pi == \sum_{k=0}^{\infty} \frac{1}{16^k} \left(-\frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} + \frac{4}{8k+1} \right)$$

02.03.06.0013.01

$$\pi == 2\sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k+1)\,(2k)!!}$$

02.03.06.0014.01

$$\pi == \frac{5}{4}\sqrt{5}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{1}{2k+1}F_{2k+1}\,16^{-k}$$

02.03.06.0015.01

$$\pi == \sqrt{5}\sum_{k=0}^{\infty}\frac{(-1)^k\,2^{2k+3}\,F_{2k+1}}{(2k+1)\left(\sqrt{5}+3\right)^{2k+1}}$$

02.03.06.0016.01

$$\pi == 20\sum_{k=0}^{\infty}\frac{(-1)^k\,F_{2k+1}^2}{(2k+1)\left(\sqrt{10}+3\right)^{2k+1}}$$

02.03.06.0017.01

$$\pi == 12\sqrt{5}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(\frac{2\left(2-\sqrt{3}\right)}{\sqrt{16\left(2-\sqrt{3}\right)+1}+\sqrt{5}}\right)^{2k+1}F_{2k+1}$$

02.03.06.0018.01

$$\pi == 4\sum_{k=1}^{\infty}\tan^{-1}\left(\frac{1}{F_{2k+1}}\right)$$

02.03.06.0019.01

$$\pi == \sum_{k=1}^{\infty}\frac{\left(3^k-1\right)\zeta(k+1)}{4^k}$$

02.03.06.0044.01

$$\pi == -3\sqrt{3}+\frac{9}{2}\sqrt{3}\sum_{k=1}^{\infty}\frac{k}{\binom{2k}{k}}$$

02.03.06.0045.01

$$\pi == -\frac{18\sqrt{3}}{5}+\frac{27}{10}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^2}{\binom{2k}{k}}$$

02.03.06.0046.01

$$\pi == -\frac{135\sqrt{3}}{37}+\frac{81}{74}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^3}{\binom{2k}{k}}$$

02.03.06.0047.01

$$\pi == -\frac{432\sqrt{3}}{119}+\frac{81}{238}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^4}{\binom{2k}{k}}$$

02.03.06.0048.01

$$\pi == -\frac{243\sqrt{3}}{67} + \frac{81}{938}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^5}{\binom{2k}{k}}$$

02.03.06.0049.01

$$\pi == -\frac{23814\sqrt{3}}{6565} + \frac{243}{13130}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^6}{\binom{2k}{k}}$$

02.03.06.0050.01

$$\pi == -\frac{42795\sqrt{3}}{11797} + \frac{81}{23594}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^7}{\binom{2k}{k}}$$

02.03.06.0051.01

$$\pi == -\frac{2355156\sqrt{3}}{649231} + \frac{729}{1298462}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^8}{\binom{2k}{k}}$$

02.03.06.0052.01

$$\pi == -\frac{48314475\sqrt{3}}{13318583} + \frac{2187}{26637166}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^9}{\binom{2k}{k}}$$

02.03.06.0053.01

$$\pi == -\frac{365306274\sqrt{3}}{100701965} + \frac{2187}{201403930}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^{10}}{\binom{2k}{k}}$$

02.03.06.0054.01

$$\pi == -\frac{99760005\sqrt{3}}{27500287} + \frac{6561}{5005052234}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^{11}}{\binom{2k}{k}}$$

02.03.06.0055.01

$$\pi == -\frac{245273327208\sqrt{3}}{67613135957} + \frac{19683}{135226271914}\sqrt{3}\sum_{k=1}^{\infty}\frac{k^{12}}{\binom{2k}{k}}$$

02.03.06.0056.01

$$\pi = 4 - 2\sum_{k=0}^{\infty}\frac{k!}{\prod_{j=0}^{k}(2j+3)}$$

Candido Otero Ramos (2007)

Expansions for $1/\pi$

02.03.06.0020.01

$$\frac{1}{\pi} == \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!\,(26390k+1103)}{k!^4\,396^{4k}}$$

02.03.06.0021.01

$$\frac{1}{\pi} == \sum_{k=0}^{\infty} \frac{42\,k+5}{2^{12\,k+4}} \binom{2\,k}{k}^3$$

02.03.06.0022.01

$$\frac{1}{\pi} == 12 \sum_{k=0}^{\infty} \frac{(-1)^k\,(6\,k)!\,(545\,140\,134\,k+13\,591\,409)}{k!^3\,(3\,k)!\,\left(640\,320^3\right)^{k+\frac{1}{2}}}$$

The above Chudnovsky`s formula is used for the numerical computation of π in *Mathematica*.

