

# Root

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## Notations

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### Traditional name

The  $k$ th root of the polynomial equation  $\sum_{j=0}^n a_j z^j = 0$

### Traditional notation

$$\left( z; \sum_{j=0}^n a_j z^j \right)_k^{-1}$$

### Mathematica StandardForm notation

$$\text{Root}\left[\text{Function}\left[z, \sum_{j=0}^n a_j z^j\right], k\right]$$

## Primary definition

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$\left( z; \sum_{j=0}^n a_j z^j \right)_k^{-1}$  is the  $k$ th root of the polynomial equation  $\sum_{j=0}^n a_j z^j = 0$ .

01.33.02.0001.01

$$\left( \sum_{j=0}^n a_j z^j /; z = \left( z; \sum_{j=0}^n a_j z^j \right)_k^{-1} \right) = 0 /; k \in \mathbb{Z} \wedge 1 \leq k \leq n$$

## Specific values

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### Specialized values

01.33.03.0001.01

$$\left( z; a_0 + a_1 z \right)_1^{-1} = -\frac{a_0}{a_1}$$

01.33.03.0002.01

$$\left( z; a_0 + a_1 z + a_2 z^2 \right)_k^{-1} = -\frac{a_1}{2a_2} + \frac{(-1)^k}{2} \sqrt{\frac{a_1^2 - 4a_2 a_0}{a_2^2}} /; 1 \leq k \leq 2$$

01.33.03.0003.01

$$\left( z; z^2 - a \right)_2^{-1} = \sqrt{a}$$

01.33.03.0004.01

$$(z; a_0 + a_1 z + a_2 z^2 + a_3 z^3)_k^{-1} = -\frac{a_2}{3a_3} + \frac{e^{\frac{2i\pi(k-1)}{3}} p}{3 \cdot 2^{1/3} a_3} - \frac{2^{1/3} e^{-\frac{2i\pi(k-1)}{3}} r}{3 p a_3} /;$$

$$p = \sqrt[3]{q + \sqrt{4r^3 + q^2}} \quad \wedge \quad q = 9a_1 a_3 a_2 - 2a_2^3 - 27a_0 a_3^2 \quad \wedge \quad r = 3a_1 a_3 - a_2^2 \quad \wedge \quad 1 \leq k \leq 3$$

01.33.03.0005.01

$$(z; a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4)_k^{-1} = -\frac{a_3}{4a_4} + \left(2 \left\lfloor \frac{k-1}{2} \right\rfloor - 1\right) \epsilon_1 + (-1)^k \epsilon_2 (1 - \theta(k-3)) - (-1)^k (\theta(2-k) - 1) \epsilon_3 /;$$

$$\epsilon_1 = \frac{1}{2} \sqrt{\frac{\sqrt[3]{2} (a_2^2 - 3a_1 a_3 + 12a_0 a_4)}{3s a_4} + \frac{3a_2^2 + 2 \cdot 2^{2/3} s a_4 - 8a_2 a_4}{12a_4^2}} \quad \wedge$$

$$\epsilon_2 = \frac{\sqrt{u+v}}{2} \quad \wedge \quad \epsilon_3 = \frac{\sqrt{u-v}}{2} \quad \wedge \quad u = 8\epsilon_1^2 - \frac{\sqrt[3]{2} s^2 + 2a_2^2 - 6a_1 a_3 + 24a_0 a_4}{2^{2/3} s a_4} \quad \wedge$$

$$v = \frac{a_3^3 - 4a_2 a_4 a_3 + 8a_1 a_4^2}{8a_4^3 \epsilon_1} \quad \wedge \quad s = \sqrt[3]{2a_2^3 - 9(a_1 a_3 + 8a_0 a_4) a_2 + t + 27(a_4 a_1^2 + a_0 a_3^2)} \quad \wedge$$

$$t = \sqrt{\left((2a_2^3 - 9(a_1 a_3 + 8a_0 a_4) a_2 + 27(a_4 a_1^2 + a_0 a_3^2))^2 - 4(a_2^2 - 3a_1 a_3 + 12a_0 a_4)^3\right)} \quad \wedge \quad 1 \leq k \leq 4$$

01.33.03.0006.01

$$(z; z^n - a)_n^{-1} = a^{1/n} /; n = 2 \vee n - 3 \in \mathbb{N} \wedge a \notin (0, \infty)$$

## Values at fixed points

01.33.03.0007.01

$$(z; 1 + z^2)_2^{-1} = i$$

01.33.03.0008.01

$$(z; -1 - z + z^2)_2^{-1} = \varphi$$

## General characteristics

### Domain and analyticity

$(z; \sum_{j=0}^n a_j z^j)_k^{-1}$  is an analytic function of the  $a_j$  ( $0 \leq j \leq n$ ) in  $\mathbb{C}^{n+1}$ .

01.33.04.0001.01

$$(a_0 * a_1 * \dots * a_n) \rightarrow \left(z; \sum_{j=0}^n a_j z^j\right)_k^{-1} :: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$$

### Symmetries and periodicities

#### Symmetry

No symmetry

**Periodicity**

No periodicity

**Poles and essential singularities**

In most cases there are no poles, but poles up to order  $n - 1$  can be present.

**Branch points**

For generic values  $a_j$ , it has  $n - 1$  branchpoints of order 2. At most  $(z; \sum_{j=0}^n a_j z^j)_k^{-1}$  can have one branch point of order  $n$ .

**Branch cuts**

The location of the branch cuts is complicated. Generically the branch cuts run asymptotically radially outwards and do not connect branch points.