Expansions for π^2

02.03.06.0023.01

$$\pi^2 == 6 \sum_{k=1}^{\infty} \frac{1}{k^2}$$

02.03.06.0024.01

$$\pi^2 == 8 \sum_{k=0}^{\infty} \frac{1}{(2\,k+1)^2}$$

02.03.06.0025.01

$$\pi^2 == 12 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}$$

02.03.06.0026.01

$$\pi^2 == 18 \sum_{k=1}^{\infty} \frac{k!^2}{k^2\,(2\,k)!}$$

02.03.06.0027.01

$$\pi^2 == 36 \sum_{k=0}^{\infty} \frac{(2\,k)!!}{(2\,k+1)!!\,2^{2\,k+2}\,(k+1)}$$

02.03.06.0028.01

$$\pi^2 == \sum_{k=1}^{\infty} \frac{1}{k^3} \left(\frac{45}{2\,(2\,k+1)} + \frac{384}{k+2} - \frac{1215}{2\,(2\,k+3)} - \frac{12}{k+1}\right)$$

02.03.06.0029.01

$$\pi^2 == 18 \sum_{k=0}^{\infty} \left(-\frac{3}{2\,(6\,k+2)^2} - \frac{1}{2\,(6\,k+3)^2} - \frac{3}{8\,(6\,k+4)^2} + \frac{1}{16\,(6\,k+5)^2} + \frac{1}{(6\,k+1)^2}\right) 64^{-k}$$

Expansions for π^3

02.03.06.0030.01

$$\pi^3 == 32 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2\,k-1)^3}$$

02.03.06.0057.01

$$\pi^3 = \frac{1}{16}\sum_{k=0}^{\infty}\frac{(-1)^k}{1024^k}\left(\frac{8}{(4k+2)^3}+\frac{1}{(4k+3)^3}+\frac{32}{(4k+1)^3}\right)+ \frac{5}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{64^k}\left(-\frac{192}{(12k+2)^3}+\frac{88}{(12k+3)^3}-\frac{8}{(12k+5)^3}+\frac{84}{(12k+6)^3}-\frac{4}{(12k+7)^3}+\frac{11}{(12k+9)^3}-\frac{12}{(12k+10)^3}+\frac{1}{(12k+11)^3}+\frac{32}{(12k+1)^3}\right)$$

G.Huvent (2006)

Expansions for π^4

02.03.06.0031.01

$$\pi^4 = 90\sum_{k=1}^{\infty}\frac{1}{k^4}$$

02.03.06.0032.01

$$\pi^4 = 96\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}$$

02.03.06.0058.01

$$\pi^4 = \frac{27}{164}\sum_{k=0}^{\infty}\frac{1}{2^{12k}}\left(-\frac{38912}{(24k+2)^4}+\frac{81920}{(24k+3)^4}-\frac{2048}{(24k+4)^4}-\frac{512}{(24k+5)^4}-\frac{23552}{(24k+6)^4}+\frac{256}{(24k+7)^4}-\frac{27648}{(24k+8)^4}-\frac{10240}{(24k+9)^4}-\frac{2432}{(24k+10)^4}-\frac{64}{(24k+11)^4}-\frac{3584}{(24k+12)^4}-\frac{32}{(24k+13)^4}-\frac{608}{(24k+14)^4}-\frac{1280}{(24k+15)^4}-\frac{1728}{(24k+16)^4}+\frac{8}{(24k+17)^4}-\frac{368}{(24k+18)^4}-\frac{4}{(24k+19)^4}-\frac{8}{(24k+20)^4}+\frac{160}{(24k+21)^4}-\frac{38}{(24k+22)^4}+\frac{1}{(24k+23)^4}+\frac{2048}{(24k+1)^4}\right)$$

G.Huvent (2006)

02.03.06.0059.01

$$\pi^4 = \frac{3240}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}$$

Expansions for π^6

02.03.06.0033.01

$$\pi^6 = 945\sum_{k=1}^{\infty}\frac{1}{k^6}$$

02.03.06.0034.01

$$\pi^6 = 960\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6}$$

Expansions for π^{2n}

02.03.06.0035.01

$$\pi^{2n} = \frac{(-1)^{n-1}(2n)!}{2^{2n-1}B_{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}} /; n \in \mathbb{N}^+$$

02.03.06.0060.01

$$\pi^{2n} = \frac{(-1)^n 2^{1-2n}(2n)!}{B_{2n}\left(\frac{1}{2}\right)}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^{2n}} /; n \in \mathbb{N}^+$$

02.03.06.0036.01

$$\pi^{2n} = \frac{(-1)^{n-1}\,2\,(2n)!}{(4^n-1)B_{2n}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n}} /; n \in \mathbb{N}^+$$

Expansions for π^{2n-1}

02.03.06.0061.01

$$\pi^{2n-1} = \frac{(-1)^{n-1}2^{2n}(2n-2)!}{E_{2n-2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{2n-1}} /; n \in \mathbb{N}^+$$

Exponential Fourier series

02.03.06.0042.01

$$\pi = x + 2\sum_{k=1}^{\infty}\frac{\sin(k x)}{k} /; x \in \mathbb{R} \land x > 0$$

02.03.06.0043.01

$$\pi = 4\sum_{k=0}^{\infty}\frac{(-1)^k\cos((2k+1)x)}{2k+1} /; x \in \mathbb{R}$$

02.03.06.0062.01

$$\pi = 4 - 2\sum_{k=0}^{\infty}\frac{k!}{\prod_{j=0}^{k}(2j+3)}$$

Candido Otero Ramos (2007)

02.03.06.0063.01

$$\pi = \lim_{n\to\infty} 2^{n+2} f(n) /; f(0) = 1 \bigwedge f(n) = \frac{f(n-1)}{1+\sqrt{1+f(n-1)^2}} \bigwedge n \in \mathbb{N}^+$$

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Other series representations

02.03.06.0040.01

$$\pi^{2n} = (2n+1)!\sum_{k_1=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}\prod_{j=1}^{n}\frac{1}{\left(\sum_{l=1}^{n}k_l\right)^2} /; n \in \mathbb{N}^+$$

Integral representations

On the real axis

Of the direct function

02.03.07.0001.01

$$\pi == 2\int_0^\infty \frac{1}{t^2+1}\,dt$$

02.03.07.0002.01

$$\pi == 4\int_0^1 \sqrt{1-t^2}\,dt$$

02.03.07.0003.01

$$\pi == 2\int_0^1 \frac{1}{\sqrt{1-t^2}}\,dt$$

02.03.07.0004.01

$$\pi == 2\int_0^\infty \frac{\sin(t)}{t}\,dt$$

02.03.07.0005.01

$$\pi == 2\int_0^\infty \frac{\sin^2(t)}{t^2}\,dt$$

02.03.07.0006.01

$$\pi == \frac{8}{3}\int_0^\infty \frac{\sin^3(t)}{t^3}\,dt$$

02.03.07.0007.01

$$\pi == 3\int_0^\infty \frac{\sin^4(t)}{t^4}\,dt$$

02.03.07.0008.01

$$\pi == \frac{384}{115}\int_0^\infty \frac{\sin^5(t)}{t^5}\,dt$$

02.03.07.0009.01

$$\pi == \frac{40}{11}\int_0^\infty \frac{\sin^6(t)}{t^6}\,dt$$

02.03.07.0010.01

$$\pi == \left(2^n\int_0^\infty \frac{\sin^n(t)}{t^n}\,dt\right)\Bigg/\left(n\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\frac{(-1)^k\,(n-2\,k)^{n-1}}{k!\,(n-k)!}\right)\,/;\,n\in\mathbb{N}^+$$

Involving the direct function

02.03.07.0011.01

$$\sqrt{\pi} == 2\int_0^\infty e^{-t^2}\,dt$$

Gaussian probability density integral

02.03.07.0012.01

$$\sqrt{\pi} = 2\sqrt{2}\int_0^\infty \sin(t^2)\,dt$$

Fresnel integral

02.03.07.0013.01

$$\sqrt{\pi} = 2\sqrt{2}\int_0^\infty \cos(t^2)\,dt$$

Fresnel integral

02.03.07.0014.01

$$\sqrt{\pi} = 2\int_0^1 \log^{\frac{1}{2}}\left(\frac{1}{t}\right)dt$$

02.03.07.0015.01

$$\sqrt{\pi} = \int_0^1 \frac{1}{\log^{\frac{1}{2}}\left(\frac{1}{t}\right)}\,dt$$