**Limit representations**

01.33.09.0001.01

$$\left(z; \sum_{j=0}^n a_j z^j\right)_k^{-1} = \lim_{m \rightarrow \infty} w_k^m /; w_k^{m+1} = w_k^m - \sum_{j=0}^n \frac{a_j (w_k^m)^j}{\prod_{i=1}^m \prod_{j=1}^m \text{If}[i \neq j, w_i^m - w_j^m, 1]} \wedge w_k^0 \in \mathbb{C} \wedge w_k^0 \neq w_i^0$$

**Differentiation**

**Low-order differentiation**

With respect to  $a_k$

01.33.20.0001.01

$$\frac{\partial}{\partial a_k} \left(z; \sum_{j=0}^n a_j z^j\right)_k^{-1} = - \frac{R^k}{k a_k R^{k-1} + \sum_{j=1}^n (1 - \delta_{k,j}) j a_j R^{j-1}} /; R = \left(z; \sum_{j=0}^n a_j z^j\right)_k^{-1}$$

**Representations through equivalent functions**

**With related functions**

01.33.27.0001.01

$$(x; x^2 - z)_2^{-1} = \sqrt{z}$$

01.33.27.0002.01

$$(z; z^n - a)_n^{-1} = a^{1/n} /; n = 2 \vee n - 3 \in \mathbb{N} \wedge a \notin (0, \infty)$$

01.33.27.0003.01

$$\sum_{j=0}^n a_j z^j = \prod_{k=1}^n (z - \alpha_k) /;$$

$$a_0 = (-1)^n \prod_{k=1}^n \alpha_k \wedge a_1 = (-1)^{n-1} \sum_{k=1}^n \prod_{j=1, j \neq k}^n \text{If}[j \neq k, \alpha_j, 1] \wedge \dots \wedge a_{n-2} = \sum_{k=1}^n \sum_{j=1}^{k-1} \text{If}[j \neq k, \alpha_j, \alpha_k, 0] \wedge a_{n-1} = -\sum_{k=1}^n \alpha_k$$

## Inequalities

01.33.29.0001.01

$$\left| \left( z; \sum_{j=0}^n a_j z^j \right)_k^{-1} \right| \leq 1 + \max \left( \left| \frac{a_0}{a_n} \right|, \left| \frac{a_1}{a_n} \right|, \dots, \left| \frac{a_{n-1}}{a_n} \right| \right)$$

## Theorems

### Salem number

The smallest known Salem number (a real algebraic integer greater than 1 whose conjugates have absolute value 1, at least one conjugate having absolute value =1) is given by  $(z; 1 + z - z^3 - z^4 - z^5 - z^6 - z^7 + z^9 + z^{10})_2^{-1} \propto 1.17628\dots$

### Pisot numbers

Raising simple algebraic numbers to a high power can yield numbers very near to integers. Example:

$$\left( (z; -1 - z + z^3)_1^{-1} \right)^{10000} \text{ is within } 10^{-611} \text{ of an integer.}$$

### Unequality for rational numbers

For any algebraic  $\alpha$  of degree greater than 1 there exists a  $c(\alpha)$  such that  $\left| \frac{p}{q} - \alpha \right| > \frac{c(\alpha)}{q^2}$  for all rational numbers  $\frac{p}{q}$ .

### The Lagrange points of the restricted three-body problem

The Lagrange points  $L_1, L_2,$  and  $L_3$  of the restricted three-body problem with potential

$V(x, y) = -\frac{1}{2}(x^2 + y^2) - \frac{1-\mu}{\sqrt{y^2+(x-x_1)^2}} - \frac{\mu}{\sqrt{y^2+(x-x_2)^2}}$  are given by  $\{\tilde{x}_i, 0\}$ , where the  $\tilde{x}_i$  are the real solutions of the quintic polynomial equation

$$x^5 - 2(x_1 + x_2)x^4 + (x_1^2 + 4x_2x_1 + x_2^2)x^3 - (2x_1x_2(x_1 + x_2) + 1)x^2 + (x_1^2x_2^2 + 2\mu x_1 - 2(\mu - 1)x_2)x + (\mu - 1)x_2^2 - \mu x_1^2 = 0.$$

### Gauss-Lucas theorem

$$\max \left( \left| \left( z; \sum_{j=0}^n a_j z^j \right)_1^{-1} \right|, \left| \left( z; \sum_{j=0}^n a_j z^j \right)_2^{-1} \right|, \dots, \left| \left( z; \sum_{j=0}^n a_j z^j \right)_n^{-1} \right| \right) \geq$$

$$\max \left( \left| \left( z; \frac{\partial}{\partial z} \sum_{j=0}^n a_j z^j \right)_1^{-1} \right|, \left| \left( z; \frac{\partial}{\partial z} \sum_{j=0}^n a_j z^j \right)_2^{-1} \right|, \dots, \left| \left( z; \frac{\partial}{\partial z} \sum_{j=0}^{n-1} a_j z^j \right)_{n-1}^{-1} \right| \right).$$

### The hard hexagon entropy constant

The hard hexagon entropy constant is an algebraic number of degree 24.

### History

- J. L. Lagrange (1767-1772)
- P. Ruffini (1799)
- C. F. Gauss (1801)
- N. H. Abel (1826)
- J. C. F. Sturm (1829)
- E. Galois (1832)
- G. Eisenstein (1844)
- J. Cockle, J. K. Thomae (1869)
- C. Jordan (1870)
- F. von Lindermann (1884,1892)
- L. Kronecker (1890,1891)
- R. H. Mellin (1915)
- R. Birkeland (1905-1925)

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