Involving related functions

02.03.07.0016.01

$$\pi = 2\,e\int_0^\infty \frac{\cos(t)}{t^2+1}\,dt$$

Product representations

02.03.08.0001.01

$$\pi = 2\prod_{k=1}^\infty \frac{4k^2}{(2k-1)(2k+1)}$$

02.03.08.0002.01

$$\pi = \frac{4}{\sqrt{2}}\prod_{k=1}^\infty \frac{4^{\left\lfloor\frac{k+1}{2}\right\rfloor}}{2k+1}$$

02.03.08.0003.01

$$\pi = 2\prod_{k=2}^\infty \sec\left(\frac{\pi}{2^k}\right)$$

02.03.08.0004.01

$$\pi = 3\prod_{k=0}^\infty \sec\left(\frac{\pi}{12\,2^k}\right)$$

02.03.08.0008.01

$$\pi = 2\,e\prod_{k=1}^\infty \left(1+\frac{2}{k}\right)^{(-1)^{k+1}k}$$

02.03.08.0009.01

$$\pi = \frac{6\,e}{\prod_{k=2}^\infty \left(1+\frac{2}{k}\right)^{(-1)^k k}}$$

02.03.08.0005.01

$$\frac{6}{\pi^2} = \prod_{k=1}^{\infty}\left(1-\frac{1}{p_k^2}\right) /;\ p_k \in \mathbb{P}$$

02.03.08.0006.01

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \times \ldots$$

02.03.08.0007.01

$$\frac{2}{\pi} = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{1}{2} \operatorname{Nest}\left[\sqrt{2+\#1}\ \&,\ 0,\ k\right]$$

Limit representations

02.03.09.0001.01

$$\pi = \lim_{n\to\infty} \left(2^{4n} \Big/ \left(n \binom{2n}{n}^2\right)\right)$$

02.03.09.0002.01

$$\pi = \lim_{n\to\infty} \frac{4}{n^2} \sum_{k=0}^{n} \sqrt{n^2-k^2}$$

02.03.09.0003.01

$$\pi = 4 \lim_{n\to\infty} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \frac{\sqrt{n-k^2}}{n}$$

02.03.09.0004.01

$$\pi = \lim_{n\to\infty} \frac{2^{4n+1}\, n!^4}{(2n+1)\,(2n)!^2}$$

02.03.09.0006.01

$$\pi = \lim_{n\to\infty} \frac{2\,(2n)!!^2}{(2n+1)\,(2n-1)!!^2}$$

02.03.09.0012.01

$$\pi = \lim_{n\to\infty} (-1)^n\, 2^{2-2n} \sum_{k=0}^{2n} (-1)^k \binom{4n}{2k+1} H_{2k+1} /;\ n \in \mathbb{N}$$

02.03.09.0013.01

$$\pi = \lim_{n\to\infty} \frac{n!^2\,(n+1)^{2n^2+n}}{2\,n^{2n^2+3n+1}}$$

Pete Koupriyanov

02.03.09.0007.01

$$\pi = 16 \lim_{n\to\infty} (n+1) \prod_{k=1}^{n} \frac{k^2}{(2k+1)^2}$$

02.03.09.0014.01

$$\pi = \lim_{n\to\infty} \sqrt{6\,\frac{\log\left(\prod_{k=1}^{n} F_k\right)}{\log(\operatorname{lcm}(F_1, F_2, \ldots, F_n))}} \;/;\; n \in \mathbb{N}$$

02.03.09.0008.01

$$\pi = -12 \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left|\cot(k\,\alpha)\right| \log(\left|\cos(k\,\alpha)\right|) \;/;\; \alpha \in \mathbb{R}$$

02.03.09.0009.01

$$\pi = \lim_{n\to\infty} \left(\alpha\, n \Big/ \left(\sum_{k=0}^{n} \delta_{\operatorname{sgn}(\cos(k\,\alpha)),-\operatorname{sgn}(\cos((k+1)\,\alpha))}\right)\right) \;/;\; 0 \leq \alpha \leq \pi$$

02.03.09.0010.01

$$\pi = \lim_{n\to\infty} \frac{2\,a_k^2}{s_k} \;/;$$

$$a_k = \frac{1}{2}(a_{k-1} + b_{k-1}) \bigwedge b_k = \sqrt{a_{k-1}\, b_{k-1}} \bigwedge s_k = s_{k-1} - 2^k\, c_k \bigwedge c_k = a_k^2 - b_k^2 \bigwedge a_0 = 1 \bigwedge b_0 = \frac{1}{\sqrt{2}} \bigwedge s_0 = \frac{1}{2}$$

02.03.09.0011.01

$$\pi = \lim_{n\to\infty} \frac{1}{\alpha_n} \;/;$$

$$\alpha_{n+1} = \left(\beta^{n+1} + 1\right)^4 \alpha_n - 2^{2\,n+3}\, \beta_{n+1} \left(\beta_{n+1}^2 + \beta_{n+1} + 1\right) \bigwedge \beta_{n+1} = \frac{1 - \sqrt[4]{1 - \beta_n^4}}{1 + \sqrt[4]{1 - \beta_n^4}} \bigwedge \alpha_0 = 6 - 4\sqrt{2} \bigwedge \beta_0 = \sqrt{2} - 1$$

02.03.09.0015.01

$$\pi = \lim_{n\to\infty} \frac{2^{n+1}}{2 - b_1} \left(\frac{b_n}{2} \sqrt{2 + b_{n-1} \sqrt{2 + b_{n-2} \sqrt{2 + \ldots + b_2 \sqrt{2 + \sin\left(\frac{\pi\, b_1}{4}\right)}}}}\right) \;/;$$

$$b_n = 1 \land b_{n-1} = -1 \land (b_k = 1 \land 2 \leq k \leq n-2 \land k \in \mathbb{N}) \land b_1 \in \mathbb{R} \land -2 \leq b_1 \leq 2$$

L. D. Servi: Nested Square Roots of 2 American Mathematical Monthly 110, 326-329 (2003)

02.03.09.0016.01

$$\pi = \lim_{n\to\infty} A(n) \;/;\; A(0) = 4 \bigwedge B(0) = \frac{1}{\sqrt{2}} \bigwedge A(n) = \frac{2\,A(n-1)\,B(n-1)}{B(n-1)+1} \bigwedge B(n) = \sqrt{\frac{1}{2}(B(n-1)+1)} \bigwedge n \in \mathbb{N}^+$$

Candido Otero Ramos (2007)

Continued fraction representations

02.03.10.0001.01

$$\pi == 3+1\left/\left(7+1\left/\left(15+1\left/\left(1+1\left/\left(292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{14+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\ldots}}}}}}}}}}}}\right)\right.\right)\right.\right)\right.\right)\right.$$

02.03.10.0002.01

$$\pi == 3+\cfrac{1}{6+\cfrac{9}{6+\cfrac{25}{6+\cfrac{49}{6+\cfrac{81}{6+\cfrac{121}{6+\ldots}}}}}}$$

02.03.10.0003.01

$$\pi == 3+\mathrm{K}_k\left((2\,k-1)^2,\,6\right)_1^{\infty}$$

02.03.10.0004.01

$$\frac{\pi}{2} == 1 - \frac{1}{3 - \frac{6}{1 - \frac{2}{3 - \frac{20}{1 - \frac{12}{3 - \frac{42}{1 - \frac{30}{3 - \ldots}}}}}}}$$

02.03.10.0005.01

$$\frac{\pi}{2} == 1 - \frac{1}{3 + \mathrm{K}_k\left(-\left(k - (-1)^k\right)\left(k - (-1)^k + 1\right), 2 + (-1)^k\right)_1^\infty}$$

02.03.10.0006.01

$$\frac{4}{\pi} == 1 + \frac{1}{3 + \frac{4}{5 + \frac{9}{7 + \frac{16}{9 + \frac{25}{11 + \frac{36}{13 + \ldots}}}}}}$$

02.03.10.0007.01

$$\frac{4}{\pi} == 1 + \mathrm{K}_k\left(k^2, 2\,k + 1\right)_1^\infty$$

02.03.10.0008.01

$$\frac{4}{\pi} == 1 + \frac{1}{2 + \frac{9}{2 + \frac{25}{2 + \frac{49}{2 + \frac{81}{2 + \frac{121}{2 + \ldots}}}}}}$$

02.03.10.0009.01

$$\frac{4}{\pi} == 1 + \mathrm{K}_k\left((2\,k - 1)^2, 2\right)_1^\infty$$

02.03.10.0010.01

$$\frac{12}{\pi^2} == 1 + \frac{1}{3 + \frac{16}{5 + \frac{81}{7 + \frac{256}{9 + \frac{625}{11 + \frac{1296}{13 + \ldots}}}}}}$$

02.03.10.0011.01

$$\frac{12}{\pi^2} == 1 + \underset{k=1}{\overset{\infty}{K}}\left(k^4, 2k+1\right)$$

02.03.10.0012.01

$$\frac{6}{\pi^2 - 6} == 1 + \cfrac{1}{1 + \cfrac{2}{1 + \cfrac{4}{1 + \cfrac{6}{1 + \cfrac{9}{1 + \cfrac{12}{1 + \ldots}}}}}}$$

02.03.10.0013.01

$$\frac{6}{\pi^2 - 6} == 1 + \underset{k=1}{\overset{\infty}{K}}\left(\left\lfloor \frac{k+1}{2} \right\rfloor \left\lfloor \frac{k+2}{2} \right\rfloor, 1\right)$$

Complex characteristics

Real part

02.03.19.0001.01

$$\operatorname{Re}(\pi) == \pi$$

Imaginary part

02.03.19.0002.01

$$\operatorname{Im}(\pi) == 0$$

Absolute value

02.03.19.0003.01

$$|\pi| == \pi$$

Argument

02.03.19.0004.01

$$\arg(\pi) == 0$$

Conjugate value

02.03.19.0005.01

$$\overline{\pi} == \pi$$

Signum value

02.03.19.0006.01

$$\operatorname{sgn}(\pi) == 1$$

Differentiation

Low-order differentiation

02.03.20.0001.01

$$\frac{\partial \pi}{\partial z} == 0$$

Fractional integro-differentiation

02.03.20.0002.01

$$\frac{\partial^{\alpha} \pi}{\partial z^{\alpha}} == \frac{z^{-\alpha} \pi}{\Gamma(1-\alpha)}$$

Integration

Indefinite integration

02.03.21.0001.01

$$\int \pi \, dz == \pi z$$

02.03.21.0002.01

$$\int z^{\alpha-1} \pi \, dz == \frac{z^{\alpha} \pi}{\alpha}$$

Integral transforms

Fourier exp transforms

02.03.22.0001.01

$$\mathcal{F}_t[\pi](z) == \sqrt{2} \, \pi^{3/2} \, \delta(z)$$

Inverse Fourier exp transforms

02.03.22.0002.01

$$\mathcal{F}_t^{-1}[\pi](z) == \sqrt{2} \, \pi^{3/2} \, \delta(z)$$

Fourier cos transforms

02.03.22.0003.01

$$\mathcal{F}c_t[\pi](z) == \frac{\pi^{3/2}}{\sqrt{2}} \delta(z)$$

Fourier sin transforms

02.03.22.0004.01

$$\mathcal{F}s_t[\pi](z) == \frac{\sqrt{2 \pi}}{z}$$

Laplace transforms

02.03.22.0005.01

$$\mathcal{L}_t[\pi](z) = \frac{\pi}{z}$$

Inverse Laplace transforms

02.03.22.0006.01

$$\mathcal{L}_t^{-1}[\pi](z) = \pi\,\delta(z)$$

Representations through more general functions

Through Meijer *G*

02.03.26.0014.01

$$\pi = \pi\, G_{0,1}^{1,0}(z \mid 0) + \pi\, G_{1,2}^{1,1}\left(z \,\middle|\, \begin{matrix} 1 \\ 1,\ 0 \end{matrix}\right)$$

Through other functions

02.03.26.0001.01

$$\pi = 4\left(4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)\right)$$

02.03.26.0008.01

$$\pi = 4\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{1}{3}\right)$$

02.03.26.0009.01

$$\pi = 8\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{1}{7}\right)$$

02.03.26.0010.01

$$\pi = 4\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{1}{5}\right) + 4\tan^{-1}\left(\frac{1}{8}\right)$$

02.03.26.0013.01

$$\pi = 4\left(6\tan^{-1}\left(\frac{1}{8}\right) + 2\tan^{-1}\left(\frac{1}{57}\right) + \tan^{-1}\left(\frac{1}{239}\right)\right)$$

Jeff Reid

02.03.26.0015.01

$$\pi = 4\left(\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{47}\right)\right)$$

Adam Bui (2007)

02.03.26.0016.01

$$\pi = 4\left(\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right)\right)$$

Adam Bui (2007)

02.03.26.0017.01

$$\pi = -\frac{4\left(\cot^{-1}(a) + 2\tan^{-1}\left(\frac{a-1}{a+\sqrt{2}\,\sqrt{a^2+1}+1}\right)\right)}{\frac{2\sqrt{a^2}}{a} - 2\sqrt{\frac{a}{a+i}}\,\sqrt{\frac{a+i}{a}} + 2\sqrt{\frac{a-i}{a}}\,\sqrt{\frac{a}{a-i}} - 1} \;/;\; a^2 \neq 1$$

Adam Bui &O.I. Marichev (2007)

02.03.26.0002.01

$$\pi = 88\tan^{-1}\left(\frac{1}{28}\right) + 8\tan^{-1}\left(\frac{1}{443}\right) - 20\tan^{-1}\left(\frac{1}{1393}\right) - 40\tan^{-1}\left(\frac{1}{11\,018}\right)$$

02.03.26.0011.01

$$\pi = 48\tan^{-1}\left(\frac{1}{18}\right) + 12\tan^{-1}\left(\frac{1}{70}\right) + 20\tan^{-1}\left(\frac{1}{99}\right) + 32\tan^{-1}\left(\frac{1}{307}\right)$$

02.03.26.0012.01

$$\pi = 640\tan^{-1}\left(\frac{1}{200}\right) - 4\tan^{-1}\left(\frac{1}{239}\right) - 16\tan^{-1}\left(\frac{1}{515}\right) - 32\tan^{-1}\left(\frac{1}{4030}\right) - 64\tan^{-1}\left(\frac{1}{50\,105}\right) - 64\tan^{-1}\left(\frac{1}{62\,575}\right) - 128\tan^{-1}\left(\frac{1}{500\,150}\right) - 320\tan^{-1}\left(\frac{1}{4\,000\,300}\right)$$

02.03.26.0003.01

$$\pi = 4\left(\tan^{-1}\left(\frac{p}{q}\right) + \tan^{-1}\left(\frac{q-p}{p+q}\right)\right) \;/;\; p \in \mathbb{N}^+ \land q \in \mathbb{N}^+$$

02.03.26.0004.01

$$\pi = 2\,K(0)$$

02.03.26.0005.01

$$\pi = 2\,E(0)$$

02.03.26.0006.01

$$\pi = \sqrt{6\,\mathrm{Li}_2(1)}$$

02.03.26.0007.01

$$\pi = \Gamma\left(\frac{1}{2}\right)^2$$

Representations through equivalent functions

02.03.27.0001.01

$$\pi = 180\,^{\circ}$$

02.03.27.0002.01

$$\pi = -i\log(-1)$$

02.03.27.0003.01

$$\pi = 2\,i\log\left(\frac{1-i}{1+i}\right)$$

02.03.27.0004.01

$$e^{\pi i} = -1$$

identity due to L. Euler

02.03.27.0005.01

$e^{2\pi i} = 1$

02.03.27.0006.01

$e^{\pi i k} = (-1)^k \;/;\; k \in \mathbb{Z}$

02.03.27.0007.01

$e^{-\frac{\pi}{2}} = i^i$

02.03.27.0008.01

$e^{\frac{\pi}{2} i k} = i^k \;/;\; k \in \mathbb{Z}$

Inequalities

02.03.29.0001.01

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}$$

B.C. Archimedes

02.03.29.0002.01

$$\left|\pi - \frac{q}{p}\right| > \frac{1}{q^{14.65}} \;/;\; p \in \mathbb{N}^+ \land q \in \mathbb{N}^+$$

02.03.29.0003.01

$e^{\pi} \geq \pi^{e}$

Theorems

Volume of an *n*-dimensional sphere

Volume V_n of an n-dimensional sphere of radius r:

$$V_{2k} = \frac{\pi^k}{k!} r^{2k}; \qquad V_{2k+1} = \frac{\pi^k 2^{2k+1} k!}{(2k+1)!} r^{2k+1}.$$

For instance, the area of a circle with radius r is πr^2 and the volume of a sphere with radius r is $\frac{4\pi}{3} r^3$.

Above general formulas can be joined into one $V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)} r^n$.

Surface area of an *n*-dimensional sphere

Surface area S_n of n-dimensional sphere of radius r:

$$S_{2k} = \frac{2\pi^k}{(k-1)!} r^{2k-1}; \qquad S_{2k+1} = \frac{\pi^k 2^{2k+1} k!}{(2k)!} r^{2k}.$$

For instance, the circumference of circle with radius r is $2\pi r$, and the surface area of a sphere with radius r is $4\pi r^2$.

Above general formulas can be joined into one $S_n == \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1}$.

Volume of an *n*-dimensional cylinder ??

Volume V_n of an n-dimensional cylinder of radius r and height h :

$$V_{2k} == \frac{\pi^k}{k!} r^{2k} h; \qquad V_{2k+1} == \frac{\pi^k 2^{2k+1} k!}{(2k+1)!} r^{2k+1} h.$$

For instance, the volume of a cylinder with radius r and height h is $\frac{4}{3} \pi r^3 h$.

Above general formulas can be joined into one $V_n == \frac{\pi^{\frac{n-1}{2}}}{n\,\Gamma\left(\frac{n+1}{2}\right)} r^{n-1} h$.

Surface area of an *n*-dimensional cylinder ??

Surface area S_n of n-dimensional cone of radius r and height h :

$$S_{2k} == \frac{2\pi^k}{k!} r^{2k-1}(h k + r); \qquad S_{2k+1} == \frac{2^{2k+1} \pi^k k!}{(2k+1)!} r^{2k} (h(2k+1) + 2r).$$

For instance, the volume of a cylinder with radius r and height h is $\frac{1}{3} \pi r^2 h$.

For instance, the surface area of a cylinder with radius r and height h is $\pi r\left(r + \sqrt{h^2 + r^2}\right)$.

Above general formulas can be joined into one $S_n == \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \left(r + \sqrt{h^2 + r^2}\right) r^{n-2}$.

Volume of an *n*-dimensional cone

Volume V_n of an n-dimensional cone of radius r and height h :

$$V_{2k} == \frac{2^{2k-1} \pi^{k-1} (k-1)!}{(2k)!} r^{2k-1} h; \qquad V_{2k+1} == \frac{\pi^k}{(2k+1)\,k!} r^{2k} h.$$

For instance, the volume of a cone with radius r and height h is $\frac{1}{3} \pi r^2 h$.

Above general formulas can be joined into one $V_n == \frac{\pi^{\frac{n-1}{2}}}{n\,\Gamma\left(\frac{n+1}{2}\right)} r^{n-1} h$.

Surface area of an *n*-dimensional cone

Surface area S_n of n-dimensional cone of radius r and height h:

$$S_{2k} == \frac{4^k \pi^{k-1} k!}{(2k)!} \left(r + \sqrt{h^2 + r^2}\right) r^{2k-2}; \qquad S_{2k+1} == \frac{\pi^k \left(r + \sqrt{h^2 + r^2}\right)}{k!} r^{2k-1}.$$

For instance, the volume of a cone with radius r and height h is $\frac{1}{3} \pi r^2 h$.

For instance, the surface area of a cone with radius r and height h is $\pi r \left(r + \sqrt{h^2 + r^2}\right)$.

Above general formulas can be joined into one $S_n == \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \left(r + \sqrt{h^2 + r^2}\right) r^{n-2}$.

Probability of two random integers being relatively prime

The probability that two integers picked at random are relatively prime is $\frac{6}{\pi^2}$.

History

– The design of Egyptian pyramids (c. 3000 BC) incorporated π as $3 + 1/7$ ($\sim 3.142857\ldots$);
– Egyptians (Rhind Papyrus, c. 2000 BC) gave π as $(16/9)^2 \sim 3.16045\ldots$
– China (c. 1200 BC) gave π as 3
– The Biblical verse I Kings 7:23 (c. 950 BC) gave π as $30/10 == 3.0$
– Archimedes (Greece, c. 240 BC) knew that $3 + 10/71 < \pi < 3 + 1/7$ and gave π as $3.1418\ldots$
– W. Jones (1706) introduced the symbol π
– C. Goldbach (1742) also used the symbol π
– J. H. Lambert (1761) established that π is an irrational number
– F. Lindemann (1882) proved that π is transcendental

The constant π is the most frequently encountered classical constant in mathematics and the natural sciences.

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